Math 132, Test 3, November 10, 2009 Name: .

No graphing calculators or electronic communication of any kind. If you need extra paper please ask. Credit will be awarded for correct content and clarity of presentation. Prepare for math battle. This test has 1500 points. There is a take-home bonus problem at the end with instructions. Make sure to at least attempt each part except the bonus. If I ask for the "complete power series representation" for some function that means I want you to give your answer in the general format

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

where the coefficients c_n are simplified as much as is possible. Of course in the event that the even and odd terms are different you might write two summations and/or write a few low order terms separately if they break from some pattern. Finally, please assume the all functions are analytic on the interior or their respective domains. I do not expect you to prove that any function on this test has a power series which converges to the function on some domain.

1.[150pts.] Find the IOC and ROC for the power series given below: (justify your answer with appropriate tests as was done in lecture, if you use a particular test then name it.)

$$g(x) = \sum_{n=1}^{\infty} n^2(2x-3)^n$$

$$L = \lim_{n\to\infty} \left| \frac{(n+1)^2(2x-3)^{n+1}}{n^2(2x-3)^n} \right|$$

$$= \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^2 |2x-3|$$

$$= \lim_{n\to\infty} \left(1+\underbrace{\frac{1}{n}}_{0}\right)^2 |2x-3|$$

$$= |2x-3| < 1 \Rightarrow -1 < 2x-3 < 1$$
$$\Rightarrow 2 < 2x < 4$$
$$\Rightarrow 1 < x < 2 \quad \therefore \quad (1,2) \subseteq \text{I.O.C.}$$

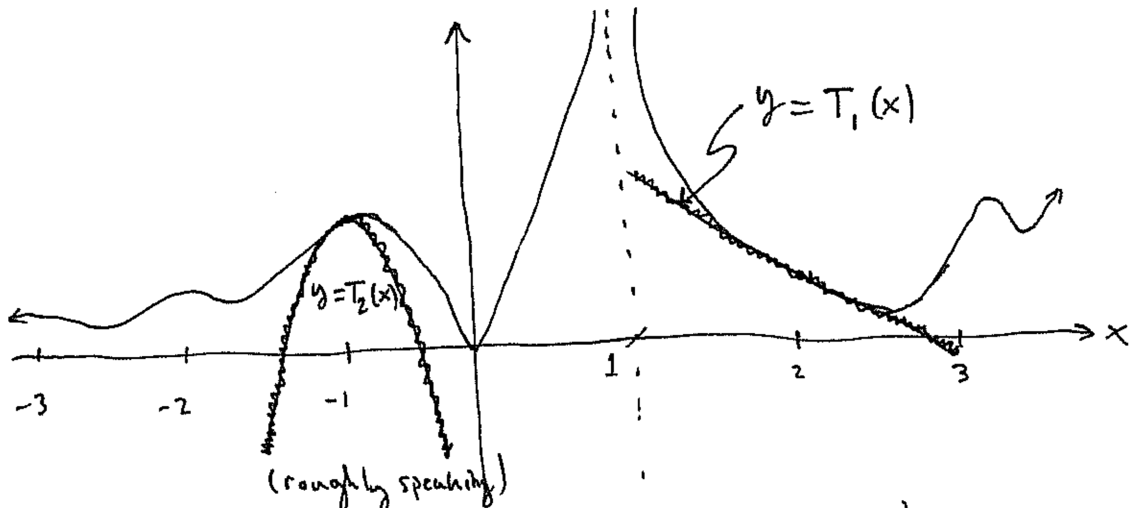
Check Endpts.

$$g(1) = \sum_{n=1}^{\infty} n^2(-1)^n$$
$$g(2) = \sum_{n=1}^{\infty} n^2(1)^n$$

both diverge by n^{th} term test

$$\therefore \boxed{\begin{array}{l} \text{I.O.C.} = (1,2) \\ \text{R.O.C.} = 1/2 \end{array}}$$

2.[100pts] Given the graph of $y = f(x)$ below answer the questions below the graph.



(YES THE GRAPH HAS A SHARP CORNER AT (0,0))

(a.) what appears to be the largest IOC for a power series representation of the function centered at $a = 2$?

Cannot cross the vertical asymptote at $x = 1$ hence $(1, 3)$ is largest I.O.C. possible.

(b.) what appears to be the largest IOC for a power series representation of the function centered at $a = -1$?

