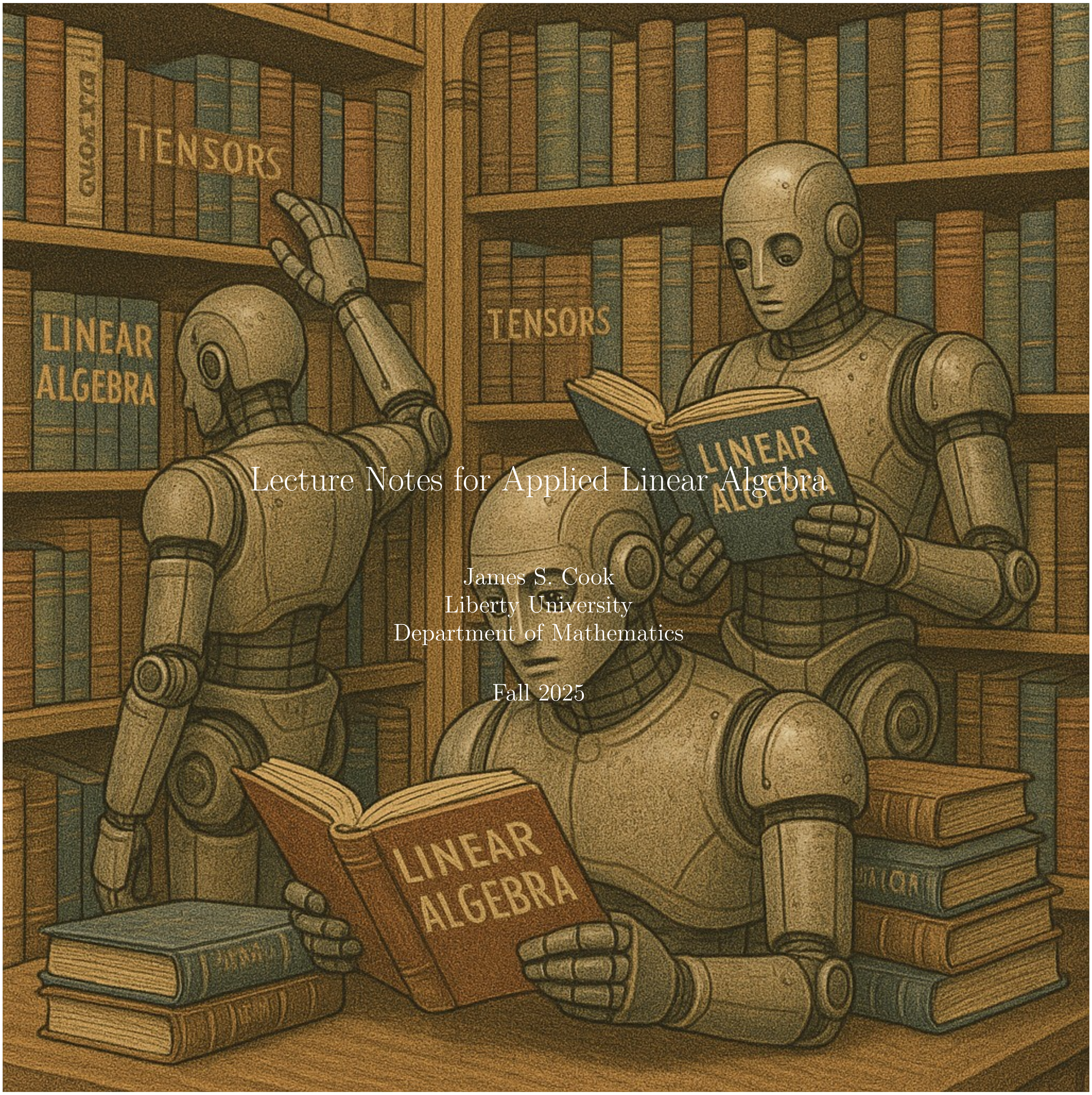

Lecture Notes for Applied Linear Algebra

James S. Cook
Liberty University
Department of Mathematics

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preface

These notes contain many examples and thoughts I will not have time for in lecture. Sometimes I may refer to the notes rather than write out certain lengthy theorems or definitions. It would be good if you had a copy of these notes for your convenience.

I try to add interesting examples in lecture which are not found in these notes. Often I present something in general in the notes whereas I present a simple low-dimensional application for class. In addition, I always am interested in interesting questions about the material. A properly engaged class will bring questions which add depth to these notes, the text, even the homework. Nothing makes me happier then learning something from my students. Ideally, I teach to learn.

Doing the homework is doing the course. I cannot overemphasize the importance of thinking through the homework. I would be happy if you left this course with a working knowledge of:

- ✓ how to solve a system of linear equations by Gaussian elimination the $rref(A)$
- ✓ concrete and abstract matrix calculations
- ✓ determinants
- ✓ the calculational utility of linear independence, spanning, coordinates and bases
- ✓ column, row and null spaces for a matrix, how to find their bases and use them
- ✓ matrix of linear transformation and change of basis
- ✓ eigenvalues and eigenvectors and diagonalization
- ✓ orthogonal bases and the Gram-Schmidt algorthim
- ✓ least squares fitting of experimental data,
- ✓ how to solve a system of linear differential equations
- ✓ principle axis theorems for conic sections and quadric surfaces
- ✓ Matlab's matrix computational commands

I use my usual conventions in these notes: green is used for definitions whereas remarks appear in red and propositions or theorems etc... appear in blue . Many proofs are omitted in this version. However, not all. I have included some since I believe that some exposure to proofs is still worthwhile for those students who only care about the real world. I happen to believe in the reality of mathematicians and you may find yourself needing to converse with one someday. Most of the theory I do insist on maintaining in this course is largely for the purpose of langauge. Mathematics is just that, it is a precise langauge which allows us to quickly get across abstract ideas to people with similar training.

To the weaker among you, to those who the proofs frighten, my apologies for their appearance in the notes. Please note that I have marked certain sections with a *. This indicates there is some significant part of that section which is beyond the core of this course and/or it involves concepts (like partial differentiation or methods of differential equations) which are not pre-requisite to this course. As a general point of order, if I don't cover something in class it is probably not as important as the parts which I do cover in class. There are a few sections and discussions in these notes which are beyond the scope of the course in most semesters. I have included many proofs which I would not typically test over in this course, but proofs do matter since they help students understand how things work. That said, many proofs are omitted in this treatment. If you seek additional insights, please ask and/or take Math 321.

The purpose of these notes is to supplement your notes from lecture. It is important that you come to class, pay attention, and keep track of what we cover. It is even more important that you regularly work on the homeworks with the goal of understanding both lecture and these notes.

I hope you find this course engaging. The availability of computers has made linear algebra the most important mathematics for business and applications. Almost without exception when we ask graduates who work in industry what courses were most important we hear linear algebra towards the top if not at the top of the list. It is my sincere desire that when you leave this course you are ready to start applying the calculations of this course to solve real world problems.

James Cook, July 12, 2025

Note on required text: as of Fall 2025 we have adopted an open source textbook as the official text for Math 221 at Liberty University. You do not need to buy anything from the bookstore for this course. In particular, if I assign problems from the text, this indicates the problem is from *Matrix Theory and Linear Algebra* by Peter Selinger

https://www.mathstat.dal.ca/ selinger/linear-algebra/

I don't generally expect you read the text above, but it might be helpful as a supplement to these notes. Notice a hardcopy can be purchased at a low cost if you wish to read the required text off screen. On the other hand, the pdf of the required text (much like the pdf of the lecture notes you are currently reading) are easily searched with the CTRL-F option. This stands is marked contrast to the textbook readers offered from publishers. Such readers are often inferior in form, function and cost.

Note on technology: You can use Matlab or Mathematica to calculate, but it should be an appropriate use. Inappropriate use would be where the purpose of the homework is averted through the use. For example, at the start of the course when we are calculating $rref(A)$ then you need to work our the row-reductions step-by-step since you will be expected to do the same on the exam.

On the other hand, later in the midst of a least squares problem using the rref command to simplify a system would be totally appropriate. It is always appropriate to check answers with technology! If in doubt, ask.

Bibliographical notes:

1. references to David Lay's *Linear Algebra and Its Applications* refer to the second edition. Incidentally, this an excellent text which you can purchase for about ten dollars and shipping.

2. references to Anton and Rorres refer to either ed. 9 or 10. I have ed. 9 if you want to look at it in office hours. I do expect you have (or share with someone) the tenth ed.

3. Insel Spence and Friedberg authored two texts on linear algebra. The text I primarily reference is titled *Elementary Linear Algebra* the 2nd edition. Their other text *Linear Algebra* is a beautiful text I would love to see us use for Math 321.

4. previous editions of my linear algebra notes from Math 321 are posted at my webpage www.supermath.info.

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Chapter 1

Gauss-Jordan elimination

Gauss-Jordan elimination is an optimal method for solving a system of linear equations. Logically it may be equivalent to methods you are already familar with but the matrix notation is by far the most efficient method. This is important since throughout this course we will be faced with the problem of solving linear equations. Additionally, the Gauss-Jordan produces the *reduced row echelon form*(rref) of the matrix. Given a particular matrix the rref is unique. This is of particular use in theoretical applications.

1.1 systems of linear equations

Let me begin with a few examples before I state the general definition.

Example 1.1.1. *Consider the following system of 2 equations and 2 unknowns,*

$$x + y = 2$$

$$x - y = 0$$

Adding equations reveals $2x = 2$ *hence* $x = 1$*. Then substitute that into either equation to deduce* $y = 1$*. Hence the solution* $(1, 1)$ *is* **unique**

Example 1.1.2. *Consider the following system of 2 equations and 2 unknowns,*

$$x + y = 2$$

$$3x + 3y = 6$$

We can multiply the second equation by $1/3$ *to see that it is equivalent to* $x + y = 2$ *thus our two equations are in fact the same equation. There are infinitely many equations of the form* (x, y) *where* $x + y = 2$*. In other words, the solutions are* $(x, 2 - x)$ *for all* $x \in \mathbb{R}$*.*

Both of the examples thus far were **consistent**.

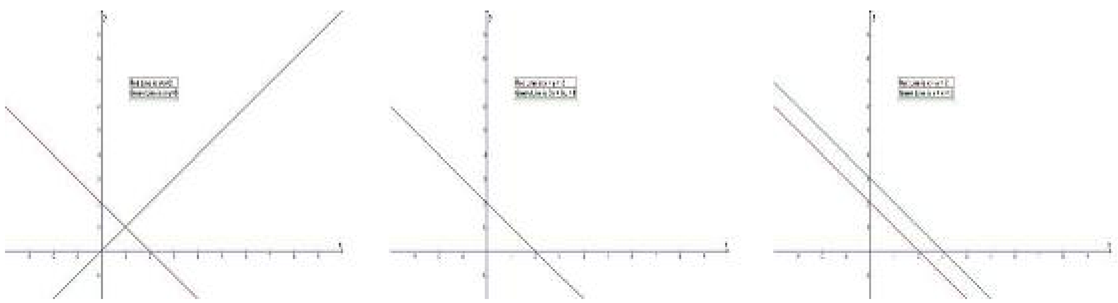
Example 1.1.3. *Consider the following system of 2 equations and 2 unknowns,*

$$x + y = 2$$

$$x + y = 3$$

These equations are inconsistent. Notice substracting the second equation yields that $0 = 1$*. This system has no solutions, it is* **inconsistent**

It is remarkable that these three simple examples reveal the general structure of solutions to linear systems. Either we get a unique solution, infinitely many solutions or no solution at all. For our examples above, these cases correspond to the possible graphs for a pair of lines in the plane. A pair of lines may intersect at a point (unique solution), be the same line (infinitely many solutions) or be paralell (inconsistent).[1]



Remark 1.1.4.

It is understood in this course that $i, j, k, l, m, n, p, q, r, s$ are in $\mathbb{N}$. I will not belabor this point. Please ask if in doubt.

Definition 1.1.5. *system of* m*-linear equations in* n*-unknowns*

Let $x_1, x_2, \dots, x_m$ be m variables and suppose $b_i, A_{ij} \in \mathbb{R}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$ then

$$A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n = b_1$$

$$A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n = b_2$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n = b_m$$

is called a **system of linear equations**. If $b_i = 0$ for $1 \leq i \leq m$ then we say the system is **homogeneous**. The **solution set** is the set of all $(x_1, x_2, \dots, x_n) \in \mathbb{R}^n$ which satisfy all the equations in the system simultaneously.

[1] I used the *Graph* program to generate these graphs. It makes nice pictures, these are ugly due to user error.

Remark 1.1.6.

We use variables $x_1, x_2, \dots, x_n$ mainly for general theoretical statements. In particular problems and especially for applications we tend to defer to the notation x, y, z etc...

Definition 1.1.7.

The augmented coefficient matrix is an array of numbers which provides an abbreviated notation for a system of linear equations.

$$\left[\begin{array}{c} A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n = b_1 \\ A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n = b_2 \\ \vdots \quad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n = b_m \end{array}\right] \quad \text{abbreviated by} \quad \left[\begin{array}{cccc|c} A_{11} & A_{12} & \cdots & A_{1n} & b_1 \\ A_{21} & A_{22} & \cdots & A_{2n} & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} & b_m \end{array}\right].$$

The vertical bar is optional, I include it to draw attention to the distinction between the matrix of coefficients A_{ij} and the nonhomogeneous terms b_i. Let's revisit my three simple examples in this new notation. I illustrate the Gauss-Jordan method for each.

Example 1.1.8. *The system $x + y = 2$ and $x - y = 0$ has augmented coefficient matrix:*

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 0 \end{array}\right] \xrightarrow{r_2 - r_1 \to r_2} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & -2 & -2 \end{array}\right]$$
$$\xrightarrow{r_2/-2 \to r_2} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right] \xrightarrow{r_1 - r_2 \to r_1} \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$$

The last augmented matrix represents the equations $x = 1$ and $y = 1$. Rather than adding and subtracting equations we added and subtracted rows in the matrix. Incidentally, the last step is called the **backward pass** *whereas the first couple steps are called the* **forward pass**. *Gauss is credited with figuring out the forward pass then Jordan added the backward pass. Calculators can accomplish these via the commands ref (Gauss' row echelon form) and $rref$ (Jordan's reduced row echelon form). In particular,*

$$ref\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 1 & 1 \end{array}\right] \qquad rref\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 1 \end{array}\right]$$

Example 1.1.9. *The system $x + y = 2$ and $3x + 3y = 6$ has augmented coefficient matrix:*

$$\left[\begin{array}{cc|c} 1 & 1 & 2 \\ 3 & 3 & 6 \end{array}\right] \xrightarrow{r_2 - 3r_1 \to r_2} \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

The nonzero row in the last augmented matrix represents the equation $x + y = 2$. In this case we cannot make a backwards pass so the ref and $rref$ are the same.

Example 1.1.10. *The system $x + y = 3$ and $x + y = 2$ has augmented coefficient matrix:*

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & 1 & 2 \end{array}\right] \xrightarrow{r_2 - 3r_1 \rightarrow r_2} \left[\begin{array}{cc|c} 1 & 1 & 1 \\ 0 & 0 & 1 \end{array}\right]$$

The last row indicates that $0x+0y = 1$ which means that there is no solution since $0 \neq 1$. Generally, when the bottom row of the $rref(A|b)$ is zeros with a 1 in the far right column then the system $Ax = b$ is inconsistent because there is no solution to the equation. In this case the solution set is the empty set $\emptyset$

1.2 Gauss-Jordan algorithm

To begin we need to identify three basic operations we do when solving systems of equations. I'll define them for system of 3 equations and 3 unknowns, but it should be obvious this generalizes to m equations and n unknowns without much thought. The following operations are called **Elementary Row Operations**.

(1.) scaling row 1 by nonzero constant c

$$\left[\begin{array}{ccc|c} A_{11} & A_{12} & A_{13} & b_1 \\ A_{21} & A_{22} & A_{23} & b_2 \\ A_{31} & A_{32} & A_{33} & b_3 \end{array}\right] \xrightarrow{cr_1 \rightarrow r_1} \left[\begin{array}{ccc|c} cA_{11} & cA_{12} & cA_{13} & cb_1 \\ A_{21} & A_{22} & A_{23} & b_2 \\ A_{31} & A_{32} & A_{33} & b_3 \end{array}\right]$$

(2.) replace row 1 with the sum of row 1 and row 2

$$\left[\begin{array}{ccc|c} A_{11} & A_{12} & A_{13} & b_1 \\ A_{21} & A_{22} & A_{23} & b_2 \\ A_{31} & A_{32} & A_{33} & b_3 \end{array}\right] \xrightarrow{r_1 + r_2 \rightarrow r_1} \left[\begin{array}{ccc|c} A_{11}+A_{21} & A_{12}+A_{22} & A_{13}+A_{23} & b_1+b_2 \\ A_{21} & A_{22} & A_{23} & b_2 \\ A_{31} & A_{32} & A_{33} & b_3 \end{array}\right]$$

(3.) swap rows 1 and 2

$$\left[\begin{array}{ccc|c} A_{11} & A_{12} & A_{13} & b_1 \\ A_{21} & A_{22} & A_{23} & b_2 \\ A_{31} & A_{32} & A_{33} & b_3 \end{array}\right] \xrightarrow{r_1 \longleftrightarrow r_2} \left[\begin{array}{ccc|c} A_{21} & A_{22} & A_{23} & b_2 \\ A_{11} & A_{12} & A_{13} & b_1 \\ A_{31} & A_{32} & A_{33} & b_3 \end{array}\right]$$

Each of the operations above corresponds to an allowed operation on a system of linear equations. When we make these operations we will not change the solution set. Notice the notation tells us what we did and also where it is going. I do expect you to use the same notation[2] I also expect you can figure out what is meant by $cr_2 \rightarrow r_2$ or $r_1 - 3r_2 \rightarrow r_1$. We are only allowed to make a finite number of the operations (1.),(2.) and (3.). The Gauss-Jordan algorithm tells us which order to make these operations in order to reduce the matrix to a particularly simple format called the "reduced row echelon form" (I abbreviate this rref most places).

[2]there is an abbrevation which is convenient and I will use to save some writing. Instead of $\xrightarrow{r_1 + r_2 \longrightarrow r_1}$ we can just write $\xrightarrow{r_1 + r_2}$. In contrast, $\xrightarrow{r_1 + r_2 \longrightarrow r_2}$ we can just write $\xrightarrow{r_2 + r_1}$. Whichever row appears first in the formula is modified.

Definition 1.2.1. Gauss-Jordan Algorithm.

Given an m by n matrix A the following sequence of steps is called the Gauss-Jordan algorithm or Gaussian elimination. I define terms such as **pivot column** and **pivot position** as they arise in the algorithm below.

Step 1: Determine the leftmost nonzero column. This is a **pivot column** and the topmost position in this column is a **pivot position**.

Step 2: Perform a row swap to bring a nonzero entry of the pivot column to the topmost row which does not already contain a pivot position. (in the first iteration this will be the top row of the matrix)

Step 3: Add multiples of the pivot row to create zeros **below** the pivot position. This is called "clearing out the entries below the pivot position".

Step 4: If there are no more nonzero rows below the last pivot row then go to step 5. Otherwise, there is a nonzero row below the pivot row and the new pivot column is the next nonzero column to the right of the old pivot column. Go to step 2.

Step 5: the leftmost entry in each nonzero row is called the **leading entry** (these are the entries in the pivot positions). Scale the bottommost nonzero row to make the leading entry 1 and use row additions to clear out any remaining nonzero entries **above** the leading entries.

Step 6: If step 5 was performed on the top row then stop, otherwise apply Step 5 to the next row up the matrix.

Steps (1.)-(4.) are called the **forward pass**. A matrix produced by a foward pass is called the $r$educed echelon form of the matrix and it is denoted ref(A). Steps (5.) and (6.) are called the **backwards pass**. The matrix produced by completing Steps (1.)-(6.) is called the $r$educed row echelon form of A and it is denoted $rref(A)$.

The $ref(A)$ is not unique because there may be multiple choices for how Step 2 is executed. On the other hand, it turns out that $rref(A)$ is unique. The proof of uniqueness can be found in Appendix E of the text *Elementary Linear Algebra: A Matrix Approach*, 2nd ed. by Spence, Insel and Friedberg. The backwards pass takes the ambiguity out of the algorithm. Notice the forward pass goes down the matrix while the backwards pass goes up the matrix.

