

Name:

MATH 221 BOSS FIGHT 2: LINEAR MAPS, COORDINATES & ORTHONORMALITY (200+20PTS)

You may use a non-graphing calculator on this test and you are allowed a page of notes front and back.

Problem 1: Let $T(x, y, z, w) = (x + 2y + 3z + 4w, 2x - 3w, 2y + z)$

(a.) (8pts) Calculate the standard matrix $[T] \in \mathbb{R}^{3 \times 4}$ since $T: \mathbb{R}^4 \rightarrow \mathbb{R}^3$

$$[T] = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 0 & 0 & -3 \\ 0 & 2 & 1 & 0 \end{bmatrix}$$

(b.) (4pts) Calculate $\text{rref}[T]$

$$\begin{aligned} [T] &\xrightarrow{r_2 - 2r_1} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & -4 & -6 & -11 \\ 0 & 2 & 1 & 0 \end{bmatrix} \xrightarrow{\substack{r_1 - r_3 \\ r_2 + 2r_3}} \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 0 & -4 & -11 \\ 0 & 2 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & 4 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1/4 \end{bmatrix} \\ &\xrightarrow{r_1 - 2r_3} \begin{bmatrix} 1 & 0 & 0 & 4 - 2 \cdot 1/4 \\ 0 & 2 & 0 & -1/4 \\ 0 & 0 & 1 & 1/4 \end{bmatrix} \xrightarrow{r_2 - r_3} \begin{bmatrix} 1 & 0 & 0 & 4 - 2 \cdot 1/4 \\ 0 & 2 & 0 & -1/4 \\ 0 & 0 & 1 & 1/4 \end{bmatrix} \sim \boxed{\begin{bmatrix} 1 & 0 & 0 & -3/2 \\ 0 & 1 & 0 & -11/8 \\ 0 & 0 & 1 & 1/4 \end{bmatrix}} = \text{rref}[T] \end{aligned}$$

(c.) (4pt) Is T an surjective mapping?

$$\text{YES, } \text{rank}[T] = 3 = \dim(\mathbb{R}^3).$$

(d.) (4pt) Is T an injective mapping?

$$\text{NO, } \text{nullity}[T] = 1 \neq 0 \therefore \text{more than one input maps to zero.}$$

Problem 2: (10pts) Suppose A is a 2×2 matrix with $\det(3A) = 900$ and B is a 3×3 matrix with $\det(B) = 5$. If $M = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B^{-1} \end{array} \right]$ then calculate $\det(M)$.

$$\det(3A) = 3^2 \det(A) = 900 \therefore \underline{\det(A) = 100}.$$

$$\begin{aligned} \det(M) &= \det(A \oplus B^{-1}) \\ &= \det(A) \det(B^{-1}) \\ &= \frac{\det(A)}{\det(B)} \\ &= \frac{100}{5} = \boxed{20} \end{aligned}$$

Problem 3: (30pts) Let $S = \{(1, 0, 1, 0), (0, 3, 0, 4), (1, -4, 1, 3)\}$. Find the following:

(a.) (10pts) an orthonormal basis β_1 for $\text{span}(S)$,

Observe $V_1 \cdot V_2 = 0$ so we can begin with V_3 , oh, $V_2 \cdot V_3 = 0$ also!

$$V_3' = V_3 - \left(\frac{V_1 \cdot V_3}{V_1 \cdot V_1}\right)V_1 - \left(\frac{V_2 \cdot V_3}{V_2 \cdot V_2}\right)V_2$$

$$= (1, -4, 1, 3) - \frac{2}{2}(1, 0, 1, 0) - 0V_2$$

$$= (0, -4, 0, 3) \leftarrow \text{we maybe could have guessed this.}$$

$$\therefore \beta_1 = \left\{ \frac{1}{\sqrt{2}}(1, 0, 1, 0), \frac{1}{5}(0, 3, 0, 4), \frac{1}{5}(0, -4, 0, 3) \right\}$$

(b.) (10pts) an orthonormal basis for β_2 for S^\perp

Observe $(1, 0, -1, 0) \perp \beta_1$, if we didn't see this,

calculate $\text{Null}[\beta_1']^T = \text{Null} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & 4 \\ 0 & -4 & 0 & 3 \end{bmatrix}$

$$[\beta_1']^T \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 12 & 0 & 16 \\ 0 & -12 & 0 & 9 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 12 & 0 & 16 \\ 0 & 0 & 0 & 25 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \therefore \begin{matrix} x_2 = -x_3 \\ x_2 = x_4 = 0 \end{matrix} \hookrightarrow \left\{ \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} \right\} = \beta_2$$

