Name:

MATH 221	BOSS FIGHT 1: MATRICES, LINEAR EQNS, RANK AND NULLITY (200+20PTS)

Show your work where appropriate and box answers. **Standard parametric form** means use non-pivotal variables as parameters for solution set. Thanks! You may use a non-graphing calculator on this test and you are allowed a 3×5 inch card of notes.

Problem 1: (24pts) Find the solution of $\left\{ \begin{array}{c} x+y-z=7 \\ 2x+y-2z=8 \\ 3x-4y+z=7 \end{array} \right\}$

$$\left[\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 2 & 1 & -2 & 8 \\ 3 & -4 & 1 & 7 \end{array}\right] \xrightarrow[r_3-3r_1]{r_2-2r_1} \left[\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 0 & -1 & 0 & -6 \\ 0 & -7 & 4 & -14 \end{array}\right] \xrightarrow[r_3-7r_2]{r_1+r_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & -1 & 0 & -6 \\ 0 & 0 & 4 & 28 \end{array}\right]$$

$$\xrightarrow[r_3/4]{-r_2} \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 7 \end{array}\right] \xrightarrow{r_1+r_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 8 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & 7 \end{array}\right] \quad \begin{array}{l} x=8 \\ y=6 \\ z=7 \end{array}$$

$$\boxed{(8,6,7)}$$

Problem 2: (16pts) Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 9 & 9 & 9 \end{bmatrix}$ let e_1, e_2, e_3, e_4 denote the standard basis for $\mathbb{R}^4$ and $\bar{e}_1, \bar{e}_2, \bar{e}_3$ denote the standard basis for $\mathbb{R}^3$. Calculate the following:

(a.) $A[e_1|e_4] = [Ae_1 \mid Ae_4] = [\text{col}_1(A) \mid \text{col}_4(A)] = \boxed{\begin{bmatrix} 1 & 4 \\ 5 & 8 \\ 9 & 9 \end{bmatrix}}$

(b.) $(A^T[\bar{e}_1|\bar{e}_2])^T = [\bar{e}_1 | \bar{e}_2]^T (A^T)^T = \left[\begin{array}{c} \bar{e}_1^T \\ \hline \bar{e}_2^T \end{array}\right] A = \left[\begin{array}{c} \bar{e}_1^T A \\ \hline \bar{e}_2^T A \end{array}\right] = \boxed{\begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}}$

Problem 3: (30pt) Let $a, b, c \in \mathbb{R}$. Calculate $\begin{bmatrix} 1 & -2 & 0 \\ 1 & -2 & 1 \\ 1 & -1 & 0 \end{bmatrix}^{-1}$ and solve $\left\{ \begin{array}{rl} x_1 - 2x_2 & = a \\ x_1 - 2x_2 + x_3 & = b \\ x_1 - x_2 & = c \end{array} \right\}$.

$$\left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow[r_3 - r_1]{r_2 - r_1} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right] \xrightarrow{r_1 + 2r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 2 \\ 0 & 0 & 1 & -1 & 1 & 0 \\ 0 & 1 & 0 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 0 & 2 \\ 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 1 & 0 \end{array}\right] \Rightarrow \boxed{\begin{bmatrix} 1 & -2 & 0 \\ 1 & -2 & 1 \\ 1 & -1 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} -1 & 0 & 2 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & -2 & 0 \\ 1 & -2 & 1 \\ 1 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 2 \\ -1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$\therefore \boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -a + 2c \\ -a + c \\ -a + b \end{bmatrix}}$$

Problem 4: Let $v_1 = (1, 2, 2, 1)$ and $v_2 = (1, 3, 2, 4)$ and $w = (1, 0, 6, 4)$.

(a.) (10pts) Determine if $w \in span\{v_1, v_2\}$

$$x V_1 + y V_2 = W \iff [V_1 | V_2] \begin{bmatrix} x \\ y \end{bmatrix} = W$$

$$[V_1 | V_2 | W] = \left[\begin{array}{c|c|c} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 2 & 2 & 6 \\ 1 & 4 & 4 \end{array}\right] \xrightarrow[\substack{r_3 - 2r_1 \\ r_4 - r_1}]{r_2 - 2r_1} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 4 \\ 0 & 3 & 3 \end{array}\right] \longrightarrow$$ inconsistent system $\therefore W \notin span\{V_1, V_2\}$

(b.) (10pts) Determine if $\{v_1, v_2, w\}$ is a LI set.