Example 1.2.2. *Given* $A = \left[\begin{smallmatrix} 1 & 2 & -3 & 1 \\ 2 & 4 & 0 & 7 \\ -1 & 3 & 2 & 0 \end{smallmatrix}\right]$ *calculate* $rref(A)$.

$$A = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 2 & 4 & 0 & 7 \\ -1 & 3 & 2 & 0 \end{array}\right] \xrightarrow{r_2 - 2r_1 \to r_2} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 0 & 6 & 5 \\ -1 & 3 & 2 & 0 \end{array}\right] \xrightarrow{r_1 + r_3 \to r_3}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 0 & 6 & 5 \\ 0 & 5 & -1 & 1 \end{array}\right] \xrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 5 & -1 & 1 \\ 0 & 0 & 6 & 5 \end{array}\right] = ref(A)$$

that completes the forward pass. We begin the backwards pass,

$$ref(A) = \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 5 & -1 & 1 \\ 0 & 0 & 6 & 5 \end{array}\right] \xrightarrow{r_3 \to \frac{1}{6}r_3} \left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 5 & -1 & 1 \\ 0 & 0 & 1 & 5/6 \end{array}\right] \xrightarrow{r_2 + r_3 \to r_2}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 0 & 5 & 0 & 11/6 \\ 0 & 0 & 1 & 5/6 \end{array}\right] \xrightarrow{r_1 + 3r_3 \to r_1} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 21/6 \\ 0 & 5 & 0 & 11/6 \\ 0 & 0 & 1 & 5/6 \end{array}\right] \xrightarrow{\frac{1}{5}r_2 \to r_2}$$

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 21/6 \\ 0 & 1 & 0 & 11/30 \\ 0 & 0 & 1 & 5/6 \end{array}\right] \xrightarrow{r_1 - 2r_2 \to r_1} \boxed{\left[\begin{array}{ccc|c} 1 & 0 & 0 & 83/30 \\ 0 & 1 & 0 & 11/30 \\ 0 & 0 & 1 & 5/6 \end{array}\right] = rref(A)}$$

Example 1.2.3. *Given* $A = \left[\begin{smallmatrix} 1 & -1 & 1 \\ 3 & -3 & 0 \\ 2 & -2 & -3 \end{smallmatrix}\right]$ *calculate* $rref(A)$.

$$A = \left[\begin{array}{ccc} 1 & -1 & 1 \\ 3 & -3 & 0 \\ 2 & -2 & -3 \end{array}\right] \xrightarrow{r_2 - 3r_1 \to r_2} \left[\begin{array}{ccc} 1 & -1 & 1 \\ 0 & 0 & -3 \\ 2 & -2 & -3 \end{array}\right] \xrightarrow{r_3 - 2r_1 \to r_3}$$

$$\left[\begin{array}{ccc} 1 & -1 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & -5 \end{array}\right] \begin{array}{c} \xrightarrow{3r_3 \to r_3} \\ \xrightarrow{5r_2 \to r_2} \end{array} \left[\begin{array}{ccc} 1 & -1 & 1 \\ 0 & 0 & -15 \\ 0 & 0 & -15 \end{array}\right] \begin{array}{c} \xrightarrow{r_3 - r_2 \to r_3} \\ \xrightarrow{\frac{-1}{15}r_2 \to r_2} \end{array}$$

$$\left[\begin{array}{ccc} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right] \xrightarrow{r_1 - r_2 \to r_1} \boxed{\left[\begin{array}{ccc} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right] = rref(A)}$$

Note it is customary to read multiple row operations from top to bottom if more than one is listed between two of the matrices. The multiple arrow notation should be used with caution as it has great potential to confuse. Also, you might notice that I did not strictly-speaking follow Gauss-Jordan in the operations $3r_3 \to r_3$ *and* $5r_2 \to r_2$*. It is sometimes convenient to modify the algorithm slightly in order to avoid fractions.*

Example 1.2.4. *easy examples are sometimes disquieting, let $r \in \mathbb{R}$,*

$$v = \begin{bmatrix} 2 & -4 & 2r \end{bmatrix} \xrightarrow{\frac{1}{2}r_1 \to r_1} \boxed{\begin{bmatrix} 1 & -2 & r \end{bmatrix} = rref(v)}$$

Example 1.2.5. *here's another easy example,*

$$v = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \xrightarrow{r_3 - 3r_1 \to r_3} \boxed{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = rref(v)}$$

Example 1.2.6. *Find the rref of the matrix A given below:*

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{r_2 - r_1 \to r_2} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -1 & 0 \\ -1 & 0 & 1 & 1 & 1 \end{bmatrix} \xrightarrow{r_3 + r_1 \to r_3}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -1 & 0 \\ 0 & 1 & 2 & 2 & 2 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 2 \\ 0 & -2 & 0 & -1 & 0 \end{bmatrix} \xrightarrow{r_3 + 2r_2 \to r_3}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 2 & 2 \\ 0 & 0 & 4 & 3 & 4 \end{bmatrix} \begin{array}{c} \xrightarrow{4r_1 \to r_1} \\ \xrightarrow{2r_2 \to r_2} \end{array} \begin{bmatrix} 4 & 4 & 4 & 4 & 4 \\ 0 & 2 & 4 & 4 & 4 \\ 0 & 0 & 4 & 3 & 4 \end{bmatrix} \begin{array}{c} \xrightarrow{r_2 - r_3 \to r_2} \\ \xrightarrow{r_1 - r_3 \to r_1} \end{array}$$

$$\begin{bmatrix} 4 & 4 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 4 & 3 & 4 \end{bmatrix} \xrightarrow{r_1 - 2r_2 \to r_1} \begin{bmatrix} 4 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 1 & 0 \\ 0 & 0 & 4 & 3 & 4 \end{bmatrix} \begin{array}{c} \xrightarrow{r_1/4 \to r_1} \\ \xrightarrow{r_2/2 \to r_2} \\ \xrightarrow{r_3/4 \to r_3} \end{array}$$

$$\boxed{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 3/4 & 1 \end{bmatrix} = rref(A)}$$

Example 1.2.7.

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ 4 & 4 & 4 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_2 - 2r_1 \rightarrow r_2} \xrightarrow{r_3 - 4r_1 \rightarrow r_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & -2 & 1 & 0 \\ 0 & 4 & 4 & -4 & 0 & 1 \end{array}\right] \xrightarrow{r_3 - 2r_2 \rightarrow r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & -2 & 1 & 0 \\ 0 & 0 & 4 & 0 & -2 & 1 \end{array}\right] \xrightarrow{r_2/2 \rightarrow r_2} \xrightarrow{r_3/4 \rightarrow r_3}$$

$$\boxed{\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & -1/2 & 1/4 \end{array}\right] = rref[A|I]}$$

Example 1.2.8.

$$A = \left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 3 & 2 & 0 & 0 \end{array}\right] \xrightarrow{r_4 - 3r_1 \rightarrow r_4} \left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 2 & -3 & 0 \end{array}\right] \xrightarrow{r_4 - r_2 \rightarrow r_4}$$

$$\left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & -3 & 0 \end{array}\right] \xrightarrow{r_4 + r_3 \rightarrow r_4} \left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_3 - r_4 \rightarrow r_3}$$

$$\left[\begin{array}{cccc} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_2/2 \rightarrow r_2} \xrightarrow{r_3/3 \rightarrow r_3} \xrightarrow{r_1 - r_3 \rightarrow r_1} \boxed{\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] = rref(A)}$$

Proposition 1.2.9.

If a particular column of a matrix is all zeros then it will be unchanged by the Gaussian elimination. Additionally, if we know $rref(A) = B$ then $rref[A|0] = [B|0]$ where 0 denotes one or more columns of zeros.

Proof: adding nonzero multiples of one row to another will result in adding zero to zero in the column. Likewise, if we multiply a row by a nonzero scalar then the zero column is uneffected. Finally, if we swap rows then this just interchanges two zeros. Gauss-Jordan elimination is just a finite sequence of these three basic row operations thus the column of zeros will remain zero as claimed. □

Example 1.2.10. *Use Example 1.2.3 and Proposition 1.2.9 to calculate,*

$$rref\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 3 & 2 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Similarly, use Example 1.2.5 and Proposition 1.2.9 to calculate:

$$rref\left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{cccc} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

I hope these examples suffice. One last advice, you should think of the Gauss-Jordan algorithm as a sort of road-map. It's ok to take detours to avoid fractions and such but the end goal should remain in sight. If you lose sight of that it's easy to go in circles. Incidentally, I would strongly recommend you find a way to check your calculations with technology. Mathematica, Maple or Matlab etc... will do any matrix calculation we learn. TI-85 and higher will do much of what we do with a few exceptions here and there. There are even websites which will do row operations, I provide a link on the course website.

Finally, let us once more note that the $rref$ of a matrix is unique:

Theorem 1.2.11.

Let $A \in \mathbb{R}^{m\times n}$ then if R_1 and R_2 are both Gauss-Jordan eliminations of A then $R_1 = R_2$. In other words, the reduced row echelon form of a matrix of real numbers is unique.

1.3 classification of solutions

Surprisingly Examples 1.1.8,1.1.9 and 1.1.10 illustrate all the possible types of solutions for a linear system. In this section I interpret the calculations of the last section as they correspond to solving systems of equations.

Example 1.3.1. *Solve the following system of linear equations if possible,*

$$\begin{aligned} &x + 2y - 3z = 1 \\ &2x + 4y = 7 \\ &-x + 3y + 2z = 0 \end{aligned}$$

We solve by doing Gaussian elimination on the augmented coefficient matrix (see Example 1.2.2 for details of the Gaussian elimination),

$$rref\left[\begin{array}{ccc|c} 1 & 2 & -3 & 1 \\ 2 & 4 & 0 & 7 \\ -1 & 3 & 2 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 83/30 \\ 0 & 1 & 0 & 11/30 \\ 0 & 0 & 1 & 5/6 \end{array}\right] \Rightarrow \boxed{\begin{array}{c} x = 83/30 \\ y = 11/30 \\ z = 5/6 \end{array}}$$

(We used the results of Example 1.2.2).

Remark 1.3.2.

The geometric interpretation of the last example is interesting. The equation of a plane with normal vector $\langle a, b, c\rangle$ is $ax + by + cz = d$. Each of the equations in the system of Example 1.2.2 has a solution set which is in one-one correspondance with a particular plane in $\mathbb{R}^3$. The intersection of those three planes is the single point $(83/30, 11/30, 5/6)$.

Example 1.3.3. *Solve the following system of linear equations if possible,*

$$\begin{aligned} x - y &= 1 \\ 3x - 3y &= 0 \\ 2x - 2y &= -3 \end{aligned}$$

Gaussian elimination on the augmented coefficient matrix reveals (see Example 1.2.3 for details of the Gaussian elimination)

$$rref\left[\begin{array}{cc|c} 1 & -1 & 1 \\ 3 & -3 & 0 \\ 2 & -2 & -3 \end{array}\right] = \left[\begin{array}{ccc} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array}\right]$$

which shows the system has $\boxed{\textit{no solutions}}$. *The given equations are inconsistent.*

Remark 1.3.4.

The geometric interpretation of the last example is also interesting. The equation of a line in the xy-plane is is $ax + by = c$, hence the solution set of a particular equation corresponds to a line. To have a solution to all three equations at once that would mean that there is an intersection point which lies on all three lines. In the preceding example there is no such point.

Example 1.3.5. *Solve the following system of linear equations if possible,*

$$\begin{aligned} x - y + z &= 0 \\ 3x - 3y &= 0 \\ 2x - 2y - 3z &= 0 \end{aligned}$$

Gaussian elimination on the augmented coefficient matrix reveals (see Example 1.2.10 for details of the Gaussian elimination)

$$rref\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 3 & -3 & 0 & 0 \\ 2 & -2 & -3 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \boxed{\begin{array}{c} x - y = 0 \\ z = 0 \end{array}}$$

The row of zeros indicates that we will not find a unique solution. We have a choice to make, either x or y can be stated as a function of the other. Typically in linear algebra we will solve for the *variables that correspond to the pivot columns in terms of the non-pivot column variables. In this problem the pivot columns are the first column which corresponds to the variable x and the third column which corresponds the variable z. The variables x, z are called* **basic variables** *while y is called a* **free** *variable. The solution set is* $\boxed{\{(y, y, 0) \mid y \in \mathbb{R}\}}$*; in other words, $x = y, y = y$ and $z = 0$ for all $y \in \mathbb{R}$.*

You might object to the last example. You might ask why is y the free variable and not x. This is roughly equivalent to asking the question why is y the dependent variable and x the independent variable in the usual calculus. However, the roles are reversed. In the preceding example the variable x depends on y. Physically there may be a reason to distinguish the roles of one variable over another. There may be a clear cause-effect relationship which the mathematics fails to capture. For example, velocity of a ball in flight depends on time, but does time depend on the ball's velocty ? I'm guessing no. So time would seem to play the role of independent variable. However, when we write equations such as $v = v_o - gt$ we can just as well write $t = \frac{v - v_o}{-g}$; the algebra alone does not reveal which variable should be taken as "independent". Hence, a choice must be made. In the case of infinitely many solutions, we customarily **choose** the pivot variables as the "dependent" or "basic" variables and the non-pivot variables as the "free" variables. Sometimes the word *parameter* is used instead of variable, it is synonomous.

Example 1.3.6. *Solve the following (silly) system of linear equations if possible,*

$$\begin{aligned} &x = 0 \\ &0x + 0y + 0z = 0 \\ &3x = 0 \end{aligned}$$

Gaussian elimination on the augmented coefficient matrix reveals (see Example 1.2.10 for details of the Gaussian elimination)

$$rref \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \end{array} \right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

we find the solution set $\boxed{\{(0, y, z) \mid \ y, z \in \mathbb{R}\}}$*. No restriction is placed the free variables y and z.*

Example 1.3.7. *Solve the following system of linear equations if possible,*

$$\begin{aligned} &x_1 + x_2 + x_3 + x_4 = 1 \\ &x_1 - x_2 + x_3 = 1 \\ &-x_1 + x_3 + x_4 = 1 \end{aligned}$$

Gaussian elimination on the augmented coefficient matrix reveals (see Example 1.2.6 for details of the Gaussian elimination)

$$rref \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 1 & 1 \end{array} \right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 3/4 & 1 \end{array} \right]$$

We find solutions of the form $x_1 = 0, \ x_2 = -x_4/2, \ x_3 = 1 - 3x_4/4$ *where* $x_4 \in \mathbb{R}$ *is free. The solution set is a subset of* $\mathbb{R}^4$*, namely* $\boxed{\{(0, -2s, 1-3s, 4s) \mid s \in \mathbb{R}\}}$ *(I used* $s = 4x_4$ *to get rid of the annoying fractions).*

Remark 1.3.8.

The geometric interpretation of the last example is difficult to visualize. Equations of the form $a_1x_1 + a_2x_2 + a_3x_3 + a_4x_4 = b$ represent volumes in $\mathbb{R}^4$, they're called *h*yperplanes. The solution is parametrized by a single free variable, this means it is a line. We deduce that the three hyperplanes corresponding to the given system intersect along a line. Geometrically solving two equations and two unknowns isn't too hard with some graph paper and a little patience you can find the solution from the intersection of the two lines. When we have more equations and unknowns the geometric solutions are harder to grasp. Analytic geometry plays a secondary role in this course so if you have not had calculus III then don't worry too much. I should tell you what you need to know in these notes.

Example 1.3.9. *Solve the following system of linear equations if possible,*

$$\begin{aligned} &x_1 + x_4 = 0 \\ &2x_1 + 2x_2 + x_5 = 0 \\ &4x_1 + 4x_2 + 4x_3 = 1 \end{aligned}$$

Gaussian elimination on the augmented coefficient matrix reveals (see Example 1.2.7 for details of the Gaussian elimination)

$$rref \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ 4 & 4 & 4 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & -1/2 & 1/4 \end{array}\right]$$

Consequently, x_4, x_5 *are free and solutions are of the form*

$$\boxed{\begin{aligned} x_1 &= -x_4 \\ x_2 &= x_4 - \tfrac{1}{2}x_5 \\ x_3 &= \tfrac{1}{4} + \tfrac{1}{2}x_5 \end{aligned}}$$

for all $x_4, x_5 \in \mathbb{R}$*.*

Example 1.3.10. *Solve the following system of linear equations if possible,*

$$\begin{aligned} &x_1 + x_3 = 0 \\ &2x_2 = 0 \\ &3x_3 = 1 \\ &3x_1 + 2x_2 = 0 \end{aligned}$$

Gaussian elimination on the augmented coefficient matrix reveals (see Example 1.2.8 for details of the Gaussian elimination)

$$rref\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 3 & 1 \\ 3 & 2 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Therefore,there are $\boxed{\textit{no solutions}}$.

Example 1.3.11. *Solve the following system of linear equations if possible,*

$$\begin{aligned} &x_1 + x_3 = 0 \\ &2x_2 = 0 \\ &3x_3 + x_4 = 0 \\ &3x_1 + 2x_2 = 0 \end{aligned}$$

Gaussian elimination on the augmented coefficient matrix reveals (see Example 1.2.10 for details of the Gaussian elimination)

$$rref\left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 \\ 3 & 2 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$$

Therefore, the unique solution is $\boxed{x_1 = x_2 = x_3 = x_4 = 0}$. *The solution set here is rather small, it's* $\{(0, 0, 0, 0)\}$.

Remark 1.3.12.

Incidentally, you might notice that the Gauss-Jordan algorithm did not assume all the structure of the real numbers. For example, we never needed to use the ordering relations $<$ or $>$. All we needed was addition, subtraction and the ability to multiply by the inverse of a nonzero number. Any **field** of numbers will likewise work. Theorems 1.5.1 and 1.2.11 also hold for matrices of rational ($\mathbb{Q}$) or complex ($\mathbb{C}$) numbers. We will encounter problems which require calculation in $\mathbb{C}$. If you are interested in encryption then calculations over a finite field $\mathbb{Z}_p$ are necessary. In contrast, Gausssian elimination does not work for matrices of integers since we do not have fractions to work with in that context. Some such questions are dealt with in Abstract Algebra I and II.

1.4 applications to curve fitting and circuits

We can use linear algebra to solve problems which reduce to linear equations. Some problems the model itself is linear whereas others require some thought or substitution to make the linearity manifest. I let you explore some substitution problems in the homework. We consider the standard examples in this section.

Example 1.4.1. *Find a polynomial $P(x)$ whose graph $y = P(x)$ fits through the points $(0, -2.7)$, $(2, -4.5)$ and $(1, 0.97)$. We expect a quadratic polynomial will do nicely here: let A, B, C be the coefficients so $P(x) = Ax^2 + Bx + C$. Plug in the data,*

$$\begin{aligned} P(0) &= C = -2.7 \\ P(2) &= 4A + 2B + C = -4.5 \\ P(1) &= A + B + C = 0.97 \end{aligned} \quad \Rightarrow \quad \left[\begin{array}{ccc|c} A & B & C & \\ \hline 0 & 0 & 1 & -2.7 \\ 4 & 2 & 1 & -4.5 \\ 1 & 1 & 1 & 0.97 \end{array}\right]$$

I put in the A, B, C labels just to emphasize the form of the augmented matrix. We can then perform Gaussian elimination on the matrix (I omit the details) to solve the system,

$$rref\left[\begin{array}{ccc|c} 0 & 0 & 1 & -2.7 \\ 4 & 2 & 1 & -4.5 \\ 1 & 1 & 1 & 0.97 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & -4.52 \\ 0 & 1 & 0 & 8.14 \\ 0 & 0 & 1 & -2.7 \end{array}\right] \quad \Rightarrow \quad \begin{array}{l} A = -4.52 \\ B = 8.14 \\ C = -2.7 \end{array}$$

The requested polynomial is $\boxed{P(x) = -4.52x^2 + 8.14x - 2.7}$.