Cannot cross the corner because $f'(0)$ d.n.e and if our power series matched $f(x)$ past zero then we could calculate $f'(0), f''(0)$ etc.... Thus the largest I.O.C. is $(-2, 0)$.

(c.) sketch $y = T_1(x)$ centered at $a = 2$ on the given graph.

(it's a line, the tangent line, see graph)
(I made them wiggle to stand apart)

(d.) sketch $y = T_2(x)$ centered at $a = -1$ on the given graph.

(it's a parabola, see graph)

3. [150pts] Find the complete power series representation centered about zero for the function given below, state the IOC and ROC for the power series.

$$f(x) = \frac{3}{1+x^2}$$

identify $a = 3$ and $r = -x^2$ for the geometric series result,

$$f(x) = \sum_{n=0}^{\infty} 3(-x^2)^n \quad \text{for } |r|<1 \Rightarrow |-x|^2<1 \Rightarrow |x|<1$$

$$= \boxed{\sum_{n=0}^{\infty} 3(-1)^n x^{2n}} \quad \text{for } -1<x<1$$

$$\boxed{\begin{array}{l} \text{I.O.C.} = (-1,1) \\ \text{R.O.C.} = 1 \end{array}}$$

4.[150pts] Find the complete power series centered at $a = 0$ for the function $f(x) = 3 + \underbrace{\ln(x^4+1)}$.

Can use geometric series tricks, and upon a little thought the 3 doesn't spoil the trick.

$$\frac{df}{dx} = \frac{4x^3}{1+x^4} = \sum_{n=0}^{\infty} 4x^3(-x^4)^n$$

$$\Rightarrow \frac{df}{dx} = \sum_{n=0}^{\infty} 4(-1)^n x^{4n+3}$$

$$f(x) = \int \frac{df}{dx}\,dx = C + \sum_{n=0}^{\infty} \frac{4(-1)^n}{4n+4} x^{4n+4}$$

Note $f(0) = 3 + \ln(1) = C + 0 \Rightarrow \underline{C = 3}$.

$$\therefore \quad \boxed{f(x) = 3 + \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{4n+4} = 3 + x^4 - \frac{1}{2}x^8 + \dots}$$

5.[100pts] Find the complete power series solution for the following indefinite integral

$$\int x^{13} \cos(x+3)\,dx$$

Note

$$x^{13}\cos(x+3) = x^{13}\left(\cos(x)\cos(3) - \sin(x)\sin(3)\right)$$

$$= \cos(3)\,x^{13}\cos(x) - \sin(3)\,x^{13}\sin(x)$$

$$= \cos(3)\,x^{13}\left(\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n}\right) - \sin(3)\,x^{13}\left(\sum_{j=0}^{\infty}\frac{(-1)^j}{(2j+1)!}x^{2j}\right)$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n\cos(3)}{(2n)!}x^{2n+13} + \sum_{j=0}^{\infty}\frac{(-1)^{j+1}\sin(3)}{(2j+1)!}x^{2j+14}$$

Then integrate,

$$\int x^{13}\cos(x+3)\,dx = \int\left(\sum_{n=0}^{\infty}\frac{(-1)^n\cos(3)}{(2n)!}x^{2n+13} + \sum_{j=0}^{\infty}\frac{(-1)^{j+1}\sin(3)}{(2j+1)!}x^{2j+14}\right)d$$

$$\boxed{= C + \sum_{n=0}^{\infty}\frac{(-1)^n\cos(3)}{(2n+14)(2n)!}x^{2n+14} + \sum_{j=0}^{\infty}\frac{(-1)^{j+1}\sin(3)}{(2j+15)(2j+1)!}x^{2j+15}}$$

6.[500pts] Find the first three non-zero terms in the power series centered at a for the functions given below

(a.) $f(x) = \dfrac{1}{\sqrt{1-4x}}$ find the first three non-zero terms power series representation centered at $a = 0$.

$$f(x) = (1-4x)^{-1/2}$$
$$= (1+u)^k \quad \text{for } u = -4x, \; k = -1/2$$
$$= 1 + ku + \tfrac{1}{2}k(k-1)u^2 + \cdots$$
$$= 1 - \tfrac{1}{2}(-4x) + \tfrac{1}{2}\left(\tfrac{-1}{2}\right)\left(\tfrac{-3}{2}\right)(-4x)^2 + \cdots$$
$$= 1 + 2x + \tfrac{3}{8}16x^2 + \cdots$$
$$= \boxed{1 + 2x + 6x^2 + \cdots}$$