(c.) (10pts) Let $v = (a, b, c, d)$. Find $v_1 \in \text{span}(S)$ and $v_2 \in S^\perp$ such that $v = v_1 + v_2$.

$$v_2 = \text{Proj}_{S^\perp}(v) = (v \cdot u_4)u_4 = \frac{a-c}{2}(1, 0, -1, 0) = \left(\frac{a-c}{2}, 0, \frac{c-a}{2}, 0 \right)$$

$$v_1 = v - v_2 = (a, b, c, d) - \left(\frac{a-c}{2}, 0, \frac{c-a}{2}, 0 \right)$$

$$v_1 = \left(\frac{a+c}{2}, b, \frac{c+a}{2}, d \right)$$

Problem 4: (10pt) Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 0 & 0 & 0 \\ 3 & 3 & 3 & 3 & 0 \\ 3 & 2 & 1 & 0 & 0 \\ 2 & 2 & 3 & 4 & 5 \end{bmatrix}$. Calculate $\det(A)$.

$$A \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 1 & 0 & 0 & 0 \\ 3 & 3 & 3 & 3 & 0 \\ 2 & 2 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 \\ 3 & 3 & 3 & 3 & 0 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

$r_5 - r_1 \uparrow$ swapped rows 1 & 5 and 3 & 4

$$\therefore \det(A) = 15$$

Problem 5: (10pts) Find the line which is closest to the points $(-2, -6), (-1, 1), (0, 9), (1, 13)$.

Let $y = mx + c$ and evaluate the data,

$$\left. \begin{array}{l} (-2, -6) : -6 = -2m + c \\ (-1, 1) : 1 = -m + c \\ (0, 9) : 9 = c \\ (1, 13) : 13 = m + c \end{array} \right\} \rightarrow \underbrace{\begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}}_A \underbrace{\begin{bmatrix} m \\ c \end{bmatrix}}_v = \underbrace{\begin{bmatrix} -6 \\ 1 \\ 9 \\ 13 \end{bmatrix}}_b \rightarrow \underbrace{A^T A v = A^T b}_{\text{Must solve for least squares}}$$

$$A^T b = \begin{bmatrix} -2 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -6 \\ 1 \\ 9 \\ 13 \end{bmatrix} = \begin{bmatrix} 24 \\ 17 \end{bmatrix} \quad \& \quad A^T A = \begin{bmatrix} -2 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}$$

$$\begin{aligned} A^T A v = A^T b &\Rightarrow v = \begin{bmatrix} m \\ c \end{bmatrix} = (A^T A)^{-1} A^T b = \begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 24 \\ 17 \end{bmatrix} \\ &= \frac{1}{24-4} \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 24 \\ 17 \end{bmatrix} \\ &= \frac{1}{20} \begin{bmatrix} 130 \\ 150 \end{bmatrix} \therefore \boxed{y = 6.5x + 7.5} \end{aligned}$$

Problem 6: (10pts) Let $T(f) = x \frac{df}{dx} + 3xf$ for each $f = a + bx + cx^2$ in $P_2(\mathbb{R})$. This is a linear transformation. Let $\beta = \{1, x, x^2\}$ and $\gamma = \{1, x, x^2, x^3\}$. Calculate $[T]_{\beta, \gamma}$.

$$\begin{array}{ccc} \boxed{f = a + bx + cx^2} & \xrightarrow{T} & \boxed{T(f) = x(b + 2cx) + 3x(a + bx + cx^2)} \\ & & = (3a + b)x + (2c + 3b)x^2 + (3c)x^3 \\ \downarrow \Phi_\beta & & \downarrow \Phi_\gamma \\ [f]_\beta = \begin{bmatrix} a \\ b \\ c \end{bmatrix} & \xrightarrow{\text{multiply by}} & [T(f)]_\gamma = \begin{bmatrix} 0 \\ 3a + b \\ 3b + 2c \\ 3c \end{bmatrix} \\ & & [T]_{\beta, \gamma} \in \mathbb{R}^{4 \times 3} \end{array}$$

$$\therefore [T]_{\beta, \gamma} = \begin{bmatrix} 0 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 3 & 2 \\ 0 & 0 & 3 \end{bmatrix}$$

Problem 7: (25pts) Let $A = \begin{bmatrix} 3 & 6 \\ 3 & 1 \\ 3 & 6 \\ 3 & 1 \end{bmatrix}$. Find the QR-decomposition of the given matrix.