Example 1.4.2. *Find which cubic polynomial $Q(x)$ have a graph $y = Q(x)$ which fits through the points $(0, -2.7)$, $(2, -4.5)$ and $(1, 0.97)$. Let A, B, C, D be the coefficients of $Q(x) = Ax^3 + Bx^2 + Cx + D$. Plug in the data,*

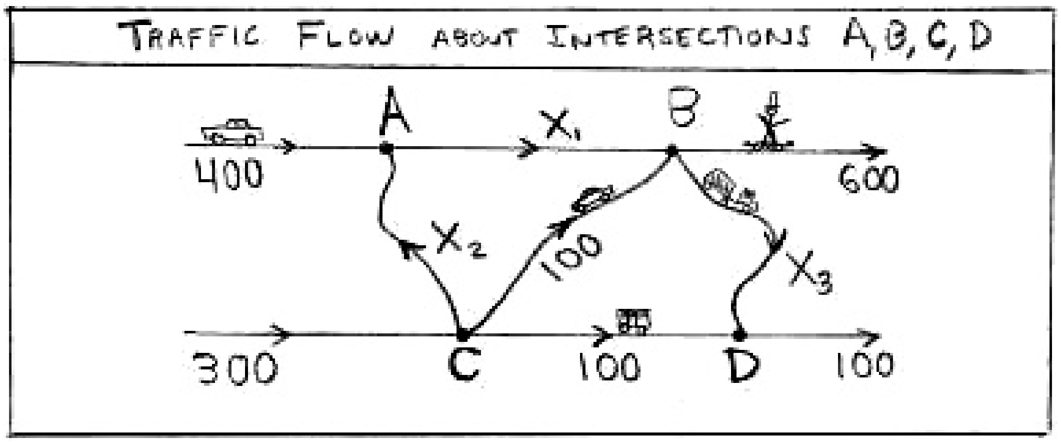
$$\begin{aligned} Q(0) &= D = -2.7 \\ Q(2) &= 8A + 4B + 2C + D = -4.5 \\ Q(1) &= A + B + C + D = 0.97 \end{aligned} \quad \Rightarrow \quad \left[\begin{array}{cccc|c} A & B & C & D & \\ \hline 0 & 0 & 0 & 1 & -2.7 \\ 8 & 4 & 2 & 1 & -4.5 \\ 1 & 1 & 1 & 1 & 0.97 \end{array}\right]$$

I put in the A, B, C, D labels just to emphasize the form of the augmented matrix. We can then perform Gaussian elimination on the matrix (I omit the details) to solve the system,

$$rref\left[\begin{array}{cccc|c} 0 & 0 & 0 & 1 & -2.7 \\ 8 & 4 & 2 & 1 & -4.5 \\ 1 & 1 & 1 & 1 & 0.97 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & -0.5 & 0 & -4.07 \\ 0 & 1 & 1.5 & 0 & 7.69 \\ 0 & 0 & 0 & 1 & -2.7 \end{array}\right] \quad \Rightarrow \quad \begin{array}{l} A = -4.07 + 0.5C \\ B = 7.69 - 1.5C \\ C = C \\ D = -2.7 \end{array}$$

It turns out there is a whole family of cubic polynomials which will do nicely. For each $C \in \mathbb{R}$ the polynomial is $\boxed{Q_C(x) = (-4.07 + 0.5C)x^3 + (7.69 - 1.5C)x^2 + Cx - 2.7}$ *fits the given points. We asked a question and found that it had infinitely many answers. Notice the choice $C = 8.14$ gets us back to the last example, in that case $Q_C(x)$ is not really a cubic polynomial.*

Example 1.4.3. *Consider the following traffic-flow pattern. The diagram indicates the flow of cars between the intersections A, B, C, D. Our goal is to analyze the flow and determine the missing pieces of the puzzle, what are the flow-rates x_1, x_2, x_3. We assume all the given numbers are cars per hour, but we omit the units to reduce clutter in the equations.*



We model this by one simple principle: conservation of vehicles

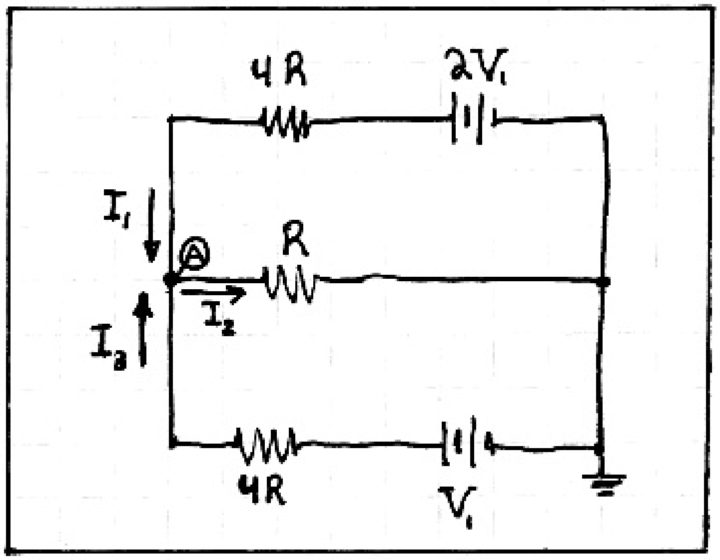
$$\begin{array}{ll} A: & x_1 - x_2 - 400 = 0 \\ B: & -x_1 + 600 - 100 + x_3 = 0 \\ C: & -300 + 100 + 100 + x_2 = 0 \\ D: & -100 + 100 + x_3 = 0 \end{array}$$

This gives us the augmented-coefficient matrix and Gaussian elimination that follows (we have to rearrange the equations to put the constants on the right and the variables on the left before we translate to matrix form)

$$rref \left[\begin{array}{ccc|c} 1 & -1 & 0 & 400 \\ -1 & 0 & 1 & -500 \\ 0 & 1 & 0 & 100 \\ 0 & 0 & 1 & 0 \end{array} \right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 500 \\ 0 & 1 & 0 & 100 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

From this we conclude, $x_3 = 0, x_2 = 100, x_1 = 500$. By the way, this sort of system is called **overdetermined** *because we have more equations than unknowns. If such a system is consistent they're often easy to solve. In truth, the rref business is completely unecessary here. I'm just trying to illustrate what can happen.*

Example 1.4.4. *Let $R = 1\Omega$ and $V_1 = 8V$. Determine the voltage V_A and currents I_1, I_2, I_3 flowing in the circuit as pictured below:*



Conservation of charge implies the sum of currents into a node must equal the sum of the currents flowing out of the node. We use Ohm's Law $V = IR$ to set-up the currents, here V should be the voltage dropped across the resistor R.

$$I_1 = \frac{2V_1 - V_A}{4R} \quad \text{Ohm's Law}$$

$$I_2 = \frac{V_A}{R} \quad \text{Ohm's Law}$$

$$I_3 = \frac{V_1 - V_A}{4R} \quad \text{Ohm's Law}$$

$$I_2 = I_1 + I_3 \quad \text{Conservation of Charge at node A}$$

Substitute the first three equations into the fourth to obtain

$$\frac{V_A}{R} = \frac{2V_1 - V_A}{4R} + \frac{V_1 - V_A}{4R}$$

Multiply by $4R$ and we find

$$4V_A = 2V_1 - V_A + V_1 - V_A \quad \Rightarrow \quad 6V_A = 3V_1 \quad \Rightarrow \quad V_A = V_1/2 = 4V.$$

Substituting back into the Ohm's Law equations we determine $I_1 = \frac{16V - 4V}{4\Omega} = 3A$, $I_2 = \frac{4V}{1\Omega} = 4A$ and $I_3 = \frac{8V - 4V}{4\Omega} = 1A$. This obvious checks with $I_2 = I_1 + I_3$. In practice it's not always best to use the full-power of the rref.

There are many other applications to consider, perhaps we shall see additional ones in lecture.

1.5 conclusions

The theorems given below form the base of our logic for this course. Proofs can be found in my Linear Algebra notes and in many other texts.

Theorem 1.5.1.

Given a system of m linear equations and n unknowns the solution set falls into one of the following cases:

1. the solution set is empty.
2. the solution set has only one element.
3. the solution set is infinite.

Theorem 1.5.2.

Suppose that $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{m\times p}$ then the first n columns of $rref[A]$ and $rref[A|B]$ are identical.

Theorem 1.5.3.

Given n-linear equations in n-unknowns $Ax = b$, a unique solution x exists iff $rref[A|b] = [I|x]$. Moreover, if $rref[A] \neq I$ then there is no unique solution to the system of equations.

There is much more to say about the meaning of particular patterns in the reduced row echelon form of the matrix. We will continue to mull over these matters in later portions of the course. Theorem 1.5.1 provides us the big picture. Again, I find it remarkable that two equations and two unknowns already revealed these patterns.

Chapter 2

matrix arithmetic I

In the preceding chapter I have used some matrix terminolgy in passing as if you already knew the meaning of such terms as "row", "column" and "matrix". I do hope you have had some previous exposure to basic matrix math, but this chapter should be self-contained. I'll start at the beginning and define all the terms.

2.1 basic terminology and notation

Definition 2.1.1.

An $m \times n$ matrix is an array of numbers with m rows and n columns. The elements in the array are called entries or components. If A is an $m \times n$ matrix then A_{ij} denotes the number in the i-th row and the j-th column. The label i is a row index and the index j is a column index in the preceding sentence. We usually denote $A = [A_{ij}]$. The set $m \times n$ of matrices with real number entries is denoted $\mathbb{R}^{m \times n}$. The set of $m \times n$ matrices with complex entries is $\mathbb{C}^{m \times n}$. If a matrix has the same number of rows and columns then it is called a **square matrix**.

Matrices can be constructed from set-theoretic arguments in much the same way as Cartesian Products. I will not pursue those matters in these notes. We will assume that everyone understands how to construct an array of numbers.

Example 2.1.2. *Suppose* $A = \left[\begin{smallmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{smallmatrix}\right]$. *We see that* A *has* 2 *rows and* 3 *columns thus* $A \in \mathbb{R}^{2\times 3}$. *Moreover,* $A_{11} = 1$, $A_{12} = 2$, $A_{13} = 3$, $A_{21} = 4$, $A_{22} = 5$, *and* $A_{23} = 6$. *It's not usually possible to find a formula for a generic element in the matrix, but this matrix satisfies* $A_{ij} = 3(i-1) + j$ *for all* i, j.

In the statement "for all i, j" it is to be understood that those indices range over their allowed values. In the preceding example $1 \leq i \leq 2$ and $1 \leq j \leq 3$.

Definition 2.1.3.

Two matrices A and B are equal if and only if they have the same size and $A_{ij} = B_{ij}$ for all i, j.

If you studied vectors before you should identify this is precisely the same rule we used in calculus III. Two vectors were equal iff all the components matched. Vectors are just specific cases of matrices so the similarity is not surprising.

Definition 2.1.4.

Let $A \in \mathbb{R}^{m\times n}$ then a submatrix of A is a matrix which is made of some rectangle of elements in A. Rows and columns are submatrices. In particular,

1. An $m \times 1$ submatrix of A is called a column vector of A. The j-th **column vector** is denoted $col_j(A)$ and $(col_j(A))_i = A_{ij}$ for $1 \leq i \leq m$. In other words,

$$col_k(A) = \begin{bmatrix} A_{1k} \\ A_{2k} \\ \vdots \\ A_{mk} \end{bmatrix} \Rightarrow A = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} = [col_1(A)|col_2(A)|\cdots|col_n(A)]$$

2. An $1 \times n$ submatrix of A is called a row vector of A. The i-th **row vector** is denoted $row_i(A)$ and $(row_i(A))_j = A_{ij}$ for $1 \leq j \leq n$. In other words,

$$row_k(A) = \begin{bmatrix} A_{k1} & A_{k2} & \cdots & A_{kn} \end{bmatrix} \Rightarrow A = \begin{bmatrix} A_{11} & A_{21} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} = \begin{bmatrix} row_1(A) \\ \hline row_2(A) \\ \hline \vdots \\ \hline row_m(A) \end{bmatrix}$$

Suppose $A \in \mathbb{R}^{m\times n}$, note for $1 \leq j \leq n$ we have $col_j(A) \in \mathbb{R}^{m\times 1}$ whereas for $1 \leq i \leq m$ we find $row_i(A) \in \mathbb{R}^{1\times n}$. In other words, an $m \times n$ matrix has n columns of length m and n rows of length m.

Example 2.1.5. *Suppose* $A = \left[\begin{smallmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{smallmatrix}\right]$. *The columns of* A *are,*

$$col_1(A) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}, \ col_2(A) = \begin{bmatrix} 2 \\ 5 \end{bmatrix}, \ col_3(A) = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

The rows of A *are*

$$row_1(A) = \begin{bmatrix} 1 & 2 & 3 \end{bmatrix}, \ row_2(A) = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}$$

Definition 2.1.6.

Let $A \in \mathbb{R}^{m \times n}$ then $A^T \in \mathbb{R}^{n \times m}$ is called the **transpose** of A and is defined by $(A^T)_{ji} = A_{ij}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$.

Example 2.1.7. *Suppose* $A = \left[\begin{smallmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{smallmatrix}\right]$ *then* $A^T = \left[\begin{smallmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{smallmatrix}\right]$. *Notice that*

$$row_1(A) = col_1(A^T), \ row_2(A) = col_2(A^T)$$

and

$$col_1(A) = row_1(A^T), \ col_2(A) = row_2(A^T), \ col_3(A) = row_3(A^T)$$

Notice $(A^T)_{ij} = A_{ji} = 3(j-1)+i$ *for all* i, j; *at the level of index calculations we just switch the indices to create the transpose.*

The preceding example shows us that we can quickly create the transpose of a given matrix by switching rows to columns. The transpose of a row vector is a column vector and vice-versa.

Remark 2.1.8. *notation, we choose* $\mathbb{R}^n$ *to be column vectors.*

It is customary in analytic geometry to denote $\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \mid x_i \in \mathbb{R} \ for \text{ all } i\}$ as the set of points in n-dimensional space. There is a natural correspondence between points and vectors. Notice that $\mathbb{R}^{1 \times n} = \{[x_1 \ x_2 \ \cdots \ x_n] \mid x_i \in \mathbb{R} \ for \text{ all } i\}$ and $\mathbb{R}^{n \times 1} = \{[x_1 \ x_2 \ \cdots \ x_n]^T \mid x_i \in \mathbb{R} \ for \text{ all } i\}$ are naturally identified with $\mathbb{R}^n$. There is a bijection between points and row or column vectors. For example, $\Phi : \mathbb{R}^{n \times 1} \to \mathbb{R}^{1 \times n}$ defined by transposition

$$\Phi[x_1 \ x_2 \ \dots \ x_n] = [x_1 \ x_2 \ \cdots x_n]^T$$

gives a one-one correspondence between row and column vectors. It is customary to use $\mathbb{R}^n$ in the place of $\mathbb{R}^{1 \times n}$ or $\mathbb{R}^{n \times 1}$ when it is convenient. This means I can express solutions to linear systems as a column vector or as a point. For example, $x + y = 2, \ x - y = 0$ has solution can be denoted by "$x = 1, y = 1$", or $(1, 1)$, or $\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$, or $[\,1 \ 1\,]$ or $[1, 1]^T$. By default I will use the convention $\mathbb{R}^n = \mathbb{R}^{n \times 1}$ and the somewhat subtle notation that

$$\boxed{[v_1, v_2, \dots, v_n]^T = (v_1, v_2, \dots, v_n).}$$

I use this convention in most everything I've written past about 2010.

2.2 addition and multiplication by scalars

Definition 2.2.1.

Let $A, B \in \mathbb{R}^{m \times n}$ then $A + B \in \mathbb{R}^{m \times n}$ is defined by $(A + B)_{ij} = A_{ij} + B_{ij}$ for all $1 \leq i \leq m$, $1 \leq j \leq n$. If two matrices A, B are not of the same size then there sum is not defined.

Example 2.2.2. *Let* $A = \left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right]$ *and* $B = \left[\begin{smallmatrix} 5 & 6 \\ 7 & 8 \end{smallmatrix}\right]$. *We calculate*

$$A + B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}.$$

Definition 2.2.3.

Let $A, B \in \mathbb{R}^{m\times n}, c \in \mathbb{R}$ then $cA \in \mathbb{R}^{m\times n}$ is defined by $(cA)_{ij} = cA_{ij}$ for all $1 \leq i \leq m$, $1 \leq j \leq n$. We call the process of multiplying A by a number c **multiplication by a scalar**. We define $A - B \in \mathbb{R}^{m\times n}$ by $A - B = A + (-1)B$ which is equivalent to $(A - B)_{ij} = A_{ij} - B_{ij}$ for all i, j.

Example 2.2.4. *Let* $A = \left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right]$ *and* $B = \left[\begin{smallmatrix} 5 & 6 \\ 7 & 8 \end{smallmatrix}\right]$. *We calculate*

$$A - B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} -4 & -4 \\ -4 & -4 \end{bmatrix}.$$

Now multiply A *by the scalar* 5,

$$5A = 5\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 5 & 10 \\ 15 & 20 \end{bmatrix}$$

Example 2.2.5. *Let* $A, B \in \mathbb{R}^{m\times n}$ *be defined by* $A_{ij} = 3i + 5j$ *and* $B_{ij} = i^2$ *for all* i, j. *Then we can calculate* $(A + B)_{ij} = 3i + 5j + i^2$ *for all* i, j.

Definition 2.2.6.

The **zero matrix** in $\mathbb{R}^{m\times n}$ is denoted 0 and defined by $0_{ij} = 0$ for all i, j. The additive inverse of $A \in \mathbb{R}^{m\times n}$ is the matrix $-A$ such that $A + (-A) = 0$. The components of the additive inverse matrix are given by $(-A)_{ij} = -A_{ij}$ for all i, j.

The zero matrix joins a long list of other objects which are all denoted by 0. Usually the meaning of 0 is clear from the context, the size of the zero matrix is chosen as to be consistent with the equation in which it is found.

Example 2.2.7. *Solve the following matrix equation,*

$$0 = \begin{bmatrix} x & y \\ z & w \end{bmatrix} + \begin{bmatrix} -1 & -2 \\ -3 & -4 \end{bmatrix}$$

Equivalently,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} x-1 & y-2 \\ z-3 & w-4 \end{bmatrix}$$

The definition of matrix equality means this single matrix equation reduces to 4 scalar equations: $0 = x - 1, 0 = y - 2, 0 = z - 3, 0 = w - 4$. *The solution is* $x = 1, y = 2, z = 3, w = 4$.

Theorem 2.2.8.

If $A \in \mathbb{R}^{m\times n}$ then

1. $0 \cdot A = 0$, (where 0 on the L.H.S. is the number zero)
2. $0A = 0$,
3. $A + 0 = 0 + A = A$.

Proof: I'll prove (2.). Let $A \in \mathbb{R}^{m\times n}$ and consider

$$(0A)_{ij} = \sum_{k=1}^{m} 0_{ik}A_{kj} = \sum_{k=1}^{m} 0A_{kj} = \sum_{k=1}^{m} 0 = 0$$

for all i, j. Thus $0A = 0$. I leave the other parts to the reader, the proofs are similar. $\square$

2.3 matrix multiplication

One very special fact about matrices is that when they are the right sizes we can multiply them.

Definition 2.3.1.

Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$ then the product of A and B is denoted by juxtaposition AB and $AB \in \mathbb{R}^{m\times p}$ is defined by:

$$\boxed{(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj}}$$

for each $1 \leq i \leq m$ and $1 \leq j \leq p$. In the case $m = p = 1$ the indices i, j are omitted in the equation since the matrix product is simply a number which needs no index.

This definition is very nice for general proofs, but pragmatically I usually think of matrix multiplication in terms of *dot-products*.

Definition 2.3.2.

Let $v, w \in \mathbb{R}^{n\times 1}$ then $v \cdot w = v_1w_1 + v_2w_2 + \cdots + v_nw_n = \sum_{k=1}^{n} v_kw_k$

Proposition 2.3.3.

Let $v, w \in \mathbb{R}^{n\times 1}$ then $v \cdot w = v^Tw$.

Proof: Since v^T is an $1 \times n$ matrix and w is an $n \times 1$ matrix the definition of matrix multiplication indicates v^Tw should be a 1×1 matrix which is a number. Note in this case the outside indices ij are absent in the boxed equation so the equation reduces to

$$v^Tw = {v^T}_1w_1 + {v^T}_2w_2 + \cdots + {v^T}_nw_n = v_1w_1 + v_2w_2 + \cdots + v_nw_n = v \cdot w.\square$$

Proposition 2.3.4.