$$\text{rref}\,[V_1 | V_2 | W] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \therefore \quad \underline{\{V_1, V_2, W\} \text{ is LI}}.$$

Problem 5: (10pts) Let $A = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 3 & 3 & 0 & 0 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -1 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 3 & 3 \end{bmatrix} \oplus \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$. Calculate A^{-1}.

$$A^{-1} = \begin{bmatrix} 3 & 1 \\ 3 & 3 \end{bmatrix}^{-1} \oplus \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}^{-1} = \left(\frac{1}{6}\begin{bmatrix} 3 & -1 \\ -3 & 3 \end{bmatrix}\right) \oplus \left(\frac{1}{6}\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}\right) = \frac{1}{6}\begin{bmatrix} 3 & -1 \\ -3 & 3 \end{bmatrix} \oplus \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$$

$$= \frac{1}{6}\begin{bmatrix} 3 & -1 & 0 & 0 \\ -3 & 3 & 0 & 0 \\ 0 & 0 & 4 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

Problem 6: (20pts) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 3 \end{bmatrix}$. Calculate:

(a) $AA^T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 1 & 3 \end{bmatrix} = \boxed{\begin{bmatrix} 3 & 5 \\ 5 & 13 \end{bmatrix}}$

(b) $A^TA = \begin{bmatrix} 1 & 2 \\ 1 & 0 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 3 \end{bmatrix} = \boxed{\begin{bmatrix} 5 & 1 & 7 \\ 1 & 1 & 1 \\ 7 & 1 & 10 \end{bmatrix}}$

Problem 7: (20pts) Find the augmented coefficient matrix $[A|b]$ for:

$$\begin{aligned} x_1 + 2x_2 + 2x_3 + 4x_4 &= 0 \\ -2x_1 - 4x_2 - 3x_3 &= -3 \end{aligned}$$

and use row operations to calculate rref$[A|b]$. Write the general solution of $Ax = b$ in standard parametric form.

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 4 & 0 \\ -2 & -4 & -3 & 0 & -3 \end{array}\right] \xrightarrow{r_2+2r_1} \left[\begin{array}{cccc|c} 1 & 2 & 2 & 4 & 0 \\ 0 & 0 & 1 & 8 & -3 \end{array}\right] \xrightarrow{r_1-2r_2} \left[\begin{array}{cccc|c} 1 & 2 & 0 & -12 & 6 \\ 0 & 0 & 1 & 8 & -3 \end{array}\right]$$

$$\begin{aligned} x_1 + 2x_2 - 12x_4 &= 6 \quad \longrightarrow \quad x_1 = -2x_2 + 12x_4 + 6 \\ x_3 + 8x_4 &= -3 \quad \longrightarrow \quad x_3 = -8x_4 - 3 \end{aligned}$$

$$\boxed{\text{Solution Set} = \{(-2x_2 + 12x_4 + 6, x_2, -8x_4 - 3, x_4) \mid x_2, x_4 \in \mathbb{R}\}}$$

$$= (6, 0, -3, 0) + \text{span}\{(-2, 1, 0, 0), (12, 0, -8, 1)\}$$

Problem 8: You are given that:

$$A = \begin{bmatrix} 1 & -3 & 2 & -1 \\ 3 & -9 & 2 & -7 \\ 3 & -9 & 2 & -7 \end{bmatrix} \quad \text{has} \quad \text{rref}(A) = \begin{bmatrix} 1 & -3 & 0 & -3 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(a.) (5pts) Find the basis for Col(A) via the CCP,

pivot columns give basis $\left\{ \begin{bmatrix} 1 \\ 3 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} \right\}$

(b.) (10pts) Find the basis for Null(A).

$$Ax = 0 \Rightarrow x_1 = 3x_2 + 3x_4 \;\;\&\;\; x_3 = -x_4$$

$$\Rightarrow x = (3x_2 + 3x_4, x_2, -x_4, x_4)$$

$$\Rightarrow x = x_2(3, 1, 0, 0) + x_4(3, 0, -1, 1)$$

$$\boxed{\text{BASIS OF Null}(A): \{(3, 1, 0, 0), (3, 0, -1, 1)\}}$$

Problem 9: If possible, write M as the product of elementary matrices. If not, explain why it is not possible.