$$V_2' = V_2 - \left(\frac{V_1 \cdot V_2}{V_1 \cdot V_1} \right) V_1 = (6, 1, 6, 1) - \frac{42}{36} (3, 3, 3, 3) = (5/2, -5/2, 5/2, -5/2)$$

Normalizing we find orthonormal basis for $\text{Col}(A)$ from dividing $(1, 1, 1, 1)$ and $(1, -1, 1, -1)$ by $\sqrt{4} = 2$

$$Q = \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix}$$

We need $A = QR$ so calculate $R = Q^T A$,

$$R = Q^T A = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 3 & 6 \\ 3 & 1 \\ 3 & 6 \\ 3 & 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 12 & 14 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 6 & 7 \\ 0 & 5 \end{bmatrix}$$

$$\therefore A = \underbrace{\begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \\ 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}}_Q \underbrace{\begin{bmatrix} 6 & 7 \\ 0 & 5 \end{bmatrix}}_R$$

Problem 8: (15pts) Suppose $W = \text{span}\{(3, 3, 3, 3), (6, 1, 6, 1)\}$. Find the point on W which is closest to (a, b, c, d) and determine how far $(0, 6, 7, 0)$ is from W .

From [P7] we have orthonormal basis $u_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $u_2 = \frac{1}{2} \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$ for W ,

$$\begin{aligned} \text{Proj}_W(a, b, c, d) &= ((a, b, c, d) \cdot u_1) u_1 + ((a, b, c, d) \cdot u_2) u_2 \\ &= \left(\frac{a+b+c+d}{4} \right) (1, 1, 1, 1) + \left(\frac{a-b+c-d}{4} \right) (1, -1, 1, -1) \\ &= \left(\frac{a+c}{2}, \frac{b+d}{2}, \frac{a+c}{2}, \frac{b+d}{2} \right) \end{aligned}$$

$$\text{Proj}_W(0, 6, 7, 0) = (7/2, 6/2, 7/2, 6/2)$$

distance to W : $\| (0, 6, 7, 0) - (7/2, 3, 7/2, 3) \| = \| (-7/2, 3, 7/2, -3) \|$

$$= \sqrt{\frac{49}{4} + 9 + \frac{49}{4} + 9} = \sqrt{\frac{85}{2}} \approx 6.52$$

Problem 9: (35pt) Let $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ -3 & 2 & 0 \end{bmatrix}$.

(a.) (10pts) Calculate $\det(A)$,

$$\det(A) = 1(8-9) - 1(4-6) + 1(3-4) = -1 + 2 - 1 = \boxed{0}$$

$\hookrightarrow A^{-1}$ d.n.e.

(b.) (10pts) Calculate $\det(B)$,

$$\det(B) = 1(0-4) + 1(0+3) = 3-4 = \boxed{-1} \leftarrow \text{nonzero} \therefore B^{-1} \text{ exists}$$

(c.) (15pts) Calculate the inverse of the invertible matrix in this problem.

$$[B|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ -3 & 2 & 0 & 0 & 0 & 1 \end{array} \right] \xrightarrow{r_3+3r_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 2 & 3 & 3 & 0 & 1 \end{array} \right] \xrightarrow{r_3-2r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & -1 & 3 & -2 & 1 \end{array} \right]$$

$$\begin{array}{l} \xrightarrow{r_1+r_3} \\ \xrightarrow{r_2+2r_3} \end{array} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -2 & 1 \\ 0 & 1 & 0 & 6 & -3 & 2 \\ 0 & 0 & -1 & 3 & -2 & 1 \end{array} \right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -2 & 1 \\ 0 & 1 & 0 & 6 & -3 & 2 \\ 0 & 0 & 1 & -3 & 2 & -1 \end{array} \right] \underbrace{\hspace{10em}}_{B^{-1}}$$

$$B^{-1} = \begin{bmatrix} 4 & -2 & 1 \\ 6 & -3 & 2 \\ -3 & 2 & -1 \end{bmatrix}$$

Problem 10: (15pts) Let $v_1 = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. Define $\beta = \{v_1, v_2, v_3\}$.