The formula given below is equivalent to the Definition 6.5.1. Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$ then

$$AB = \begin{bmatrix} row_1(A)\cdot col_1(B) & row_1(A)\cdot col_2(B) & \cdots & row_1(A)\cdot col_p(B) \\ row_2(A)\cdot col_1(B) & row_2(A)\cdot col_2(B) & \cdots & row_2(A)\cdot col_p(B) \\ \vdots & \vdots & \cdots & \vdots \\ row_m(A)\cdot col_1(B) & row_m(A)\cdot col_2(B) & \cdots & row_m(A)\cdot col_p(B) \end{bmatrix}$$

Proof: The formula above claims $(AB)_{ij} = row_i(A)\cdot col_j(B)$ for all i, j. Recall that $(row_i(A))_k = A_{ik}$ and $(col_j(B))_k = B_{kj}$ thus

$$(AB)_{ij} = \sum_{k=1}^{n} A_{ik}B_{kj} = \sum_{k=1}^{n}(row_i(A))_k(col_j(B))_k$$

Hence, using definition of the dot-product, $(AB)_{ij} = row_i(A)\cdot col_j(B)$. This argument holds for all i, j therefore the Proposition is true. $\square$

Example 2.3.5. *Let* $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ *and* $v = \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix}$ *calculate* Av.

$$Av = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} (1,2,3)\cdot(1,0,-3) \\ (4,5,6)\cdot(1,0,-3) \\ (7,8,9)\cdot(1,0,-3) \end{bmatrix} = \begin{bmatrix} -2 \\ -14 \\ -20 \end{bmatrix}.$$

Example 2.3.6. *Let* $A = \left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right]$ *and* $B = \left[\begin{smallmatrix} 5 & 6 \\ 7 & 8 \end{smallmatrix}\right]$. *We calculate*

$$\begin{aligned} AB &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \\ &= \begin{bmatrix} [1,2][5,7]^T & [1,2][6,8]^T \\ [3,4][5,7]^T & [3,4][6,8]^T \end{bmatrix} \\ &= \begin{bmatrix} 5+14 & 6+16 \\ 15+28 & 18+32 \end{bmatrix} \\ &= \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix} \end{aligned}$$

Notice the product of square matrices is square. For numbers $a, b \in \mathbb{R}$ *it we know the product of a* *and b is commutative ($ab = ba$). Let's calculate the product of A and B in the opposite order,*

$$\begin{aligned} BA &= \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\ &= \begin{bmatrix} [5,6][1,3]^T & [5,6][2,4]^T \\ [7,8][1,3]^T & [7,8][2,4]^T \end{bmatrix} \\ &= \begin{bmatrix} 5+18 & 10+24 \\ 7+24 & 14+32 \end{bmatrix} \\ &= \begin{bmatrix} 23 & 34 \\ 31 & 46 \end{bmatrix} \end{aligned}$$

Clearly $AB \neq BA$ thus matrix multiplication is **noncommutative** *or* **nonabelian**.

When we say that matrix multiplication is noncommuative that indicates that the product of two matrices does not *generally* commute. However, there are special matrices which commute with other matrices.

Example 2.3.7. *Let $I = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$. We calculate*

$$IA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Likewise calculate,

$$AI = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Since the matrix A was arbitrary we conclude that $IA = AI$ for all $A \in \mathbb{R}^{2\times 2}$.

Definition 2.3.8.

The **identity matrix** in $\mathbb{R}^{n\times n}$ is the $n\times n$ square matrix I which has components $I_{ij} = \delta_{ij}$. The notation I_n is sometimes used if the size of the identity matrix needs emphasis, otherwise the size of the matrix I is to be understood from the context.

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad I_4 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Example 2.3.9. *The product of a 3×2 and 2×3 is a 3×3*

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} = \begin{bmatrix} [1,0][4,7]^T & [1,0][5,8]^T & [1,0][6,9]^T \\ [0,1][4,7]^T & [0,1][5,8]^T & [0,1][6,9]^T \\ [0,0][4,7]^T & [0,0][5,8]^T & [0,0][6,9]^T \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 7 & 8 & 9 \\ 0 & 0 & 0 \end{bmatrix}$$

Example 2.3.10. *The product of a 3×1 and 1×3 is a 3×3*

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 4 \cdot 1 & 5 \cdot 1 & 6 \cdot 1 \\ 4 \cdot 2 & 5 \cdot 2 & 6 \cdot 2 \\ 4 \cdot 3 & 5 \cdot 3 & 6 \cdot 3 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{bmatrix}$$

Example 2.3.11. *The product of a 2×2 and 2×1 is a 2×1. Let $A = \left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right]$ and let $v = \left[\begin{smallmatrix} 5 \\ 7 \end{smallmatrix}\right]$,*

$$Av = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 \\ 7 \end{bmatrix} = \begin{bmatrix} [1,2][5,7]^T \\ [3,4][5,7]^T \end{bmatrix} = \begin{bmatrix} 19 \\ 43 \end{bmatrix}$$

Likewise, define $w = \left[\begin{smallmatrix} 6 \\ 8 \end{smallmatrix}\right]$ and calculate

$$Aw = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} [1,2][6,8]^T \\ [3,4][6,8]^T \end{bmatrix} = \begin{bmatrix} 22 \\ 50 \end{bmatrix}$$

Something interesting to observe here, recall that in Example 2.3.6 we calculated $AB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$. But these are the same numbers we just found from the two matrix-vector products calculated above. We identify that B is just the **concatenation** *of the vectors v and w; $B = [v|w] = \left[\begin{array}{c|c} 5 & 6 \\ 7 & 8 \end{array}\right]$. Observe that:*

$$AB = A[v|w] = [Av|Aw].$$

The term **concatenate** is sometimes replaced with the word **adjoin**. I think of the process as gluing matrices together. This is an important operation since it allows us to lump together many solutions into a single matrix of solutions. (I will elaborate on that in detail in a future section)

Proposition 2.3.12.

Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times p}$ then we can understand the matrix multiplication of A and B as the concatenation of several matrix-vector products,

$$AB = A[col_1(B)|col_2(B)|\cdots|col_p(B)] = [Acol_1(B)|Acol_2(B)|\cdots|Acol_p(B)]$$

Proof: left to the reader. It's not too hard, you just have to think through the notation. □

Example 2.3.13. *Consider A, v, w from Example 2.3.11.*

$$v + w = \begin{bmatrix} 5 \\ 7 \end{bmatrix} + \begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 11 \\ 15 \end{bmatrix}$$

Using the above we calculate,

$$A(v+w) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 11 \\ 15 \end{bmatrix} = \begin{bmatrix} 11 + 30 \\ 33 + 60 \end{bmatrix} = \begin{bmatrix} 41 \\ 93 \end{bmatrix}.$$

In constrast, we can add Av and Aw,

$$Av + Aw = \begin{bmatrix} 19 \\ 43 \end{bmatrix} + \begin{bmatrix} 22 \\ 50 \end{bmatrix} = \begin{bmatrix} 41 \\ 93 \end{bmatrix}.$$

Behold, $A(v + w) = Av + Aw$ for this example. It turns out this is true in general.

2.3.1 multiplication of matrix by a vector is linear combination of columns

Observe that the general definition of the matrix product gives the following: If $A = [A_{ij}] \in \mathbb{R}^{k\times n}$ and $v = [v_j] \in \mathbb{R}^n$ then $(Av)_i = \sum_{j=1}^n A_{ij}v_j$ for each $i = 1, 2, \dots, k$. But, $col_j(A) = [A_{ij}] \in \mathbb{R}^k$ thus $(col_j(A))_i = A_{ij}$ and we find for each $i = 1, 2, \dots, k$.

$$(Av)_i = \sum_{j=1}^{n} A_{ij}v_j = \sum_{j=1}^{n} v_j(col_j(A))_i = \left(\sum_{j=1}^{n} v_j col_j(A)\right)_i$$

Proposition 2.3.14.

If $A \in \mathbb{R}^{k\times n}$ and $v \in \mathbb{R}^n$ then Av is a linear combination of the columns of A weighted by the components of v;

$$\boxed{Av = v_1 col_1(A) + v_2 col_2(A) + \cdots + v_n col_n(A)}$$

.

The formula above is sometimes taken as the definition of the matrix-vector product. For example, see the elementary text by Insel Spence and Friedberg.

Example 2.3.15. *Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $v = \begin{bmatrix} x \\ y \end{bmatrix}$ then we may calculate the product Av as follows:*

$$Av = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x\begin{bmatrix} 1 \\ 3 \end{bmatrix} + y\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} x + 2y \\ 3x + 4y \end{bmatrix}.$$

Example 2.3.16. *Let $A = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \end{bmatrix}$ and $v = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ then we may calculate the product Av as follows:*

$$Av = \begin{bmatrix} 1 & 1 & 1 \\ a & b & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\begin{bmatrix} 1 \\ a \end{bmatrix} + y\begin{bmatrix} 1 \\ b \end{bmatrix} + z\begin{bmatrix} 1 \\ c \end{bmatrix} = \begin{bmatrix} x + y + z \\ ax + by + cz \end{bmatrix}.$$

If you prefer then you can use the dot-product rule we first introduced. However, the pattern we considered in this subsection gives us great insight to the question of spanning later in this course.

2.3.2 rules of matrix algebra

I collect all my favorite properties for matrix multiplication in the theorem below. To summarize, matrix math works as you would expect with the exception that matrix multiplication is not commutative. We must be careful about the order of letters in matrix expressions.

Theorem 2.3.17.

If $A, B, C \in \mathbb{R}^{m\times n}$, $X, Y \in \mathbb{R}^{n\times p}$, $Z \in \mathbb{R}^{p\times q}$ and $c_1, c_2 \in \mathbb{R}$ then

1. $(A + B) + C = A + (B + C)$,
2. $(AX)Z = A(XZ)$,
3. $A + B = B + A$,
4. $c_1(A + B) = c_1A + c_2B$,
5. $(c_1 + c_2)A = c_1A + c_2A$,
6. $(c_1c_2)A = c_1(c_2A)$,
7. $(c_1A)X = c_1(AX) = A(c_1X) = (AX)c_1$,
8. $1A = A$,
9. $I_mA = A = AI_n$,
10. $A(X + Y) = AX + AY$,
11. $A(c_1X + c_2Y) = c_1AX + c_2AY$,
12. $(A + B)X = AX + BX$,

Proof: I will prove a couple of these and relegate most of the rest to the Problem Set. They actually make pretty fair proof-type test questions. Nearly all of these properties are proved by breaking the statement down to components then appealing to a property of real numbers. Just a reminder, we assume that it is known that $\mathbb{R}$ is an ordered field. Multiplication of real numbers is commutative, associative and distributes across addition of real numbers. Likewise, addition of real numbers is commutative, associative and obeys familar distributive laws when combined with addition.

Proof of (1.): assume A, B, C are given as in the statement of the Theorem. Observe that

$$\begin{aligned}
((A + B) + C)_{ij} &= (A + B)_{ij} + C_{ij} && \text{defn. of matrix add.} \\
&= (A_{ij} + B_{ij}) + C_{ij} && \text{defn. of matrix add.} \\
&= A_{ij} + (B_{ij} + C_{ij}) && \text{assoc. of real numbers} \\
&= A_{ij} + (B + C)_{ij} && \text{defn. of matrix add.} \\
&= (A + (B + C))_{ij} && \text{defn. of matrix add.}
\end{aligned}$$

for all i, j. Therefore $(A + B) + C = A + (B + C)$. $\square$
Proof of (6.): assume c_1, c_2, A are given as in the statement of the Theorem. Observe that

$$\begin{aligned} ((c_1c_2)A)_{ij} &= (c_1c_2)A_{ij} && \text{defn. scalar multiplication.} \\ &= c_1(c_2A_{ij}) && \text{assoc. of real numbers} \\ &= (c_1(c_2A))_{ij} && \text{defn. scalar multiplication.} \end{aligned}$$

for all i, j. Therefore $(c_1c_2)A = c_1(c_2A)$. $\square$
Proof of (10.): assume A, X, Y are given as in the statement of the Theorem. Observe that

$$\begin{aligned} ((A(X+Y))_{ij} &= \textstyle\sum_k A_{ik}(X+Y)_{kj} && \text{defn. matrix multiplication,} \\ &= \textstyle\sum_k A_{ik}(X_{kj} + Y_{kj}) && \text{defn. matrix addition,} \\ &= \textstyle\sum_k (A_{ik}X_{kj} + A_{ik}Y_{kj}) && \text{dist. of real numbers,} \\ &= \textstyle\sum_k A_{ik}X_{kj} + \sum_k A_{ik}Y_{kj}) && \text{prop. of finite sum,} \\ &= (AX)_{ij} + (AY)_{ij} && \text{defn. matrix multiplication}(\times\ 2), \\ &= (AX + AY)_{ij} && \text{defn. matrix addition,} \end{aligned}$$

for all i, j. Therefore $A(X + Y) = AX + AY$. $\square$

The proofs of the other items are similar, we consider the i, j-th component of the identity and then apply the definition of the appropriate matrix operation's definition. This reduces the problem to a statement about real numbers so we can use the properties of real numbers at the level of components. Then we reverse the steps. Since the calculation works for arbitrary i, j it follows the the matrix equation holds true. This Theorem provides a foundation for later work where we may find it convenient to prove a statement without resorting to a proof by components. Which method of proof is best depends on the question. However, I can't see another way of proving most of 2.3.17.

2.4 n-dimensional space and the standard basis

Two dimensional space is $\mathbb{R}^2 = \mathbb{R} \times \mathbb{R}$. To obtain n-dimensional space we just take the Cartesian product of n-copies of $\mathbb{R}$.

Definition 2.4.1.

Let $n \in \mathbb{N}$, we define $\mathbb{R}^n = \{(x_1, x_2, \dots, x_n) \mid x_j \in \mathbb{R} \ for\ j = 1, 2, \dots, n\}$. If $v \in \mathbb{R}^n$ then we say v is an **n-vector**. The numbers in the vector are called the **components**; $v = (v_1, v_2, \dots, v_n)$ has j-th component v_j.

Notice, a consequence of the definition above and the construction of the Cartesian product[1] is that two vectors v and w are equal iff $v_j = w_j$ for all j. Equality of two vectors is only true if all components are found to match. Addition and scalar multiplication are naturally generalized from the $n = 2$ case.

[1]see my Math 200 notes or ask me if interested, it's not entirely trivial

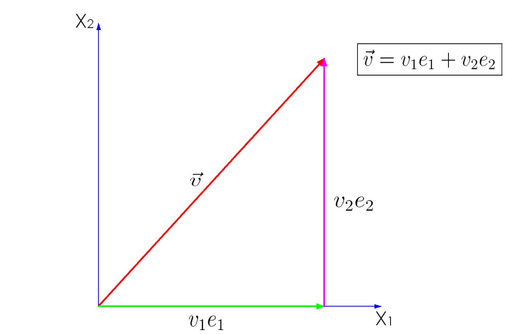
Definition 2.4.2.

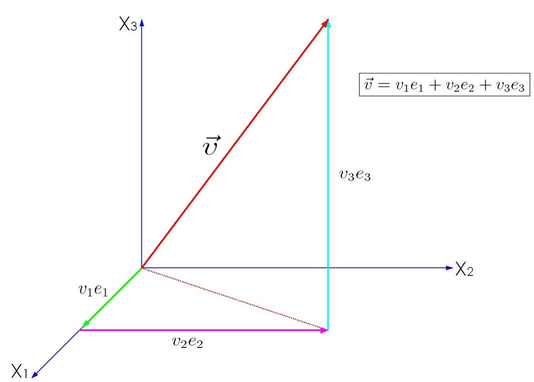
Define functions $+ : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ and $\cdot : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ by the following rules: for each $v, w \in \mathbb{R}^n$ and $c \in \mathbb{R}$:

$$(1.)\ \ (v + w)_j = v_j + w_j \qquad\qquad (2.)\ \ (cv)_j = cv_j$$

for all $j \in \{1, 2, \dots, n\}$. The operation $+$ is called **vector addition** and it takes two vectors $v, w \in \mathbb{R}^n$ and produces another vector $v + w \in \mathbb{R}^n$. The operation $\cdot$ is called **scalar multiplication** and it takes a number $c \in \mathbb{R}$ and a vector $v \in \mathbb{R}^n$ and produces another vector $c \cdot v \in \mathbb{R}^n$. Often we simply denote $c \cdot v$ by juxtaposition cv.

If you are a gifted at visualization then perhaps you can add three-dimensional vectors in your mind. If you're mind is really unhinged maybe you can even add 4 or 5 dimensional vectors. The beauty of the definition above is that we have no need of pictures. Instead, algebra will do just fine. That said, let's draw a few pictures.





Notice these pictures go to show how you can break-down vectors into component vectors which point in the direction of the coordinate axis. Vectors of length[2] one which point in the coordinate

[2]the length of vectors is an important concept which we mine in depth later in the course

directions make up what is called the **standard basis**[3] It is convenient to define special notation for the standard basis. First I define a useful shorthand,

Definition 2.4.3.

The symbol $\delta_{ij} = \begin{cases} 1 & , i = j \\ 0 & , i \neq j \end{cases}$ is called the **Kronecker delta**.

For example, $\delta_{22} = 1$ while $\delta_{12} = 0$.

Definition 2.4.4.

Let $e_i \in \mathbb{R}^{n\times 1}$ be defined by $(e_i)_j = \delta_{ij}$. The size of the vector e_i is determined by context. We call e_i the i-th standard basis vector.

Example 2.4.5. *Let me expand on what I mean by "context" in the definition above:*
In $\mathbb{R}$ *we have* $e_1 = (1) = 1$ *(by convention we drop the brackets in this case)*
In $\mathbb{R}^2$ *we have* $e_1 = (1, 0)$ *and* $e_2 = (0, 1)$.
In $\mathbb{R}^3$ *we have* $e_1 = (1, 0, 0)$ *and* $e_2 = (0, 1, 0)$ *and* $e_3 = (0, 0, 1)$.
In $\mathbb{R}^4$ *we have* $e_1 = (1, 0, 0, 0)$ *and* $e_2 = (0, 1, 0, 0)$ *and* $e_3 = (0, 0, 1, 0)$ *and* $e_4 = (0, 0, 0, 1)$.

Example 2.4.6. *Any vector in* $\mathbb{R}^n$ *can be written as a sum of these basic vectors. For example,*

$$\begin{aligned} v = (1, 2, 3) &= (1, 0, 0) + (0, 2, 0) + (0, 0, 3) \\ &= 1(1, 0, 0) + 2(0, 1, 0) + 3(0, 0, 1) \\ &= e_1 + 2e_2 + 3e_3. \end{aligned}$$

We say that v *is a* **linear combination** *of* e_1, e_2 *and* e_3.

The concept of a linear combination is very important.

Definition 2.4.7.

A **linear combination** of objects $A_1, A_2, \dots, A_k$ is a sum

$$c_1A_1 + c_2A_2 + \cdots + c_kA_k = \sum_{i=1}^{k} c_iA_i$$

where $c_i \in \mathbb{R}$ for each i.

We will look at linear combinations of vectors, matrices and even functions in this course. If $c_i \in \mathbb{C}$ then we call it a *complex linear combination*. The proposition below generalizes the calculation from Example 2.4.6.

[3]for now we use the term "basis" without meaning, in Chapter 5 we make a great effort to refine the concept.

Proposition 2.4.8.

Every vector in $\mathbb{R}^n$ is a linear combination of $e_1, e_2, \dots, e_n$.

Proof: Let $v = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$. By the definition of vector addition:

$$\begin{aligned} v &= (v_1, v_2, \dots, v_n) \\ &= (v_1, 0, \dots, 0) + (0, v_2, \dots, v_n) \\ &= (v_1, 0, \dots, 0) + (0, v_2, \dots, 0) + \cdots + (0, 0, \dots, v_n) \\ &= (v_1, 0 \cdot v_1, \dots, 0 \cdot v_1) + (0 \cdot v_2, v_2, \dots, 0 \cdot v_2) + \cdots + (0 \cdot v_n, 0 \cdot v_n, \dots, v_n) \end{aligned}$$

In the last step I rewrote each zero to emphasize that the each entry of the k-th summand has a v_k factor. Continue by applying the definition of scalar multiplication to each vector in the sum above we find,

$$\begin{aligned} v &= v_1(1, 0, \dots, 0) + v_2(0, 1, \dots, 0) + \cdots + v_n(0, 0, \dots, 1) \\ &= v_1 e_1 + v_2 e_2 + \cdots + v_n e_n. \end{aligned}$$

Therefore, every vector in $\mathbb{R}^n$ is a linear combination of $e_1, e_2, \dots, e_n$. For each $v \in \mathbb{R}^n$ we have $v = \sum_{i=1}^n v_n e_n$. $\square$

Vector notation gives us another way to look at systems of equations. We will learn much more about this way of thinking in future sections.

Example 2.4.9. *Suppose $x + y + z = 3$, $x + y = 2$ and $x - y - z = -1$. This system can be written as a single vector equation by simply stacking these equations into a column vector:*

$$\begin{bmatrix} x + y + z \\ x + y \\ x - y - z \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

Furthermore, we can break up the vector of variables into linear combination where the coefficients in the sum are the variables x, y, z:

$$x \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + z \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix}$$

Note that the solution to the system is $x = 1, y = 1, z = 1$.