(b.) find . the first three non-zero terms power series centered at $a = 0$ for $f(x) = \tan(x+3)$

$$\Rightarrow f(0) = \tan(3)$$
$$f'(x) = \sec^2(x+3) \quad \Rightarrow f'(0) = \sec^2(3)$$
$$f''(x) = 2\sec(x+3)\cdot\sec(x+3)\tan(x+3) \quad \Rightarrow f''(0) = 2\sec^2(3)\tan(3)$$

$$\therefore \quad \tan(x+3) = f(0) + f'(0)x + \tfrac{1}{2}f''(0)x^2 + \cdots$$
$$= \boxed{\tan(3) + \sec^2(3)x + \sec^2(3)\tan(3)x^2 + \cdots}$$

Aternatively, could divide series, (less fun).

(c.) find the first three non-zero terms power series centered at $a = 0$ for $f(x) = \sin(x)e^x + \dfrac{1}{1-x}$: notice we'll have nonzero terms at each order so clearly we only need upto x^2 or so.

$$= \left(x - \frac{1}{6}x^3 + \cdots\right)\left(1 + x + \frac{1}{2}x^2 + \cdots\right) + 1 + x + x^2 + \cdots$$

$$= x + x^2 + \frac{1}{2}x^3 - \frac{1}{6}x^3 + \cdots + 1 + x + x^2 + x^3 + \cdots$$

$$= \boxed{1 + 2x + 2x^2 + \cdots}$$

As a check,

$$f(0) = \sin(0)e^0 + \frac{1}{1-0} = 1.$$

$$f'(0) = \cos(0)e^0 + \sin(0)e^0 + \frac{1}{(1-0)^2} = 1 + 1 = 2$$

$$f''(0) = -\sin(0)e^0 + \cos(0)e^0 + \cos(0)e^0 + \sin(0)e^0 + \frac{2}{(1-0)^3} = 1 + 1 + 2 = 4$$

and $\dfrac{f''(0)}{2!} = \dfrac{4}{2} = 2.$ (good!)

(d.) find the first ~~three~~ four non-zero terms power series centered at $a = 1$ for $f(x) = 2x^3 + 4x^2 - x + 1.$

$f(1) = 2 + 4 - 1 + 1 = 6.$

$f'(x) = 6x^2 + 8x - 1$ $\qquad$ $f'(1) = 6 + 8 - 1 = 13.$

$f''(x) = 12x + 8$ $\qquad$ $f''(1) = 12 + 8 = 20.$

$f'''(x) = 12$ $\qquad$ $f'''(1) = 12.$

$f^{(n)}(x) = 0 \quad n \geq 4.$

Hence, noting $\frac{20}{2!} = 10$ and $\frac{12}{3!} = \frac{12}{6} = 2,$

$$\boxed{f(x) = 6 + 13(x-1) + 10(x-1)^2 + 2(x-1)^3}$$

7.[100pts] The binomial series applied to the function $\sqrt{1+x}$ yields:

$$\sqrt{1+x} = 1 + \tfrac{1}{2}x - \tfrac{1}{8}x^2 + \tfrac{1}{16}x^3 + \cdots.$$

for $x \in (-1,1)$. Place an upper bound on the error in the approximation

$$\sqrt{1+\tfrac{1}{2}} = 1 + \tfrac{1}{4} - \tfrac{1}{32} + error$$

$$\sqrt{1+\tfrac{1}{2}} = 1 + \tfrac{1}{2}\left(\tfrac{1}{2}\right) - \tfrac{1}{8}\left(\tfrac{1}{2}\right)^2 + \tfrac{1}{16}\left(\tfrac{1}{2}\right)^3 + \cdots$$

alternating series

$$\Rightarrow \quad |error| \le \frac{1}{16}\left(\frac{1}{2}\right)^3 = \boxed{\frac{1}{128}}.$$

estimation Th$^{\underline{m}}$

(0.007813)