Suppose $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ has $T(v_1) = v_1$ and $T(v_2) = 2v_2$ and $T(v_3) = 3v_3$. Calculate $[T]_{\beta, \beta}$ and express $[T]$ as a product of three explicit matrices.

$$\begin{aligned} [T]_{\beta, \beta} &= \left[[T(v_1)]_{\beta} \mid [T(v_2)]_{\beta} \mid [T(v_3)]_{\beta} \right] \\ &= \left[[v_1]_{\beta} \mid [2v_2]_{\beta} \mid [3v_3]_{\beta} \right] \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \end{aligned}$$

$$\boxed{P_9} \quad [v_1 | v_2 | v_3]$$

We found $[T]_{\beta, \beta} = [\beta]^{-1} [T] [\beta]$, notice $B = [\beta]$

$$\therefore [\beta] [T]_{\beta, \beta} [\beta]^{-1} = [\beta] [\beta]^{-1} [T] [\beta] [\beta]^{-1} = [T]$$

$$\therefore [T] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ -3 & 2 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 4 & -2 & 1 \\ 6 & -3 & 2 \\ -3 & 2 & -1 \end{bmatrix}$$

Choose your adventure: work either (a.) or (b.) for credit

Problem 11: (10pts) Prove W given below is a subspace by use of an appropriate theorem from lecture:

(a.) $W = \{A \in \mathbb{R}^{n \times n} \mid A^T = A\}$.

(b.) $W = \{a + bx^2 + cx^4 \mid a, b, c \in \mathbb{R}\}$

(a.) Notice $0^T = 0$ thus $0 \in W \neq \emptyset$. Let $A, B \in W$ and $c \in \mathbb{R}$ and observe $(cA)^T = cA^T = cA$ and $(A+B)^T = A^T + B^T = A+B$. Thus $cA, A+B \in W$ and we conclude $W \leq \mathbb{R}^{n \times n}$ by subspace test Th^m //

(b.) Notice $W = \{a + bx^2 + cx^4 \mid a, b, c \in \mathbb{R}\} = \text{span}\{1, x^2, x^4\}$ thus W is a subspace of $\underbrace{P(\mathbb{R}) = \mathbb{R}[x]}_{\text{space of all polynomials over } \mathbb{R}}$.

Problem 12: (10pts) Let $A \in \mathbb{R}^{3 \times 3}$. Suppose $T(x) = Ax$ defines $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$. You're given that

$$T(v_1) = (1, 2, 3), \quad T(v_2) = (3, 4, 4), \quad T(v_3) = (4, 6, 7).$$

Suppose $\det[v_1 | v_2 | v_3] = 6$ and $\text{tr}[v_1 | v_2 | v_3] = 3$. Calculate the determinant and trace of A .

my apologies, there is not enough info to find this

$$[T(v_1) | T(v_2) | T(v_3)] = [Av_1 | Av_2 | Av_3] = A[v_1 | v_2 | v_3]$$

$$\begin{bmatrix} 1 & 3 & 4 \\ 2 & 4 & 6 \\ 3 & 4 & 7 \end{bmatrix} = A[v_1 | v_2 | v_3] \Rightarrow 0 = \det(A) \det[v_1 | v_2 | v_3] = \det(A) 6$$

$\therefore \det(A) = 0$

Problem 13: (20pts) Suppose $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ is a linear transformation for which

$$T\left(\underbrace{\begin{bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{bmatrix}}_{u_1}\right) = \underbrace{\begin{pmatrix} 2 \\ 2 \\ 1 \end{pmatrix}}_{v_1} \quad \& \quad T\left(\underbrace{\begin{bmatrix} 2/3 \\ -1/3 \\ -2/3 \end{bmatrix}}_{u_2}\right) = \underbrace{\begin{pmatrix} -1 \\ 2 \\ -2 \end{pmatrix}}_{v_2} \quad \& \quad T\left(\underbrace{\begin{bmatrix} -1/3 \\ 2/3 \\ -2/3 \end{bmatrix}}_{u_3}\right) = \underbrace{\begin{pmatrix} 2 \\ -1 \\ -2 \end{pmatrix}}_{v_3}$$

Find the standard matrix of T .

$$[[T]u_1 \mid [T]u_2 \mid [T]u_3] = [T][u_1 \mid u_2 \mid u_3] = [v_1 \mid v_2 \mid v_3]$$

$$[T] = [v_1 \mid v_2 \mid v_3][u_1 \mid u_2 \mid u_3]^{-1} \rightarrow \{u_1, u_2, u_3\} \text{ orthonormal!}$$

$$= \frac{1}{3} \begin{bmatrix} 2 & -1 & 2 \\ 2 & 2 & -1 \\ 1 & -2 & -2 \end{bmatrix} \begin{bmatrix} 2 & 2 & 1 \\ 2 & -1 & -2 \\ -1 & 2 & -2 \end{bmatrix}$$

$$= \frac{1}{3} \begin{bmatrix} 0 & 9 & 0 \\ 9 & 0 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 3 & 0 \\ 3 & 0 & 0 \\ 0 & 0 & 3 \end{bmatrix}}$$