Example 2.4.10. *(taken from Lay's homework, §1.6#7) Alka Seltzer makes fizzy soothing bubbles through a chemical reaction of the following type:*

$$\underbrace{NaHCO_3}_{sodium\ bicarbonate} + \underbrace{H_3C_6H_5O_7}_{citric\ acid} \rightarrow \underbrace{Na_3C_6H_5O_7}_{sodium\ citrate} + \underbrace{H_2O}_{water} + \underbrace{CO_2}_{carbon\ dioxide}$$

The reaction above is **unbalanced** *because it lacks weights to describe the relative numbers of the various molecules involved in a particular reaction. To* **balance** *the equation we seek integers x_1, x_2, x_3, x_4, x_5 such that the following reaction is balanced.*

$$x_1(NaHCO_3) + x_2(H_3C_6H_5O_7) \quad \rightarrow \quad x_3(Na_3C_6H_5O_7) + x_4(H_2O) + x_5(CO_2)$$

In a chemical reaction the atoms the enter the reaction must also leave the reaction. Atoms are neither created nor destroyed in chemical reactions[4]*. It follows that the number of sodium(Na), hydrogen(H), carbon(C) and oxygen(O) atoms must be conserved in the reaction. Each element can be represented by a component in a 4-dimensional vector;* (Na, H, C, O)*. Using this notation the equation to balance the reaction is simply:*

$$x_1 \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 8 \\ 6 \\ 7 \end{bmatrix} = x_3 \begin{bmatrix} 3 \\ 5 \\ 6 \\ 7 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

In other words, solve

$$\begin{array}{l} x_1 = 3x_3 \\ x_1 + 8x_2 = 5x_3 + 2x_4 \\ x_1 + 6x_2 = 6x_3 + x_5 \\ 3x_1 + 7x_2 = 7x_3 + x_4 + 2x_5 \end{array} \quad \Rightarrow \quad \left[\begin{array}{ccccc|c} 1 & 0 & -3 & 0 & 0 & 0 \\ 1 & 8 & -5 & -2 & 0 & 0 \\ 1 & 6 & -6 & 0 & -1 & 0 \\ 3 & 7 & -7 & -1 & -2 & 0 \end{array}\right]$$

After a few row operations we will deduce,

$$rref \left[\begin{array}{ccccc|c} 1 & 0 & -3 & 0 & 0 & 0 \\ 1 & 8 & -5 & -2 & 0 & 0 \\ 1 & 6 & -6 & 0 & -1 & 0 \\ 3 & 7 & -7 & -1 & -2 & 0 \end{array}\right] = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 & \frac{-1}{3} & 0 \\ 0 & 0 & 1 & 0 & \frac{-1}{3} & 0 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

Therefore, $x_1 = x_5, x_2 = x_5/3, x_3 = x_5/3$ *and* $x_4 = x_5$*. Atoms are indivisible (in this context) hence we need to choose* $x_5 = 3k$ *for* $k \in \mathbb{N}$ *to assure integer solutions. The basic reaction follows from* $x_5 = 3$,

$$\boxed{3NaHCO_3 + H_3C_6H_5O_7 \quad \rightarrow \quad Na_3C_6H_5O_7 + 3H_2O + 3CO_2}$$

Finding integer solutions to chemical reactions is more easily solved by the method I used as an undergraduate. You guess and check and adjust. Because the solutions are integers it's not too hard to work out. That said, if you don't want to guess then we have a method via Gaussian elimination. Chemists have more to worry about than just this algebra. If you study reactions carefully then there are a host of other considerations involving energy transfer and ultimately quantum mechanics.

2.5 all your base are belong to us (e_i and E_{ij} that is)

The purpose of this section is to introduce some compact notation that allows for elegant proofs of certain statements in n-dimensions. When we face general question these are nice results since they allow us to trade equations about matrices for simple, but arbitrary, scalar equations. This type of calculation is an example of tensorial calculation. Tensors are of great importance to modern

[4] chemistry is based on electronic interactions which do not possess the mechanisms needed for alchemy, transmutation is in fact accomplished in nuclear physics. Ironically, alchemy, while known, is not economical.

physics and engineering [5].

We defined $e_i \in \mathbb{R}^n$ by $(e_i)_j = \delta_{ij} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$. We call e_i the i-th standard basis vector. Proposition 2.4.8 reveals that every vector in $\mathbb{R}^n$ is a linear combination of $e_1, e_2, \dots, e_n$. We can define a standard basis for matrices of arbitrary size in much the same manner.

Definition 2.5.1.

The ij-th **standard basis matrix** for $\mathbb{R}^{m\times n}$ is denoted E_{ij} for $1 \leq i \leq m$ and $1 \leq j \leq n$. The matrix E_{ij} is zero in all entries except for the (i,j)-th slot where it has a 1. In other words, we define $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$.

Proposition 2.5.2.

Every matrix in $\mathbb{R}^{m\times n}$ is a linear combination of the E_{ij} where $1 \leq i \leq m$ and $1 \leq j \leq n$.

Proof: Let $A \in \mathbb{R}^{m\times n}$ then

$$\begin{aligned} A &= \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & A_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{bmatrix} \\ &= A_{11}\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix} + A_{12}\begin{bmatrix} 0 & 1 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix} + \cdots + A_{mn}\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \cdots & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix} \\ &= A_{11}E_{11} + A_{12}E_{12} + \cdots + A_{mn}E_{mn}. \end{aligned}$$

The calculation above follows from repeated mn-applications of the definition of matrix addition and another mn-applications of the definition of scalar multiplication of a matrix. We can restate the final result in a more precise langauge,

$$A = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}E_{ij}.$$

As we claimed, any matrix can be written as a linear combination of the E_{ij}. $\square$

The term "basis" has a technical meaning which we will discuss at length in due time. For now, just think of it as part of the names of e_i and E_{ij}. These are the basic building blocks for matrix theory.

[5] I discuss tensors and their calculus in the advanced calculus course when the opportunity presents itself.

Example 2.5.3. *Suppose $A \in \mathbb{R}^{m\times n}$ and $e_i \in \mathbb{R}^n$ is a standard basis vector,*

$$(Ae_i)_j = \sum_{k=1}^{n} A_{jk}(e_i)k = \sum_{k=1}^{n} A_{jk}\delta_{ik} = A_{ji}$$

Thus, $\boxed{[Ae_i] = col_i(A)}$. We find that multiplication of a matrix A by the standard basis e_i yields the $i-th$ column of A.

Example 2.5.4. *Suppose $A \in \mathbb{R}^{m\times n}$ and $e_i \in \mathbb{R}^{m\times 1}$ is a standard basis vector,*

$$({e_i}^T A)_j = \sum_{k=1}^{n} (e_i)_k A_{kj} = \sum_{k=1}^{n} \delta_{ik} A_{kj} = A_{ij}$$

Thus, $\boxed{[{e_i}^T A] = row_i(A)}$. We find multiplication of a matrix A by the transpose of standard basis e_i yields the $i-th$ row of A.

Example 2.5.5. *Again, suppose $e_i, e_j \in \mathbb{R}^n$ are standard basis vectors. The product ${e_i}^T e_j$ of the $1\times n$ and $n \times 1$ matrices is just a 1×1 matrix which is just a number. In particular consider,*

$${e_i}^T e_j = \sum_{k=1}^{n} ({e_i}^T)_k (e_j)_k = \sum_{k=1}^{n} \delta_{ik}\delta_{jk} = \delta_{ij}$$

The product is zero unless the vectors are identical.

Example 2.5.6. *Suppose $e_i \in \mathbb{R}^{m\times 1}$ and $e_j \in \mathbb{R}^n$. The product of the $m \times 1$ matrix e_i and the $1 \times n$ matrix ${e_j}^T$ is an $m \times n$ matrix. In particular,*

$$(e_i {e_j}^T)_{kl} = ({e_i}^T)_k (e_j)_k = \delta_{ik}\delta_{jk} = (E_{ij})_{kl}$$

Thus we can construct the standard basis matrices by multiplying the standard basis vectors; $E_{ij} = e_i {e_j}^T$.

Example 2.5.7. *What about the matrix E_{ij}? What can we say about multiplication by E_{ij} on the right of an arbitrary matrix? Let $A \in \mathbb{R}^{m\times n}$ and consider,*

$$(AE_{ij})_{kl} = \sum_{p=1}^{n} A_{kp}(E_{ij})_{pl} = \sum_{p=1}^{n} A_{kp}\delta_{ip}\delta_{jl} = A_{ki}\delta_{jl}$$

Notice the matrix above has zero entries unless $j = l$ which means that the matrix is mostly zero except for the j-th column. We can select the j-th column by multiplying the above by e_j, using Examples 2.5.5 and 2.5.3,

$$(AE_{ij}e_j)_k = (Ae_i {e_j}^T e_j)_k = (Ae_i\delta_{jj})_k = (Ae_i)_k = (col_i(A))_k$$

This means,

$$AE_{ij} = \begin{bmatrix} & & & \text{column } j & & \\ 0 & 0 & \cdots & A_{1i} & \cdots & 0 \\ 0 & 0 & \cdots & A_{2i} & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & A_{mi} & \cdots & 0 \end{bmatrix}$$

Right multiplication of matrix A by E_{ij} moves the i-th column of A to the j-th column of AE_{ij} and all other entries are zero. It turns out that left multiplication by E_{ij} moves the j-th row of A to the i-th row and sets all other entries to zero.

Example 2.5.8. *Let $A = \left[\begin{smallmatrix} 1 & 2 \\ 3 & 4 \end{smallmatrix}\right]$ consider multiplication by E_{12},*

$$AE_{12} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \left[\begin{array}{c|c} 0 & 1 \\ 0 & 3 \end{array}\right] = \left[\begin{array}{c|c} 0 & col_1(A) \end{array}\right]$$

Which agrees with our general abstract calculation in the previous example. Next consider,

$$E_{12}A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 0 & 0 \end{bmatrix} = \left[\begin{array}{c} row_2(A) \\ \hline 0 \end{array}\right].$$

Example 2.5.9. *Calculate the product of E_{ij} and E_{kl}.*

$$(E_{ij}E_{kl})_{mn} = \sum_p (E_{ij})_{mp}(E_{kl})_{pn} = \sum_p \delta_{im}\delta_{jp}\delta_{kp}\delta_{ln} = \delta_{im}\delta_{jk}\delta_{ln}$$

For example,

$$(E_{12}E_{34})_{mn} = \delta_{1m}\delta_{23}\delta_{4n} = 0.$$

In order for the product to be nontrivial we must have $j = k$,

$$(E_{12}E_{24})_{mn} = \delta_{1m}\delta_{22}\delta_{4n} = \delta_{1m}\delta_{4n} = (E_{14})_{mn}.$$

We can make the same identification in the general calculation,

$$(E_{ij}E_{kl})_{mn} = \delta_{jk}(E_{il})_{mn}.$$

Since the above holds for all m, n,

$$\boxed{E_{ij}E_{kl} = \delta_{jk}E_{il}}$$

this is at times a very nice formula to know about.

Remark 2.5.10.

You may find the general examples in this portion of the notes a bit too much to follow. If that is the case then don't despair. Focus on mastering the numerical examples to begin with then come back to this section later. These examples are actually not that hard, you just have to get used to index calculations. The proofs in these examples are much longer if written without the benefit of index notation.

Example 2.5.11. *Let $A \in \mathbb{R}^{m\times n}$ and suppose $e_i \in \mathbb{R}^{m\times 1}$ and $e_j \in \mathbb{R}^n$. Consider,*

$$(e_i)^T A e_j = \sum_{k=1}^{m} ((e_i)^T)_k (Ae_j)_k = \sum_{k=1}^{m} \delta_{ik}(Ae_j)_k = (Ae_j)_i = A_{ij}$$

This is a useful observation. If we wish to select the (i,j)-entry of the matrix A then we can use the following simple formula,

$$\boxed{A_{ij} = (e_i)^T A e_j}$$

This is analogus to the idea of using dot-products to select particular components of vectors in analytic geometry; (reverting to calculus III notation for a moment) recall that to find v_1 of $\vec{v}$ we learned that the dot product by $\hat{i} = \langle 1, 0, 0\rangle$ selects the first components $v_1 = \vec{v}\cdot\hat{i}$. The following theorem is simply a summary of our results for this section.

Theorem 2.5.12.

Assume $A \in \mathbb{R}^{m\times n}$ and $v \in \mathbb{R}^n$ and define $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$ and $(e_i)_j = \delta_{ij}$ as we previously discussed,

$$v = \sum_{i=1}^{n} v_n e_n \qquad A = \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}E_{ij}.$$

$$[e_i{}^T A] = row_i(A) \qquad [Ae_i] = col_i(A) \qquad A_{ij} = (e_i)^T A e_j$$

$$E_{ij}E_{kl} = \delta_{jk}E_{il} \qquad E_{ij} = e_i e_j{}^T \qquad e_i{}^T e_j = \delta_{ij}$$

Chapter 3

spanning, LI, and the CCP

The span of a set of vectors is the set of all possible linear combinations. LI is short for **linear independence**. Finally, CCP stands for the **Column Correspondence Property** which is also known as the **linear correspondence**. If a set is both a spanning set and a LI set then the set is called a basis. The number of vectors in a basis is known as the dimension of the space. We'll see how the CCP aids in a number of questions concerning the calculation of bases, LI and spanning.

3.1 systems of linear equations revisited

In the previous chapter we found that systems of equations could be efficiently solved by doing row operations on the augmented coefficient matrix. Let's return to that central topic now that we know more about matrix addition and multiplication. The proof of the proposition below is simply matrix multiplication.

Proposition 3.1.1.

Let $x_1, x_2, \dots, x_m$ be m variables and suppose $b_i, A_{ij} \in \mathbb{R}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$ then

$$A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n = b_1$$

$$A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n = b_2$$

$$\vdots \quad \vdots \quad \vdots \quad \vdots$$

$$A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n = b_m$$

is called a **system of linear equations**. We define the **coefficient matrix** A, the **inhomogeneous term** b and the **vector solution** x as follows:

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \qquad x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$$

then the system of equations is equivalent to **matrix form of the system** $Ax = b$. (sometimes we may use v or $\vec{x}$ if there is danger of confusion with the scalar variable x)

Definition 3.1.2.

Let $Ax = b$ be a system of m equations and n-unknowns and x is in the solution set of the system. In particular, we denote the solution set by $Sol_{[A|b]}$ where

$$Sol_{[A|b]} = \{x \in \mathbb{R}^{\,n\times 1} \mid Ax = b\}$$

We learned how to find the solutions to a system $Ax = b$ in the last Chapter by performing Gaussian elimination on the augmented coefficient matrix $[A|b]$. We'll discuss the structure of matrix solutions further in the next Chapter. To give you a quick preview, it turns out that solutions to $Ax = b$ have the decomposition $x = x_h + x_p$ where the homogeneous term x_h satisfies $Ax_h = 0$ and the nonhomogeneous term x_p solves $Ax_p = b$.

Example 3.1.3. *We found that the system in Example 1.3.1,*

$$\begin{aligned} &x + 2y - 3z = 1 \\ &2x + 4y = 7 \\ &-x + 3y + 2z = 0 \end{aligned}$$

has the unique solution $x = 83/30, y = 11/30$ *and* $z = 5/6$*. This means the matrix equation* $Av = b$ *where*

$$Av = \underbrace{\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & 0 \\ -1 & 3 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{v} = \underbrace{\begin{bmatrix} 1 \\ 7 \\ 0 \end{bmatrix}}_{b} \quad \text{has vector solution} \quad \boxed{v = \begin{bmatrix} 83/30 \\ 11/30 \\ 5/6 \end{bmatrix}}.$$

Example 3.1.4. *Consider the following generic system of two equations and three unknowns,*

$$\begin{aligned} ax + by + cz &= d \\ ex + fy + gz &= h \end{aligned}$$

in matrix form this system of equations is $Av = b$ *where*

$$Av = \underbrace{\begin{bmatrix} a & b & c \\ e & f & g \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \\ z \end{bmatrix}}_{v} = \begin{bmatrix} (a, b, c) \cdot (x, y, z) \\ (e, f, g) \cdot (x, y, z) \end{bmatrix} = \begin{bmatrix} ax + by + cz \\ ex + fy + gz \end{bmatrix} = \underbrace{\begin{bmatrix} d \\ h \end{bmatrix}}_{b}$$

Example 3.1.5. *We can rewrite the following system of linear equations*

$$\begin{aligned} x_1 + x_4 &= 0 \\ 2x_1 + 2x_2 + x_5 &= 0 \\ 4x_1 + 4x_2 + 4x_3 &= 1 \end{aligned}$$

in matrix form this system of equations is $Av = b$ *where*

$$Av = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 & 1 \\ 4 & 4 & 4 & 0 & 0 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}}_{v} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{b}.$$

Gaussian elimination on the augmented coefficient matrix reveals (see Example 1.2.7 for details of the Gaussian elimination)

$$rref\left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ 4 & 4 & 4 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & -1/2 & 1/4 \end{array}\right].$$

Consequently, x_4, x_5 *are free and solutions are of the form*

$$\begin{aligned} x_1 &= -x_4 \\ x_2 &= x_4 - \tfrac{1}{2}x_5 \\ x_3 &= \tfrac{1}{4} + \tfrac{1}{2}x_5 \end{aligned}$$

for all $x_4, x_5 \in \mathbb{R}$. The vector form of the solution is as follows:

$$v = \begin{bmatrix} -x_4 \\ x_4 - \frac{1}{2}x_5 \\ \frac{1}{4} + \frac{1}{2}x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{4} \\ 0 \\ 0 \end{bmatrix}.$$

Remark 3.1.6.

You might ask the question: what is the geometry of the solution set above ? Let $S = Sol_{[A|b]} \subset \mathbb{R}^5$, we see S is formed by tracing out all possible linear combinations of the vectors $v_1 = (-1, 1, 0, 1, 0)$ and $v_2 = (0, -\frac{1}{2}, \frac{1}{2}, 0, 1)$ based from the point $p_o = (0, 0, \frac{1}{4}, 0, 0)$. In other words, this is a two-dimensional plane containing the vectors v_1, v_2 and the point p_o. This plane is placed in a 5-dimensional space, this means that at any point on the plane you could go in three different directions away from the plane.

3.1.1 concatenation for solving many systems at once

If we wish to solve $Ax = b_1$ and $Ax = b_2$ we use a concatenation trick to do both at once. In fact, we can do it for $k \in \mathbb{N}$ problems which share the same coefficient matrix but possibly differing inhomogeneous terms.

Proposition 3.1.7.

Let $A \in \mathbb{R}^{m\times n}$. Vectors $v_1, v_2, \dots, v_k$ are solutions of $Av = b_i$ for $i = 1, 2, \dots k$ iff $V = [v_1|v_2|\cdots|v_k]$ solves $AV = B$ where $B = [b_1|b_2|\cdots|b_k]$.

Proof: Let $A \in \mathbb{R}^{m\times n}$ and suppose $Av_i = b_i$ for $i = 1, 2, \dots k$. Let $V = [v_1|v_2|\cdots|v_k]$ and use the concatenation Proposition 2.3.12,

$$AV = A[v_1|v_2|\cdots|v_k] = [Av_1|Av_2|\cdots|Av_k] = [b_1|b_2|\cdots|b_k] = B.$$

Conversely, suppose $AV = B$ where $V = [v_1|v_2|\cdots|v_k]$ and $B = [b_1|b_2|\cdots|b_k]$ then by Proposition 2.3.12 $AV = B$ implies $Av_i = b_i$ for each $i = 1, 2, \dots k$. $\square$

Example 3.1.8. *Solve the systems given below,*

$$\begin{aligned} x + y + z &= 1 \\ x - y + z &= 0 \\ -x + z &= 1 \end{aligned} \qquad \text{and} \qquad \begin{aligned} x + y + z &= 1 \\ x - y + z &= 1 \\ -x + z &= 1 \end{aligned}$$

The systems above share the same coefficient matrix, however $b_1 = (1, 0, 1)$ whereas $b_2 = (1, 1, 1)$. We can solve both at once by making an extended augmented coefficient matrix $[A|b_1|b_2]$

$$[A|b_1|b_2] = \left[\begin{array}{ccc|c|c} 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 0 & 1 \\ -1 & 0 & 1 & 1 & 1 \end{array}\right] \qquad rref[A|b_1|b_2] = \left[\begin{array}{ccc|c|c} 1 & 0 & 0 & -1/4 & 0 \\ 0 & 1 & 0 & 1/2 & 0 \\ 0 & 0 & 1 & 3/4 & 1 \end{array}\right]$$

We use Proposition 3.1.7 to conclude that

$$\begin{array}{l} x+y+z=1 \\ x-y+z=0 \\ -x+z=1 \end{array} \qquad \textit{has solution } x=-1/4, y=1/2, z=3/4$$

$$\begin{array}{l} x+y+z=1 \\ x-y+z=1 \\ -x+z=1 \end{array} \qquad \textit{has solution } x=0, y=0, z=1.$$

3.2 elementary matrices

The concept of an elementary matrix allows us to see Gauss-Jordan elimination as matrix multiplication. Gauss Jordan elimination consists of three *elementary row operations*:

$$(1.)\ r_i + ar_j \to r_i, \qquad (2.)\ br_i \to r_i, \qquad (3.)\ r_i \leftrightarrow r_j$$

Left multiplication by **elementary matrices** will accomplish the same operation on a matrix.