8.[100pts] The Fourier coefficients for a function $f(x)$ with period $2L = 2\pi$ are

$$a_0 = 1, \quad a_n = 0, \quad b_n = \frac{1}{n}$$

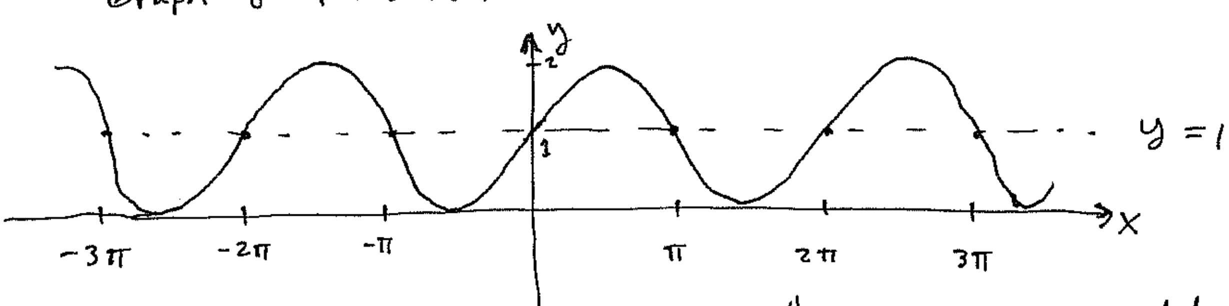
for $n = 1, 2, 3, \ldots$ Use the given coefficients to write the Fourier series for $f(x)$ and then sketch the graph of the terms up to n=1 over the interval $-3\pi \le x \le 3\pi$. Is $f(x)$ an even function, and odd function, or neither even nor odd?

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right)\right)$$

$$\Rightarrow \quad f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos(nx) + b_n \sin(nx)$$

$$\boxed{f(x) = 1 + \sum_{n=1}^{\infty} \frac{1}{n}\sin(nx) = 1 + \sin(x) + \tfrac{1}{2}\sin(2x) + \cdots}$$

Graph $y = 1 + \sin(x)$



neither even nor odd.

9.[150pts] Find the complete power series centered at $a = 0$ for the function $f(x) = \frac{x}{3+x} + xe^{2x^2}$. Also, calculate $f^{(99)}(0)$. You can leave the value of the derivative in terms of powers and factorials of particular integers, I don't want or expect a decimal answer.

$$f(x) = \sum_{n=0}^{\infty} \frac{x}{3}\left(\frac{-x}{3}\right)^n + x\sum_{n=0}^{\infty} \frac{(2x^2)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{3^{n+1}} x^{n+1} + \sum_{n=0}^{\infty} \frac{2^n}{n!} x^{2n+1}$$

: used Mac. series for exponential and also geom. series noting $\frac{x}{3+x} = \frac{x/3}{1+x/3}$ $\therefore a = \frac{x}{3},\ r = \frac{-x}{3}$

Split the sum over n into even/odd powers so we can see how to simplify the whole expression later...

$$f(x) = \underbrace{\sum_{k=0}^{\infty} \frac{(-1)^{2k}}{3^{2k+1}} x^{2k+1}}_{\text{odd powers}} + \underbrace{\sum_{j=0}^{\infty} \frac{(-1)^{2j+1}}{3^{2j+2}} x^{2j+2}}_{\text{even powers}} + \underbrace{\sum_{n=0}^{\infty} \frac{2^n}{n!} x^{2n+1}}_{\text{odd powers}}$$

$$= \sum_{k=0}^{\infty}\left(\frac{(-1)^{2k}}{3^{2k+1}} + \frac{2^k}{k!}\right)x^{2k+1} + \sum_{j=0}^{\infty} \frac{(-1)^{2j+1}}{3^{2j+2}} x^{2j+2}$$

: $n = j+1$, $j = n-1$, $2j+1 = 2n$

$$= \boxed{\sum_{k=0}^{\infty}\left(\frac{(-1)^{2k}}{3^{2k+1}} + \frac{2^k}{k!}\right)x^{2k+1} + \sum_{n=1}^{\infty} \frac{(-1)^{2n-1}}{3^{2n}} x^{2n}}$$

$$= \boxed{\sum_{k=0}^{\infty}\left(\frac{1}{3^{2k+1}} + \frac{2^k}{k!}\right)x^{2k+1} + \sum_{n=1}^{\infty}\left(\frac{-1}{3^{2n}}\right)x^{2n}}$$

We need to isolate the 99^{th} power, $2k+1 = 99 \Rightarrow k = \frac{98}{2} = 4$

thus the term $k = 49$ gives x^{99}. Taylor's Th^{m} says

$$\frac{f^{(99)}(0)}{(99)!} = \left(\frac{1}{3^{99}} + \frac{2^{49}}{(49)!}\right) \quad \therefore \boxed{f^{(99)}(0) = (99)!\left[\frac{1}{3^{99}} + \frac{2^{49}}{(49)!}\right]}$$