(a.) (5pts) $M = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{rref}(M) \neq I_3$ $\therefore M^{-1}$ d.n.e. and so M cannot be expressed as product of elementary matrices

(b.) (10pts) $M = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow[\text{1}]{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{2}]{r_1 + 2r_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$E_2 E_1 M = I \Rightarrow M = E_1^{-1} E_2^{-1} I$$

$$\boxed{M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$

$$\left(E_2^{-1} : I_3 \xrightarrow{r_1 - 2r_2} \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \right.$$

Problem 10: (10pts) Let W be the set of quadratic polynomials whose graphs contain the point $(3, 0)$. Express W as a span.

$$\begin{aligned} W &= \{ f(x) \in P_2(\mathbb{R}) \mid f(3) = 0 \} \\ &= \{ Ax^2 + Bx + C \mid (x-3) \text{ is factor} \} \\ &= \{ (x-3)(Ax + D) \mid A, D \in \mathbb{R} \} \\ &= \{ Ax(x-3) + D(x-3) \mid A, D \in \mathbb{R} \} \\ &= \boxed{\text{span}\{ x^2 - 3x, x - 3 \}} \end{aligned}$$

Choose your adventure: pick one of the following two problems for credit:

Problem 11: (15pt) Let $A \in \mathbb{R}^{m \times p}$ and $B \in \mathbb{R}^{p \times m}$. Prove $tr(AB) = tr(BA)$.

Problem 12: (15pt) Let $rref[v_1|v_2|v_3|e_3|e_2|e_1] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3 & 7 & 0 \\ 0 & 1 & 0 & 0 & 7 & 4 \\ 0 & 0 & 1 & 3 & 0 & 1 \end{array}\right]$. If $A = [v_1|v_2|v_3]$ then calculate A^{-1} in terms of v_1, v_2, v_3.

P11

$$tr(AB) = \sum_{k=1}^{m} (AB)_{kk} = \sum_{k=1}^{m}\sum_{j=1}^{p} A_{kj} B_{jk} = \sum_{j=1}^{p}\sum_{k=1}^{m} B_{jk} A_{kj} = \sum_{j=1}^{p} (BA)_{jj} = tr(BA).$$

def$^{\underline{n}}$ of trace; def$^{\underline{n}}$ of matrix multiplication; finite sums and mult. of scalars commutes; def$^{\underline{n}}$ of matrix multiplication; def$^{\underline{n}}$ of trace.

P12 From the given rref of $A = [V_1|V_2|V_3]$ we find that

$$rref[A|e_1] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right], \quad rref[A|e_2] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 7 \\ 0 & 0 & 1 & 0 \end{array}\right], \quad rref[A|e_3] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

CCP $\Rightarrow$ $A(4V_2 + V_3) = e_1$, $A(7V_1 + 7V_2) = e_2$, $A(3V_1 + 3V_3) = e_3$

$$\therefore \boxed{A^{-1} = [4V_2 + V_3 \,|\, 7V_1 + 7V_2 \,|\, 3V_1 + 3V_3]}$$

Problem 13: (15pts) Suppose $G = \{A \mid A^T J A = J\} \subseteq \mathbb{R}^{n \times n}$ where J is an invertible matrix. Show that if $A, B \in G$ then $AB \in G$ and $A^{-1} \in G$.

Let $A, B \in G$ then $A^T J A = J$ and $B^T J B = J$ then

$$(AB)^T J (AB) = B^T (A^T J A) B = B^T J B = J$$

thus $AB \in G$. We would like to show A^{-1} exists and $A^{-1} \in G$. Suppose $\exists x \neq 0$ s.t. $Ax = 0$ then if $A \in G$ then

$$A^T J A = J \Rightarrow A^T J A x = Jx = 0 \text{ for } x \neq 0 \Rightarrow J^{-1} \text{ d.n.e.}$$

which contradicts the existence of J^{-1} $\therefore Ax = 0 \Leftrightarrow x = 0$ hence A^{-1} exists. Observe then $A \in G$ implies

$$\begin{aligned} A^T J A = J &\Rightarrow (A^{-1})^T A^T J A (A^{-1}) = (A^{-1})^T J A^{-1} \\ &\Rightarrow (A^T)^{-1} A^T J I = (A^{-1})^T J A^{-1} \\ &\Rightarrow I J I = (A^{-1})^T J A^{-1} \\ &\Rightarrow (A^{-1})^T J A^{-1} = J \quad \therefore \underline{A^{-1} \in G} \; // \end{aligned}$$