Definition 3.2.1.

Let $[A:\ r_i + ar_j \to r_i]$ denote the matrix produced by replacing row i of matrix A with $row_i(A) + arow_j(A)$. Also define $[A:\ cr_i \to r_i]$ and $[A: r_i \leftrightarrow r_j]$ in the same way. Let $a, b \in \mathbb{R}$ and $b \neq 0$. The following matrices are called **elementary matrices**:

$$E_{r_i+ar_j\to r_i} = [I:\ r_i + ar_j \to r_i]$$

$$E_{br_i\to r_i} = [I:\ br_i \to r_i]$$

$$E_{r_i\leftrightarrow r_j} = [I:\ r_i \leftrightarrow r_j]$$

Example 3.2.2. *Let* $A = \left[\begin{smallmatrix} a & b & c \\ 1 & 2 & 3 \\ u & m & e \end{smallmatrix}\right]$

$$E_{r_2+3r_1\to r_2}A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ 1 & 2 & 3 \\ u & m & e \end{bmatrix} = \begin{bmatrix} a & b & c \\ 3a+1 & 3b+2 & 3c+3 \\ u & m & e \end{bmatrix}$$

$$E_{7r_2\to r_2}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ 1 & 2 & 3 \\ u & m & e \end{bmatrix} = \begin{bmatrix} a & b & c \\ 7 & 14 & 21 \\ u & m & e \end{bmatrix}$$

$$E_{r_2\to r_3}A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} a & b & c \\ 1 & 2 & 3 \\ u & m & e \end{bmatrix} = \begin{bmatrix} a & b & c \\ u & m & e \\ 1 & 2 & 3 \end{bmatrix}$$

Proposition 3.2.3.

If $A \in \mathbb{R}^{m\times n}$ then there exist elementary matrices $E_1, E_2, \dots, E_k$ such that $rref(A) = E_1 E_2 \cdots E_k A$.

Proof: Gauss Jordan elimination is accomplished by k-successive elementary row operations. Each row operation can be implemented by multiplying the corresponding elementary matrix on the left. The Theorem follows. $\square$

Example 3.2.4. *Just for fun let's see what happens if we multiply the elementary matrices on the right instead.*

$$AE_{r_2+3r_1\to r_2} = \begin{bmatrix} a & b & c \\ 1 & 2 & 3 \\ u & m & e \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a+3b & b & c \\ 1+6 & 2 & 3 \\ u+3m & m & e \end{bmatrix}$$

$$AE_{7r_2\to r_2} = \begin{bmatrix} a & b & c \\ 1 & 2 & 3 \\ u & m & e \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a & 7b & c \\ 1 & 14 & 3 \\ u & 7m & e \end{bmatrix}$$

$$AE_{r_2\to r_3} = \begin{bmatrix} a & b & c \\ 1 & 2 & 3 \\ u & m & e \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} a & c & b \\ 1 & 3 & 2 \\ u & e & m \end{bmatrix}$$

Curious, they generate column operations, we might call these elementary column operations. In our notation the row operations are more important.

In the section that follows and in the discussion of the product rule for determinants we will see that elementary matrices are needed to uncover some of the deeper results.

3.3 linear combinations and spanning

We saw that linear combinations of the standard basis will generate any vector in $\mathbb{R}^n$ in the previous section. We now set out to answer a set similar question:

PROBLEM: Given vectors $v_1, v_2, \dots, v_k$ and a vector b do there exist constants $c_1, c_2, \dots, c_k$ such that $c_1v_1 + c_2v_2 + \cdots + c_kv_k = b$? If so, how should we determine them in general?

We have all the tools we need to solve such problems. I'll show a few examples before I state the general algorithm.

Example 3.3.1. Problem: *given that $v = (2, -1, 3)$, $w = (1, 1, 1)$ and $b = (4, 1, 5)$ find values for x, y such that $xv + yw = b$* **(if possible).**

Solution: *using our column notation we find $xv + yw = b$ gives*

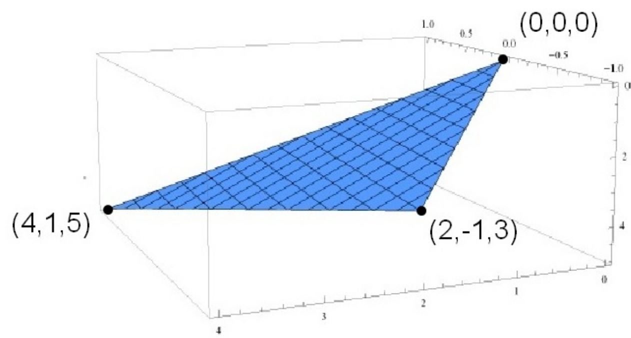
$$x\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} + y\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} 2x+y \\ -x+y \\ 3x+y \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}$$

We are faced with solving the system of equations $2x + y = 4, -x + y = 1$ and $3x + y = 5$. As we discussed in depth last chapter we can efficiently solve this type of problem in general by Gaussian elimination on the corresponding augmented coefficient matrix. In this problem, you can calculate that

$$rref\left[\begin{array}{cc|c} 2 & 1 & 4 \\ -1 & 1 & 1 \\ 3 & 1 & 5 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{array}\right]$$

hence $x = 1$ and $y = 2$. Indeed, it is easy to check that $v + 2w = b$.

The geometric question which is equivalent to the previous question is as follows: "is the vector b found in the plane which contains v and w"? Here's a picture of the calculation we just performed:



The set of all linear combinations of several vectors in $\mathbb{R}^n$ is called the *span* of those vectors. To be precise

Definition 3.3.2.

Let $S = \{v_1, v_2, \dots, v_k\} \subset \mathbb{R}^n$ be a finite set of n-vectors then $span(S)$ is defined to be the set of all linear combinations formed from vectors in S:

$$span\{v_1, v_2, \dots, v_k\} = \{\sum_{i=1}^{k} c_i v_i \mid c_i \in \mathbb{R} \text{ for } i = 1, 2, \dots, k\}$$

If $W = span(S)$ then we say that S is a **generating set** for W.

If we have one vector then it has a span which could be a line. With two vectors we might generate a plane. With three vectors we might generate a volume. With four vectors we might generate a hypervolume or 4-volume. We'll return to these geometric musings in § 3.4 and explain why I have used the word "might" rather than an affirmative "will" in these claims. For now, we return to the question of how to decide if a given vector is in the span of another set of vectors.

Example 3.3.3. Problem: *Let* $b_1 = (1, 1, 0), b_2 = (0, 1, 1)$ *and* $b_3 = (0, 1, -1)$.
Is[1] $e_3 \in span\{b_1, b_2, b_3\}$*?*

Solution: *Find the explicit linear combination of* b_1, b_2, b_3 *that produces* e_3*. We seek to find* $x, y, z \in \mathbb{R}$ *such that* $xb_1 + yb_2 + zb_3 = e_3$*,*

$$x\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + z\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x \\ x+y+z \\ y-z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Following essentially the same arguments as the last example we find this question of solving the system formed by gluing the given vectors into a matrix and doing row reduction. In particular, we can solve the vector equation above by solving the corresponding system below:

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \xrightarrow{r_2 - r_1 \to r_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 1 \end{array}\right] \xrightarrow{r_3 - r_2 \to r_3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{array}\right] \begin{array}{c} \xrightarrow{-r_3/2 \to r_3} \\ \xrightarrow{r_2 - r_3 \to r_2} \\ \xrightarrow{r_1 - r_3 \to r_1} \end{array} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & -1/2 \end{array}\right]$$

Therefore, $x = 0, y = \frac{1}{2}$ *and* $z = -\frac{1}{2}$*. We find that* $\boxed{e_3 = \frac{1}{2}b_1 + \frac{1}{2}b_2 - \frac{1}{2}b_3}$ *thus* $e_3 \in span\{b_1, b_2, b_3\}$*.*

The power of the matrix technique shines bright in the next example. Sure you could guess the last two, but as things get messy we'll want a refined efficient algorithm to dispatch spanning questions with ease.

[1]challenge: once you understand this example for e_3 try answering it for other vectors or for an arbitrary vector $v = (v_1, v_2, v_3)$. How would you calculate $x, y, z \in \mathbb{R}$ such that $v = xb_1 + yb_2 + zb_3$?

Example 3.3.4. **Problem:** *Let* $b_1 = (1, 2, 3, 4)$*,* $b_2 = (0, 1, 0, 1)$ *and* $b_3 = (0, 0, 1, 1)$*.*
Is $w = (1, 1, 4, 4) \in span\{b_1, b_2, b_3\}$*?*

Solution: *Following the same method as the last example we seek to find* x_1, x_2 *and* x_3 *such that* $x_1b_1 + x_2b_2 + x_3b_3 = w$ *by solving the aug. coeff. matrix as is our custom:*

$$[b_1|b_2|b_3|w] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 \\ 3 & 0 & 1 & 4 \\ 4 & 1 & 1 & 4 \end{array}\right] \xrightarrow[\substack{r_3 - 3r_1 \to r_3 \\ r_4 - 4r_1 \to r_4}]{r_2 - 2r_1 \to r_2} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{array}\right] \xrightarrow{r_4 - r_2 \to r_4}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{r_4 - r_3 \to r_4} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] = rref[b_1|b_2|b_3|w]$$

We find $x_1 = 1, x_2 = -1, x_3 = 1$ *thus* $\boxed{w = b_1 - b_2 + b_3}$*. Therefore,* $w \in span\{b_1, b_2, b_3\}$*.*

Pragmatically, if the question is sufficiently simple you may not need to use the augmented coefficient matrix to solve the question. I use them here to illustrate the method.

Example 3.3.5. **Problem:** *Let* $b_1 = (1, 1, 0)$ *and* $b_2 = (0, 1, 1)$*.*
Is $e_2 \in span\{b_1, b_2\}$*?*

Solution: *Attempt to find the explicit linear combination of* b_1, b_2 *that produces* e_2*. We seek to find* $x, y \in \mathbb{R}$ *such that* $xb_1 + yb_2 = e_3$*,*

$$x \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x \\ x + y \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

We don't really need to consult the augmented matrix to solve this problem. Clearly $x = 0$ *and* $y = 0$ *is found from the first and third components of the vector equation above. But, the second component yields* $x + y = 1$ *thus* $0 + 0 = 1$*. It follows that this system is inconsistent and we may conclude that* $w \notin span\{b_1, b_2\}$*. For the sake of curiousity let's see how the augmented solution matrix looks in this case: omitting details of the row reduction,*

$$rref \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

note the last row again confirms that this is an inconsistent system.

Proposition 3.3.6.

Given vectors $v_1, v_2, \dots, v_k \in \mathbb{R}^n$ and another vector $b \in \mathbb{R}^n$ we can solve the vector equation $x_1v_1 + x_2v_2 + \cdots + x_kv_k = b$ by Gaussian elimination of the corresponding matrix problem $[v_1|v_2|\cdots|v_k|b]$. Moreover, $b \in span\{v_1, v_2, \dots, v_k\}$ iff $[v_1|v_2|\cdots|v_k|b]$ represents the augmented matrix of a consistent system of equations.

Proof: note that solving the single vector equation $x_1v_1 + x_2v_2 + \cdots + x_kv_k = b$ for $x_1, x_2, \dots, x_k$ is equivalent to solving n-scalar equations

$$\begin{aligned} &x_1(v_1)_1 + x_2(v_2)_1 + \cdots + x_k(v_k)_1 = b_1 \\ &x_1(v_1)_2 + x_2(v_2)_2 + \cdots + x_k(v_k)_2 = b_2 \\ &\vdots \quad \vdots \quad \vdots \quad \vdots \\ &x_1(v_1)_n + x_2(v_2)_n + \cdots + x_k(v_k)_n = b_n. \end{aligned}$$

But, this can be solved by performing Gaussian elimination on the matrix

$$\left[\begin{array}{cccc|c} (v_1)_1 & (v_2)_1 & \cdots & (v_k)_1 & b_1 \\ (v_1)_2 & (v_2)_2 & \cdots & (v_k)_2 & b_2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ (v_1)_n & (v_2)_n & \cdots & (v_k)_n & b_n \end{array}\right].$$

Therefore, $b \in span\{v_1, v_2, \dots, v_k\}$ iff the system above reduces to a consistent system. $\square$

Remark 3.3.7.

If we are given $B = \{b_1, b_2, \dots, b_k\} \subset \mathbb{R}^n$ and $T = \{w_1, w_2, \dots, w_r\} \subset \mathbb{R}^n$ and we wish to determine if $T \subset span(B)$ then we can answer the question by examining if $[b_1|b_2|\cdots|b_k]x = w_j$ has a solution for each $j = 1, 2, \dots r$. Or we could make use of Proposition 3.1.7 and solve it in one sweeping matrix calculation;

$$rref[b_1|b_2|\cdots|b_k|w_1|w_2|\cdots|w_r]$$

If there is a row with zeros in the first k-columns and a nonzero entry in the last r-columns then this means that at least one vector w_k is not in the span of B(moreover, the vector not in the span corresponds to the nonzero entrie(s)). Otherwise, each vector is in the span of B and we can read the precise linear combination from the matrix. I will illustrate this in the example that follows.

Example 3.3.8. *Let* $W = span\{e_1+e_2, e_2+e_3, e_1-e_3\}$ *and suppose* $T = \{e_1, e_2, e_3 - e_1\}$*. Is* $T \subset W$*? If not, which vectors in* T *are not in* W*? Consider,*

$$[e_1+e_1|e_2+e_3|e_1-e_3||e_1|e_2|e_3-e_1] = \left[\begin{array}{ccc||ccc} 1 & 0 & 1 & 1 & 0 & -1 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_2 - r_1 \rightarrow r_2}$$

$$\left[\begin{array}{ccc||ccc} 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & -1 & 1 & 1 \\ 0 & 1 & -1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_3 - r_2 \rightarrow r_3}$$

$$\left[\begin{array}{ccc||ccc} 1 & 0 & 1 & 1 & 0 & -1 \\ 0 & 1 & -1 & -1 & 1 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right] \begin{array}{c} \xrightarrow{r_2 + r_3 \rightarrow r_2} \\ \xrightarrow{r_1 - r_3 \rightarrow r_1} \end{array}$$

$$\left[\begin{array}{ccc||ccc} 1 & 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

Let me summarize the calculation:

$$rref[e_1+e_2|e_2+e_3||e_1-e_3|e_1|e_2|e_3-e_1] = \left[\begin{array}{ccc||ccc} 1 & 0 & 1 & 0 & 1 & -1 \\ 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

We deduce that e_1 *and* e_2 *are not in* W*. However,* $e_1 - e_3 \in W$ *and we can read from the matrix* $-(e_1+e_2)+(e_2+e_3) = e_3 - e_1$*. I added the double vertical bar for book-keeping purposes, as usual the vertical bars are just to aid the reader in parsing the matrix.*

The main point which provides the theorem below is this: Av is a linear combination of the columns of A therefore $Av = b$ has a solution iff b is a linear combination of the columns in A. We have seen for a particular matrix A and a given vector b it may or may not be the case that $Av = b$ has a solution. It turns out that certain special matrices will have a solution for each choice of b. The theorem below is taken from Lay's text on page 43. The abbreviation **TFAE** means "The Following Are Equivalent".

Theorem 3.3.9.

Suppose $A = [A_{ij}] \in \mathbb{R}^{k \times n}$ then TFAE,

1. $Av = b$ has a solution for each $b \in \mathbb{R}^k$
2. each $b \in \mathbb{R}^k$ is a linear combination of the columns of A
3. columns of A span $\mathbb{R}^k$
4. A has a pivot position in each row.

3.4 linear independence

In the previous sections we have only considered questions based on a fixed spanning set. We asked if $b \in span\{v_1, v_2, \dots, v_n\}$ and we even asked if it was possible for all b. What we haven't thought about yet is the following:

PROBLEM: Given vectors $v_1, v_2, \dots, v_k$ and a vector $b = c_1v_1 + c_2v_2 + \cdots + c_kv_k$ for some constants c_j is it possible that b can be written as a linear combination of some subset of $\{v_1, v_2, \dots, v_k\}$? If so, how should we determine which vectors can be taken away from the spanning set? How should we decide which vectors to keep and which are redundant?

The concept of linear independence is central to answering these questions. We will examine the basics of linear independence in this section.

Definition 3.4.1.

If a vector v_k can be written as a linear combination of vectors $\{v_1, v_2, \dots, v_{k-1}\}$ then we say that the vectors $\{v_1, v_2, \dots, v_{k-1}, v_k\}$ are **linearly dependent**.
If the vectors $\{v_1, v_2, \dots, v_{k-1}, v_k\}$ are not linear dependent then they are said to be **linearly independent**.

Example 3.4.2. *Let* $v = (1, 2, 3)$ *and* $w = (2, 4, 6)$. *Clearly* v, w *are linearly dependent since* $w = 2v$.

I often quote the following proposition as the defintion of linear independence, it is an equivalent statement and as such can be used as the definition(but not by us, I already made the definition above). If this was our definition then our definition would become a proposition. Math always has a certain amount of this sort of ambiguity.

Proposition 3.4.3.

Let $v_1, v_2, \dots, v_k \in \mathbb{R}^n$. The set of vectors $\{v_1, v_2, \dots, v_k\}$ is linearly independent iff

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0 \;\Rightarrow\; c_1 = c_2 = \cdots = c_k = 0.$$

Proof: ($\Rightarrow$) Suppose $\{v_1, v_2, \dots, v_k\}$ is linearly independent. Assume that there exist constants $c_1, c_2, \dots, c_k$ such that

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$$

and at least one constant, say c_j, is nonzero. Then we can divide by c_j to obtain

$$\tfrac{c_1}{c_j}v_1 + \tfrac{c_2}{c_j}v_2 + \cdots + v_j + \cdots + \tfrac{c_k}{c_j}v_k = 0$$

solve for v_j, (we mean for $\widehat{v_j}$ to denote the deletion of v_j from the list)

$$v_j = -\tfrac{c_1}{c_j}v_1 - \tfrac{c_2}{c_j}v_2 - \cdots - \widehat{v_j} - \cdots - \tfrac{c_k}{c_j}v_k$$

but this means that v_j linearly depends on the other vectors hence $\{v_1, v_2, \dots, v_k\}$ is linearly dependent. This is a contradiction, therefore $c_j = 0$. Note j was arbitrary so we may conclude $c_j = 0$ for all j. Therefore, $c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0 \;\Rightarrow\; c_1 = c_2 = \cdots = c_k = 0$.

Proof: ($\Leftarrow$) Assume that

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0 \;\Rightarrow\; c_1 = c_2 = \cdots = c_k = 0.$$

If $v_j = b_1v_1 + b_2v_2 + \cdots + \widehat{b_jv_j} + \cdots + b_kv_k$ then $b_1v_1 + b_2v_2 + \cdots + b_jv_j + \cdots + b_kv_k = 0$ where $b_j = -1$, this is a contradiction. Therefore, for each j, v_j is not a linear combination of the other vectors. Consequently, $\{v_1, v_2, \dots, v_k\}$ is linearly independent.
Another way to characterize LI of vectors is given by the proposition below:

Proposition 3.4.4.

S is a linearly independent set of vectors iff for all $v_1, v_2, \dots, v_k \in S$,

$$a_1v_1 + a_2v_2 + \cdots + a_kv_k = b_1v_1 + b_2v_2 + \cdots + b_kv_k$$

implies $a_i = b_i$ for each $i = 1, 2, \dots, k$. In other words, we can equate coefficients of linearly indpendent vectors. And, conversely if a set of vectors allows for equating coefficients then it is linearly independent.

Proof: left to the reader. $\square$

Proposition 3.4.5.

If S is a finite set of vectors which contains the zero vector then S is linearly dependent.

Proof: Let $\{\vec{0}, v_2, \dots v_k\} = S$ and observe that

$$1\vec{0} + 0v_2 + \cdots + 0v_k = 0$$

Thus $c_1\vec{0} + c_2v_2 + \cdots + c_kv_k = 0$ does not imply $c_1 = 0$ hence the set of vectors is not linearly independent. Thus S is linearly dependent. $\square$

Proposition 3.4.6.

Let v and w be nonzero vectors.

$$v, w \text{ are linearly dependent } \Leftrightarrow \exists k \neq 0 \in \mathbb{R} \text{ such that } v = kw.$$

Proof: Suppose v, w are linearly dependent then there exist constants c_1, c_2, not all zero, such that $c_1v + c_2w = 0$. Suppose that $c_1 = 0$ then $c_2w = 0$ hence[2] $c_2 = 0$ or $w = 0$. But, this is a

[2]if the product of a scalar and a vector in $\mathbb{R}^n$ is zero then you can prove that one or both is zero by examining the components of the vector equation

contradiction since v, w are nonzero and at least one of c_1, c_2 must be nonzero. Therefore, $c_1 \neq 0$. Likewise, if $c_2 = 0$ we find a similar contradiction. Hence c_1, c_2 are both nonzero and we calculate $v = (-c_2/c_1)w$, identify that $k = -c_2/c_1$. $\square$

Remark 3.4.7.

For two vectors the term "linearly dependent" can be taken quite literally: two vectors are linearly dependent if they point along the same line. For three vectors they are linearly dependent if they point along the same line or possibly lay in the same plane. When we get to four vectors we can say they are linearly dependent if they reside in the same volume, plane or line. I don't find the geometric method terribly successful for dimensions higher than two. However, it is neat to think about the geometric meaning of certain calculations in dimensions higher than 3. We can't even draw it but we can eulicidate all sorts of information with the mathematics of linear algebra.

Example 3.4.8. *Let $v = (1, 2, 3)$ and $w = (1, 0, 0)$. Let's prove these are linearly independent. Assume that $c_1 v + c_2 w = 0$, this yields*

$$c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

thus $c_1 + c_2 = 0$ and $2c_1 = 0$ and $3c_1 = 0$. We find $c_1 = c_2 = 0$ thus v, w are linearly independent. Alternatively, you could explain why there does not exist any $k \in \mathbb{R}$ such that $v = kw$

Think about this, if the set of vectors $\{v_1, v_2, \dots, v_k\} \subset \mathbb{R}^n$ is linearly independent then the equation $c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$ has the unique solution $c_1 = 0, c_2 = 0, \dots, c_k = 0$. Notice we can reformulate the problem as a matrix equation:

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0 \;\Leftrightarrow\; [v_1|v_2|\cdots|v_k][c_1 \; c_2 \; \cdots \; c_k]^T = 0$$

The matrix $[v_1|v_2|\cdots|v_k]$ is an $n \times k$. This is great. We can use the matrix techniques we already developed to probe for linear independence of a set of vectors.

Proposition 3.4.9.

Let $\{v_1, v_2, \dots, v_k\}$ be a set of vectors in $\mathbb{R}^n$.

1. If $rref[v_1|v_2|\cdots|v_k]$ has less than k pivot columns then the set of vectors $\{v_1, v_2, \dots, v_k\}$ is linearly dependent.

2. If $rref[v_1|v_2|\cdots|v_k]$ has k pivot columns then the set of vectors $\{v_1, v_2, \dots, v_k\}$ is linearly independent.

Proof: follows from thinking through the details of Gaussian elimination in the relevant cases. $\square$. The following result is a simple consequence of the above proposition.

Corollary 3.4.10.

If $\{v_1, v_2, \dots, v_k\}$ is a set of vectors in $\mathbb{R}^n$ and $k > n$ then the vectors are linearly dependent.

Proof: Proposition 3.4.9 tells us that the set is linearly independent if there are k pivot columns in $[v_1|\cdots|v_k]$. However, that is impossible since $k > n$ this means that there will be at least one column of zeros in $rref[v_1|\cdots|v_k]$. Therefore the vectors are linearly dependent. $\square$

We may have at most 2 linearly independent vectors in $\mathbb{R}^2$, 3 in $\mathbb{R}^3$, 4 in $\mathbb{R}^4$, and so forth...

Example 3.4.11. *Determine if v_1, v_2, v_3 (given below) are linearly independent or dependent. If the vectors are linearly dependent show how they depend on each other.*

$$v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$$

We seek to use the Proposition 3.4.9. Consider then,

$$[v_1|v_2|v_3] = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 1 & 0 & 1 \end{bmatrix} \xrightarrow[r_3 - r_1 \to r_3]{r_2 - r_1 \to r_2} \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -1 \\ 0 & -2 & -2 \end{bmatrix} \xrightarrow[r_3 - 2r_2 \to r_3]{r_1 + 2r_2 \to r_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & -1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus we find that,

$$rref[v_1|v_2|v_3] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

hence the variable c_3 is free in the solution of $Vc = 0$. We find solutions of the form $c_1 = -c_3$ and $c_2 = -c_3$. This means that

$$-c_3 v_1 - c_3 v_2 + c_3 v_3 = 0$$

for any value of c_3. I suggest $c_3 = 1$ is easy to plug in,

$$-v_1 - v_2 + v_3 = 0 \quad \textit{or we could write} \quad v_3 = v_1 + v_2$$

We see clearly that v_3 is a linear combination of v_1, v_2.

Example 3.4.12. *Determine if v_1, v_2, v_3, v_4 (given below) are linearly independent or dependent.*

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad v_4 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

We seek to use the Proposition 3.4.9. Omitting details we find,

$$rref[v_1|v_2|v_3|v_4] = rref\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

In this case no variables are free, the only solution is $c_1 = 0, c_2 = 0, c_3 = 0, c_4 = 0$ *hence the set of vectors* $\{v_1, v_2, v_3, v_4\}$ *is linearly independent.*

Example 3.4.13. *Determine if* v_1, v_2, v_3 *(given below) are linearly independent or dependent. If the vectors are linearly dependent show how they depend on each other.*

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 3 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ -3 \end{bmatrix}$$

We seek to use the Proposition 3.4.9. Consider $[v_1|v_2|v_3] =$

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \\ 3 & 0 & -3 \end{bmatrix} \xrightarrow{r_4 - 3r_1 \to r_4} \begin{bmatrix} 1 & 3 & 2 \\ 0 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & -9 & -9 \end{bmatrix} \begin{array}{c} \xrightarrow{r_1 - 3r_2 \to r_1} \\ \xrightarrow{r_3 - 2r_2 \to r_3} \\ \xrightarrow{r_4 + 9r_2 \to r_4} \end{array} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = rref[V].$$

Hence the variable c_3 *is free in the solution of* $Vc = 0$. *We find solutions of the form* $c_1 = c_3$ *and* $c_2 = -c_3$. *This means that*

$$c_3v_1 - c_3v_2 + c_3v_3 = 0$$

for any value of c_3. *I suggest* $c_3 = 1$ *is easy to plug in,*

$$v_1 - v_2 + v_3 = 0 \quad \text{or we could write} \quad v_3 = v_2 - v_1$$

We see clearly that v_3 *is a linear combination of* v_1, v_2.

Example 3.4.14. *Determine if* v_1, v_2, v_3, v_4 *(given below) are linearly independent or dependent. If the vectors are linearly dependent show how they depend on each other.*

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad v_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

We seek to use the Proposition 3.4.9. Consider $[v_1|v_2|v_3|v_4] =$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{bmatrix} 1 & 1 & 2 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_1 - r_2 \to r_1} \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = rref[v_1|v_2|v_3|v_4].$$

Hence the variables c_3 *and* c_4 *are free in the solution of* $Vc = 0$. *We find solutions of the form* $c_1 = -c_3 + c_4$ *and* $c_2 = -c_3 - c_4$. *This means that*

$$(c_4 - c_3)v_1 - (c_3 + c_4)v_2 + c_3 v_3 + c_4 v_4 = 0$$

for any value of c_3 *or* c_4. *I suggest* $c_3 = 1, c_4 = 0$ *is easy to plug in,*

$$-v_1 - v_2 + v_3 = 0 \quad \textit{or we could write} \quad v_3 = v_2 + v_1$$

Likewise select $c_3 = 0, c_4 = 1$ *to find*

$$v_1 - v_2 + v_4 = 0 \quad \textit{or we could write} \quad v_4 = v_2 - v_1$$

We find that v_3 *and* v_4 *are linear combinations of* v_1 *and* v_2.

Observe that we used Proposition 3.4.9 in Examples 3.4.11, 3.4.12, 3.4.13 and 3.4.14 to ascertain the linear independence of certain sets of vectors. If you pay particular attention to those examples you may have picked up on a pattern. The columns of the $rref[v_1|v_2|\cdots|v_k]$ depend on each other in the same way that the vectors $v_1, v_2, \dots v_k$ depend on each other. These provide examples of the so-called **"column correspondence property"**. In a nutshell, the property says you can read the linear dependencies right off the $rref[v_1|v_2|\cdots|v_k]$.

Proposition 3.4.15. *Column Correspondence Property (CCP)*

Let $A = [col_1(A)|\cdots|col_n(A)] \in \mathbb{R}^{m\times n}$ and $R = rref[A] = [col_1(R)|\cdots|col_n(R)]$. There exist constants $c_1, c_2, \dots c_k$ such that $c_1 col_1(A) + c_2 col_2(A) + \cdots + c_k col_k(A) = 0$ if and only if $c_1 col_1(R) + c_2 col_2(R) + \cdots + c_k col_k(R) = 0$. If $col_j(rref[A])$ is a linear combination of other columns of $rref[A]$ then $col_j(A)$ is likewise the same linear combination of columns of A.

We prepare for the proof of the Proposition by establishing a useful Lemma.

Lemma 3.4.16.

Let $A \in \mathbb{R}^{m\times n}$ then there exists an invertible matrix E such that $col_j(rref(A)) = E col_j(A)$ for all $j = 1, 2, \dots n$.

Proof of Lemma: Recall that there exist elementary matrices $E_1, E_2, \dots E_r$ such that $A = E_1 E_2 \cdots E_r rref(A) = E^{-1} rref(A)$ where I have defined $E^{-1} = E_1 E_2 \cdots E_k$ for convenience. Recall the concatenation proposition: $X[b_1|b_2|\cdots|b_k] = [Xb_1|Xb_2|\cdots|Xb_k]$. We can unravel the Gaussian elimination in the same way,

$$\begin{aligned} EA &= E[col_1(A)|col_2(A)|\cdots|col_n(A)] \\ &= [Ecol_1(A)|Ecol_2(A)|\cdots|Ecol_n(A)] \end{aligned}$$

Observe that $EA = rref(A)$ hence we find the above equation says $col_j(rref(A)) = Ecol_j(A)$ for all j. $\square$

Proof of Proposition: Suppose that there exist constants $c_1, c_2, \dots, c_k$ such that $c_1 col_1(A) + c_2 col_2(A) + \cdots + c_k col_k(A) = 0$. By the Lemma we know there exists E such that $col_j(rref(A)) = E col_j(A)$. Multiply linear combination by E to find:

$$c_1 E col_1(A) + c_2 E col_2(A) + \cdots + c_k E col_k(A) = 0$$

which yields

$$c_1 col_1(rref(A)) + c_2 col_2(rref(A)) + \cdots + c_k col_k(rref(A)) = 0.$$

Likewise, if we are given a linear combination of columns of $rref(A)$ we can multiply by E^{-1} to recover the same linear combination of columns of A. $\square$

Example 3.4.17. *I will likely use the abbreviation "CCP" for column correspondence property. We could have deduced all the linear dependencies via the CCP in Examples 3.4.11,3.4.13 and 3.4.14. We found in 3.4.11 that*

$$rref[v_1|v_2|v_3] = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Obviously $col_3(R) = col_1(R) + col_2(R)$ hence by CCP $v_3 = v_1 + v_2$.
We found in 3.4.13 that

$$rref[v_1|v_2|v_3] = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

By inspection, $col_3(R) = col_2(R) - col_1(R)$ hence by CCP $v_3 = v_2 - v_1$.
We found in 3.4.14 that

$$rref[v_1|v_2|v_3|v_4] = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

By inspection, $col_3(R) = col_1(R) + col_2(R)$ hence by CCP $v_3 = v_1 + v_2$. Likewise by inspection, $col_4(R) = col_2(R) - col_1(R)$ hence by CCP $v_4 = v_2 - v_1$.

You should notice that the CCP saves us the trouble of expressing how the constants c_i are related. If we are only interested in how the vectors are related the CCP gets straight to the point quicker. We should pause and notice another pattern here while were thinking about these things.

Proposition 3.4.18.

> The non-pivot columns of a matrix can be written as linear combinations of the pivot columns and the pivot columns of the matrix are linearly independent.

Proof: Let A be a matrix. Notice the Proposition is clearly true for $rref(A)$. Hence, using Lemma 3.4.16 we find the same is true for the matrix A. $\square$

Proposition 3.4.19.

The rows of a matrix A can be written as linear combinations of the transposes of pivot columns of A^T, and the rows which are transposes of the pivot columns of A^T are linearly independent.

Proof: Let A be a matrix and A^T its transpose. Apply Proposition 3.4.15 to A^T to find pivot columns which we denote by $col_{i_j}(A^T)$ for $j = 1, 2, \dots k$. These columns are linearly independent and they span $Col(A^T)$. Suppose,

$$c_1 row_{i_1}(A) + c_2 row_{i_2}(A) + \cdots + c_k row_{i_k}(A) = 0.$$

Take the transpose of the equation above, use Proposition 4.2.3 to simplify:

$$c_1(row_{i_1}(A))^T + c_2(row_{i_2}(A))^T + \cdots + c_k(row_{i_k}(A))^T = 0.$$

Recall $(row_j(A))^T = col_j(A^T)$ thus,

$$c_1 col_{i_1}(A^T) + c_2 col_{i_2}(A^T) + \cdots + c_k col_{i_k}(A^T) = 0.$$

hence $c_1 = c_2 = \cdots = c_k = 0$ as the pivot columns of A^T are linearly independendent. This shows the corresponding rows of A are likewise linearly independent. The proof that these same rows span $Row(A)$ is similar. $\Box$

3.5 on computation of bases for the matrix subspaces

Sometimes we can remove a vector from the generating set and still generate the whole vector space[3]. For example,

$$span\{e_1, e_2, e_1 + e_2\} = \mathbb{R}^2$$

and we can remove any one of these vector and still span $\mathbb{R}^2$,

$$span\{e_1, e_2\} = span\{e_1, e_1 + e_2\} = span\{e_2, e_1 + e_2\} = \mathbb{R}^2$$

However, if we remove another vector then we will not span $\mathbb{R}^2$. A generating set which is just big enough is called a basis.

Definition 3.5.1.

A **basis** for a vector space V is a set of vectors S such that

(1.) $V = span(S)$,

(2.) S is linearly independent.

If a vector space V has a basis which consists of $n < \infty$ vectors then we say that V is **finite-dimensional** vector space and $dim(V) = n$. Otherwise V is said to be **infinite-dimensional**

[3] a vector space is a set which is closed under vector addition and scalar multiplication. We study abstract vector spaces briefly at the conclusion of this course and in great depth in Math 321. For the most part, a vector space in this course is some subset of column or row vectors which is closed under vector addition and scalar multiplication.

Let me remind the reader how we define column, row and null space of a matrix. We also introduce terminology for the dimension of column and null space:

Definition 3.5.2.

Let $A \in \mathbb{R}^{m \times n}$. We define

(1.) $Col(A) = span\{col_j(A)|j = 1, 2, \dots, n\}$ and $r = rank(A) = dim(Col(A))$

(2.) $Row(A) = span\{row_i(A)|i = 1, 2, \dots, m\}$

(3.) $Null(A) = \{x \in \mathbb{R}^n | Ax = 0\}$ and $\nu = nullity(A) = dim(Null(A))$

Our goal in this section is to find bases for these spaces and more generally understand how to find a basis for some subspace of $\mathbb{R}^n$.

3.5.1 how to calculate a basis for a span of row or column vectors

Given some subspace of $\mathbb{R}^n$ we would like to know how to find a basis for that space. In particular, if $V = span\{v_1, v_2, \dots, v_k\}$ then what is a basis for W? Likewise, given some set of row vectors $W = \{w_1, w_2, \dots w_k\} \subset \mathbb{R}^{1\times n}$ how can we select a basis for $span(W)$. We would like to find answers to these question since most subspaces are characterized either as spans or solution sets(see the next section on $Null(A)$). We already have the tools to answer these questions, we just need to apply them to the tasks at hand.

Proposition 3.5.3.

Let $W = span\{v_1, v_2, \dots, v_k\} \subset \mathbb{R}^n$ then a basis for W can be obtained by selecting the vectors that reside in the pivot columns of $[v_1|v_2|\cdots|v_k]$.

Proof: this is immediately obvious from Proposition 3.4.15. $\square$

The proposition that follows is also follows immediately from Proposition 3.4.15.

Proposition 3.5.4.

Let $A \in \mathbb{R}^{m\times n}$ the pivot columns of A form a basis for $Col(A)$.

Example 3.5.5. *Suppose A is given as below: (I omit the details of the Gaussian elimination)*

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} \qquad \Rightarrow \qquad rref[A] = \begin{bmatrix} 1 & 0 & 5/3 & 0 \\ 0 & 1 & 2/3 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Identify that columns 1,2 and 4 are pivot columns. Moreover,

$$Col(A) = span\{col_1(A), col_2(A), col_4(A)\}$$

In particular we can also read how the second column is a linear combination of the basis vectors.

$$\begin{aligned} col_3(A) &= \tfrac{5}{3}col_1(A) + \tfrac{2}{3}col_2(A) \\ &= \tfrac{5}{3}(1,2,0) + \tfrac{2}{3}(2,1,0) \\ &= (5/3, 10/3, 0) + (4/3, 2/3, 0) \\ &= (3,4,0). \end{aligned}$$

What if we want a basis for $Row(A)$ which consists of rows in A itself?

Proposition 3.5.6.

Let $W = span\{w_1, w_2, \dots, w_k\} \subset \mathbb{R}^{1\times n}$ and construct A by concatenating the row vectors in W into a matrix A:

$$A = \begin{bmatrix} w_1 \\ \hline w_2 \\ \hline \vdots \\ \hline w_k \end{bmatrix}$$

A basis for W is given by the transposes of the pivot columns for A^T.

Proof: this is immediately obvious from Proposition 3.4.19. □

The proposition that follows is also follows immediately from Proposition 3.4.19.

Proposition 3.5.7.

Let $A \in \mathbb{R}^{m\times n}$ the rows which are transposes of the pivot columns of A^T form a basis for $Row(A)$.

Example 3.5.8.

$$A^T = \begin{bmatrix} 1 & 2 & 0 \\ 2 & 1 & 0 \\ 3 & 4 & 0 \\ 4 & 1 & 3 \end{bmatrix} \qquad \Rightarrow \qquad rref[A^T] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Notice that each column is a pivot column in A^T thus a basis for $Row(A)$ is simply the set of all rows of A; $Row(A) = span\{[1,2,3,4],[2,1,4,1],[0,0,1,0]\}$ and the spanning set is linearly independent.

Example 3.5.9.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 4 & 0 \\ 5 & 6 & 2 \end{bmatrix} \qquad \Rightarrow \qquad A^T = \begin{bmatrix} 1 & 2 & 3 & 5 \\ 1 & 2 & 4 & 6 \\ 1 & 2 & 0 & 2 \end{bmatrix} \qquad \Rightarrow \qquad rref[A^T] = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

We deduce that rows 1 and 3 or A form a basis for $Row(A)$. Notice that $row_2(A) = 2row_1(A)$ and $row_4(A) = row_3(A) + 2row_1(A)$. We can read linear dependendcies of the rows from the corresponding linear dependencies of the columns in the rref of the transpose.

The preceding examples are nice, but what should we do if we want to find both a basis for $Col(A)$ and $Row(A)$ for some given matrix ? Let's pause to think about how elementary row operations modify the row and column space of a matrix. In particular, let A be a matrix and let A' be the result of performing an elementary row operation on A. It is fairly obvious that

$$Row(A) = Row(A').$$

Think about it. If we swap to rows that just switches the order of the vectors in the span that makes $Row(A)$. On the other hand if we replace one row with a nontrivial linear combination of itself and other rows then that will not change the span either. Column space is not so easy though. Notice that elementary row operations can change the column space. For example,

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \Rightarrow rref[A] = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

has $Col(A) = span\{(1,1)\}$ whereas $Col(rref(A)) = span((1,0))$. We cannot hope to use columns of $ref(A)$ (or $rref(A)$) for a basis of $Col(A)$. That's no big problem though because we already have the CCP-principle which helped us pick out a basis for $Col(A)$. Let's collect our thoughts:

Proposition 3.5.10.

Let $A \in \mathbb{R}^{m\times n}$ then a basis for $Col(A)$ is given by the pivot columns in A and a basis for $Row(A)$ is given by the nonzero rows in $ref(A)$.

This means we can find a basis for $Col(A)$ and $Row(A)$ by performing the forward pass on A. We need only calculate the $ref(A)$ as the pivot columns are manifest at the end of the forward pass.

Example 3.5.11.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix} \xrightarrow[r_3 - r_1 \to r_3]{r_2 - r_1 \to r_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = ref[A]$$

We deduce that $\{[1,1,1],[0,1,2]\}$ *is a basis for* $Row(A)$ *whereas* $\{(1,1,1),(1,1,2)\}$ *is a basis for* $Col(A)$. *Notice that if I wanted to reveal further linear dependencies of the non-pivot columns on the pivot columns of* A *it would be wise to calculate* $rref[A]$ *by making the backwards pass on* $ref[A]$.

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_1 - r_2 \to r_1} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = rref[A]$$

From which I can read $col_3(A) = 2col_2(A) - col_1(A)$, *a fact which is easy to verify.*

Example 3.5.12.

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 8 & 10 \\ 1 & 2 & 4 & 11 \end{bmatrix} \xrightarrow[r_3 - r_1 \to r_3]{r_2 - r_1 \to r_2} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 5 & 6 \\ 0 & 0 & 1 & 7 \end{bmatrix} = ref[A]$$

We find that $Row(A)$ has basis

$$\{[1,2,3,4],[0,1,5,6],[0,0,1,7]\}$$

and $Col(A)$ has basis

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 8 \\ 4 \end{bmatrix} \right\}$$

Proposition 3.5.10 was the guide for both examples above.

3.5.2 calculating basis of a solution set

Often a subspace is described as the solution set of some equation $Ax = 0$. How do we find a basis for $Null(A)$? If we can do that we find a basis for subspaces which are described by some equation.

Proposition 3.5.13.

Let $A \in \mathbb{R}^{m \times n}$ and define $W = Null(A)$. A basis for W is obtained from the solution set of $Ax = 0$ by writing the solution as a linear combination where the free variables appear as coefficients in the vector-sum.

Proof: $x \in W$ implies $Ax = 0$. Denote $x = (x_1, x_2, \dots, x_n)$. Suppose that $rref[A]$ has r-pivot columns (we must have $0 \leq r \leq n$). There will be $(m-r)$-rows which are zero in $rref(A)$ and $(n-r)$-columns which are not pivot columns. The non-pivot columns correspond to free-variables in the solution. Define $p = n - r$ for convenience. Suppose that $x_{i_1}, x_{i_2}, \dots, x_{i_p}$ are free whereas $x_{j_1}, x_{j_2}, \dots, x_{j_r}$ are functions of the free variables: in particular they are linear combinations of the free variables as prescribed by $rref[A]$. There exist constants b_{ij} such that

$$\begin{aligned} x_{j_1} &= b_{11}x_{i_1} + b_{12}x_{i_2} + \cdots + b_{1p}x_{i_p} \\ x_{j_2} &= b_{21}x_{i_1} + b_{22}x_{i_2} + \cdots + b_{2p}x_{i_p} \\ \vdots \quad & \quad \vdots \qquad \cdots \qquad \vdots \\ x_{j_r} &= b_{r1}x_{i_1} + b_{r2}x_{i_2} + \cdots + b_{rp}x_{i_p} \end{aligned}$$

For convenience of notation assume that the free variables are put at the end of the list. We have

$$\begin{aligned} x_1 &= b_{11}x_{r+1} + b_{12}x_{r+2} + \cdots + b_{1p}x_n \\ x_2 &= b_{21}x_{r+1} + b_{22}x_{r+2} + \cdots + b_{2p}x_n \\ \vdots \quad & \quad \vdots \qquad \cdots \qquad \vdots \\ x_r &= b_{r1}x_{r+1} + b_{r2}x_{n-p+2} + \cdots + b_{rp}x_n \end{aligned}$$

and $x_j = x_j$ for $j = r+1, r+2, \dots, r+p = n$ (those are free, we have no conditions on them, they can take any value). We find,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_r \\ x_{r+1} \\ x_{r+2} \\ \vdots \\ x_n \end{bmatrix} = x_{r+1}\begin{bmatrix} b_{11} \\ b_{21} \\ \vdots \\ b_{r1} \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_{r+2}\begin{bmatrix} b_{12} \\ b_{22} \\ \vdots \\ b_{r2} \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n\begin{bmatrix} b_{1p} \\ b_{2p} \\ \vdots \\ b_{rp} \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

We define the vectors in the sum above as $v_1, v_2, \dots, v_p$. If any of the vectors, say v_j, was linearly dependent on the others then we would find that the variable x_{r+j} was likewise dependent on the other free variables. This would contradict the fact that the variable x_{r+j} was free. Consequently the vectors $v_1, v_2, \dots, v_p$ are linearly independent. Moreover, they span the null-space by virtue of their construction. □

Didn't follow the proof above? No problem. See the examples to follow here. These are just the proof in action for specific cases. We've done these sort of calculations in §1.3. We're just adding a little more insight here.

Example 3.5.14. *Find a basis for the null space of A given below,*

$$A = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 & 1 \\ 4 & 4 & 4 & 0 & 0 \end{bmatrix}$$

Gaussian elimination on the augmented coefficient matrix reveals (see Example 1.2.7 for details of the Gaussian elimination)

$$rref\begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 2 & 2 & 0 & 0 & 1 \\ 4 & 4 & 4 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -1 & 1/2 \\ 0 & 0 & 1 & 0 & -1/2 \end{bmatrix}$$

Denote $x = (x_1, x_2, x_3, x_4, x_5)$ *in the equation* $Ax = 0$ *and identify from the calculation above that* x_4 *and* x_5 *are free thus solutions are of the form*

$$\begin{aligned} x_1 &= -x_4 \\ x_2 &= x_4 - \tfrac{1}{2}x_5 \\ x_3 &= \tfrac{1}{2}x_5 \\ x_4 &= x_4 \\ x_5 &= x_5 \end{aligned}$$

for all $x_4, x_5 \in \mathbb{R}$. We can write these results in vector form to reveal the basis for $Null(A)$,

$$x = \begin{bmatrix} -x_4 \\ x_4 - \frac{1}{2}x_5 \\ \frac{1}{2}x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_4 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$$

It follows that the basis for $Null(A)$ is simply

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{1}{2} \\ \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} \right\}$$

Of course, you could multiply the second vector by 2 if you wish to avoid fractions. In fact there is a great deal of freedom in choosing a basis. We simply show one way to do it.

Example 3.5.15. *Find a basis for the null space of A given below,*

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Gaussian elimination on the augmented coefficient matrix reveals:

$$rref \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Denote $x = (x_1, x_2, x_3, x_4)$ in the equation $Ax = 0$ and identify from the calculation above that x_2, x_3 and x_4 are free thus solutions are of the form

$$\begin{aligned} x_1 &= -x_2 - x_3 - x_4 \\ x_2 &= x_2 \\ x_3 &= x_3 \\ x_4 &= x_4 \end{aligned}$$

for all $x_2, x_3, x_4 \in \mathbb{R}$. We can write these results in vector form to reveal the basis for $Null(A)$,

$$x = \begin{bmatrix} -x_2 - x_3 - x_4 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

It follows that the basis for $Null(A)$ is simply

$$\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

Proposition 3.5.16.

Let $A \in \mathbb{R}^{m\times n}$ then $dim(Row(A)) = dim(Col(A))$

Proof: By Proposition 3.5.4 we know the number of vectors in the basis for $Col(A)$ is the number of pivot columns in A. Likewise, Proposition 3.5.10 showed the number of vectors in the basis for $Row(A)$ was the number of nonzero rows in $ref(A)$. But the number of pivot columns is precisely the number of nonzero rows in $ref(A)$ therefore, $dim(Row(A)) = dim(Col(A))$. $\square$

Proposition 3.5.17. *(rank nullity theorem for matrices)*

Let $A \in \mathbb{R}^{m\times n}$ then $n = rank(A) + nullity(A)$.

Proof: The proof of Proposition 3.5.13 makes is clear that if a $m \times n$ matrix A has r-pivot columns then there will be $n - r$ vectors in the basis of $Null(A)$. It follows that

$$rank(A) + nullity(A) = r + (n - r) = n.$$

3.6 general theory of linear systems

Let $A \in \mathbb{R}^{m\times n}$ we should notice that $Null(A) \leq \mathbb{R}^n$ is only possible since homogeneous systems of the form $Ax = 0$ have the nice property that linear combinations of solutions is again a solution:

Proposition 3.6.1.

Let $Ax = 0$ denote a homogeneous linear system of m-equations and n-unknowns. If v_1 and v_2 are solutions then any linear combination $c_1v_1 + c_2v_2$ is also a solution of $Ax = 0$.

Proof: Suppose $Av_1 = 0$ and $Av_2 = 0$. Let $c_1, c_2 \in \mathbb{R}$ and recall Theorem 2.3.17 part 13,

$$A(c_1v_1 + c_2v_2) = c_1Av_1 + c_2Av_2 = c_10 + c_20 = 0.$$

Therefore, $c_1v_1 + c_2v_2 \in Sol_{[A|0]}$. $\square$
We proved this before, but I thought it might help to see it again here.

Proposition 3.6.2.

Let $A \in \mathbb{R}^{m \times n}$. If $v_1, v_2, \dots, v_k$ are solutions of $Av = 0$ then $V = [v_1|v_2|\cdots|v_k]$ is a **solution matrix** of $Av = 0$ (V a solution matrix of $Av = 0$ iff $AV = 0$)

Proof: Let $A \in \mathbb{R}^{m \times n}$ and suppose $Av_i = 0$ for $i = 1, 2, \dots k$. Let $V = [v_1|v_2|\cdots|v_k]$ and use the solution concatenation Proposition 3.1.7,

$$AV = A[v_1|v_2|\cdots|v_k] = [Av_1|Av_2|\cdots|Av_k] = [0|0|\cdots|0] = 0. \qquad \square$$

In simple terms, a solution matrix of a linear system is a matrix in which each column is itself a solution to the system. We've proved this before, (sorry the notes are not LI.)

Proposition 3.6.3.

Let $A \in \mathbb{R}^{m \times n}$. The system of equations $Ax = b$ is consistent iff $b \in Col(A)$.

The proposition below explains how to solve $Ax = b$ in general.

Proposition 3.6.4.

Let $A \in \mathbb{R}^{m \times n}$ and suppose the system of equations $Ax = b$ is consistent. We find $x \in \mathbb{R}^n$ is a solution of the system if and only if it can be written in the form

$$x = x_h + x_p = c_1 v_1 + c_2 v_2 + \cdots + c_\nu v_\nu + x_p$$

where $Ax_h = 0$, $\{v_j\}_{j=1}^{\nu}$ are a basis for $Null(A)$, and $Ax_p = b$. We call x_h the homogeneous solution and x_p is the nonhomogeneous solution.

Proof: Suppose $Ax = b$ is consistent then $b \in Col(A)$ therefore there exists $x_p \in \mathbb{R}^n$ such that $Ax_p = b$. Let x be any solution. We have $Ax = b$ thus observe

$$A(x - x_p) = Ax - Ax_p = Ax - b = 0 \quad \Rightarrow \quad (x - x_p) \in Null(A).$$

Define $x_h = x - x_p$ it follows that there exist constants c_i such that $x_h = c_1 v_1 + c_2 v_2 + \cdots + c_\nu v_\nu$ since the vectors v_i span the null space.

Conversely, suppose $x = x_p + x_h$ where $x_h = c_1 v_1 + c_2 v_2 + \cdots + c_\nu v_\nu \in Null(A)$ then it is clear that

$$Ax = A(x_p + x_h) = Ax_p + Ax_h = b + 0 = b$$

thus $x = x_p + x_h$ is a solution. $\square$

Example 3.6.5. *Consider the system of equations $x + y + z = 1, x + z = 1$. In matrix notation,*

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \Rightarrow rref[A|b] = rref \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

It follows that $x = 1 - y - z$ is a solution for any choice of $y, z \in \mathbb{R}$.

$$v = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1-y-z \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

We recognize that $v_p = (1, 0, 0)$ while $v_h = y(-1, 1, 0) + z(-1, 0, 1)$ and $\{(-1, 1, 0),\ (-1, 0, 1)\}$ is a basis for the null space of A. We call y, z parameters in the solution.

We will see that null spaces play a central part in the study of eigenvectors. In fact, about half of the calculation is finding a basis for the null space of a certain matrix. So, don't be too disappointed if I don't have too many examples here. You'll work dozens of them later. Let us conclude the theory with a simple observation about the connection between the parameters in the general solution and the dimension of the nullspace:

Proposition 3.6.6.

Let $A \in \mathbb{R}^{m \times n}$. If the system of equations $Ax = b$ is consistent then the general solution has as many parameters as the $dim(Null(A))$.

3.6.1 similarities with the general solution of linear differential equation*

A very similar story is told in differential equations. In Math 334 we spend some time unraveling the solution of $L[y] = g$ where $L = P(D)$ is an n-th order polynomial in the differentiation operator with constant coefficients. In total we learn that $y = c_1y_1 + c_2y_2 + \cdots + c_ny_n + y_p$ is the solution where y_j are the homogeneous solutions which satisfy $L[y_j] = 0$ for each $j = 1, 2, \dots, n$ and, in contrast, y_p is the so-called "particular solution" which satisfies $L[y_p] = g$. On the one hand, the results in DEqns are very different because the solutions are functions which live in the infinite-dimensional function space. However, on the other hand, $L[y] = g$ is a finite dimensional problem thanks to the fortunate fact that $Null(L) = \{f \in \mathcal{F}(\mathbb{R}) | L(f) = 0\} = span\{y_1, y_2, \dots, y_n\}$. For this reason there are n-parameters in the general solution which we typically denote by $c_1, c_2, \dots, c_n$ in the Math 334 course. The particular solution is not found by row reduction on a matrix in DEqns. Instead, we either use the annihilator method, power series techniques, or most generally the method of variation of parameters will calculate y_p. The analogy to the linear system $Av = b$ is striking:

1. $Av = b$ has solution $v = c_1v_1 + c_2v_2 + \cdots + c_kv_n + v_p$ where $v_j \in Null(A)$ and $Av_p = b$.
2. $L[y] = g$ has solution $v = c_1y_1 + c_2y_2 + \cdots + c_ky_n + y_p$ where $y_j \in Null(L)$ and $L[y_p] = b$.

The reason the DEqn $L[y] = g$ possesses such an elegant solution stems from the linearity of L. If you study nonlinear DEqns the structure is not so easily described.

Example 3.6.7. *Here's a simple differential equation you all should be able to solve: $L[y] = y'' = e^x$. Simply integrate twice to obtain $y = c_1 + c_2x + e^x$. Observe that $y_h = c_1 + c_2x$ has $L[y_h] = y_h'' = 0$ whereas $L[e^x] = (e^x)'' = e^x$ and we identify $y_p = e^x$.*

Of course, if all differential equations were so simple to solve as the last example then we'd hardly need a course for the subject!

Chapter 4

matrix arithmetic II

Let us continue our study of matrices and their properties.

4.1 invertible matrices

Definition 4.1.1.

Let $A \in \mathbb{R}^{n\times n}$. If there exists $B \in \mathbb{R}^{n\times n}$ such that $AB = I$ and $BA = I$ then we say that A is **invertible** and $A^{-1} = B$. Invertible matrices are also called **nonsingular**. If a matrix has no inverse then it is called a **noninvertible** or **singular** matrix.

Proposition 4.1.2.

Elementary matrices are invertible.

Proof: I list the inverse matrix for each below:

$$(E_{r_i+ar_j\to r_i})^{-1} = [I:\ r_i - ar_j \to r_i]$$
$$(E_{br_i\to r_i})^{-1} = [I:\ \tfrac{1}{b}r_i \to r_i]$$
$$(E_{r_i\leftrightarrow r_j})^{-1} = [I:\ r_j \leftrightarrow r_i]$$

I leave it to the reader to convince themselves that these are indeed inverse matrices. $\square$

Example 4.1.3. *Let me illustrate the mechanics of the proof above,* $E_{r_1+3r_2\to r_1} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *and* $E_{r_1-3r_2\to r_1} = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ *satisfy,*

$$E_{r_1+3r_2\to r_1}E_{r_1-3r_2\to r_1} = \begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Likewise,

$$E_{r_1-3r_2\to r_1}E_{r_1+3r_2\to r_1} = \begin{bmatrix} 1 & -3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 3 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Thus, $(E_{r_1+3r_2\to r_1})^{-1} = E_{r_1-3r_2\to r_1}$ *just as we expected.*

Theorem 4.1.4.

Let $A \in \mathbb{R}^{n\times n}$. The solution of $Ax = 0$ is unique iff A^{-1} exists.

Proof:($\Rightarrow$) Suppose $Ax = 0$ has a unique solution. Observe $A0 = 0$ thus the only solution is the zero solution. Consequently, $rref[A|0] = [I|0]$. Moreover, by Proposition 3.2.3 there exist elementary matrices $E_1, E_2, \cdots, E_k$ such that $rref[A|0] = E_1E_2\cdots E_k[A|0] = [I|0]$. Applying the concatenation Proposition 2.3.12 we find that $[E_1E_2\cdots E_kA|E_1E_2\cdots E_k0] = [I|0]$ thus $E_1E_2\cdots E_kA = I$.

It remains to show that $AE_1E_2\cdots E_k = I$. Multiply $E_1E_2\cdots E_kA = I$ on the left by ${E_1}^{-1}$ followed by ${E_2}^{-1}$ and so forth to obtain

$$ {E_k}^{-1}\cdots {E_2}^{-1}{E_1}^{-1}E_1E_2\cdots E_kA = {E_k}^{-1}\cdots {E_2}^{-1}{E_1}^{-1}I $$

this simplifies to

$$ A = {E_k}^{-1}\cdots {E_2}^{-1}{E_1}^{-1}. $$

Observe that

$$ AE_1E_2\cdots E_k = {E_k}^{-1}\cdots {E_2}^{-1}{E_1}^{-1}E_1E_2\cdots E_k = I. $$

We identify that $A^{-1} = E_1E_2\cdots E_k$ thus A^{-1} exists.

($\Leftarrow$) The converse proof is much easier. Suppose A^{-1} exists. If $Ax = 0$ then multiply by A^{-1} on the left, $A^{-1}Ax = A^{-1}0 \ \Rightarrow\ Ix = 0$ thus $x = 0$. $\square$

Proposition 4.1.5.

Let $A \in \mathbb{R}^{n\times n}$.

1. If $BA = I$ then $AB = I$.
2. If $AB = I$ then $BA = I$.

Proof of (1.): Suppose $BA = I$. If $Ax = 0$ then $BAx = B0$ hence $Ix = 0$. We have shown that $Ax = 0$ only has the trivial solution. Therefore, Theorem 4.1.4 shows us that A^{-1} exists. Multiply $BA = I$ on the left by A^{-1} to find $BAA^{-1} = IA^{-1}$ hence $B = A^{-1}$ and by definition it follows $AB = I$.

Proof of (2.): Suppose $AB = I$. If $Bx = 0$ then $ABx = A0$ hence $Ix = 0$. We have shown that $Bx = 0$ only has the trivial solution. Therefore, Theorem 4.1.4 shows us that B^{-1} exists. Multiply $AB = I$ on the right by B^{-1} to find $ABB^{-1} = IB^{-1}$ hence $A = B^{-1}$ and by definition it follows $BA = I$. $\square$

Proposition 4.1.5 shows that we don't need to check both conditions $AB = I$ and $BA = I$. If either holds the other condition automatically follows.

Proposition 4.1.6.

If $A \in \mathbb{R}^{n\times n}$ is invertible then its inverse matrix is unique.

Proof: Suppose B, C are inverse matrices of A. It follows that $AB = BA = I$ and $AC = CA = I$ thus $AB = AC$. Multiply B on the left of $AB = AC$ to obtain $BAB = BAC$ hence $IB = IC \Rightarrow B = C$. $\square$

Example 4.1.7. *In the case of a 2×2 matrix a nice formula to find the inverse is known:*

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

It's not hard to show this formula works,

$$\begin{aligned} \tfrac{1}{ad-bc}\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} &= \tfrac{1}{ad-bc}\begin{bmatrix} ad-bc & -ab+ab \\ cd-dc & -bc+da \end{bmatrix} \\ &= \tfrac{1}{ad-bc}\begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

How did we know this formula? Can you derive it? To find the formula from first principles you could suppose there exists a matrix $B = \left[\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right]$ such that $AB = I$. The resulting algebra would lead you to conclude $x = d/t, y = -b/t, z = -c/t, w = a/t$ where $t = ad - bc$. I leave this as an exercise for the reader.

There is a giant assumption made throughout the last example. What is it?

Example 4.1.8. *A counterclockwise* **rotation** *by angle θ in the plane can be represented by a matrix $R(\theta) = \left[\begin{smallmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{smallmatrix}\right]$. The inverse matrix corresponds to a rotation by angle $-\theta$ and (using the even/odd properties for cosine and sine) $R(-\theta) = \left[\begin{smallmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{smallmatrix}\right] = R(\theta)^{-1}$. Notice that $R(0) = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ thus $R(\theta)R(-\theta) = R(0) = I$. We'll talk more about geometry in a later chapter. If you'd like to see how this matrix is related to the imaginary exponential $e^{i\theta} = \cos(\theta) + i\sin(\theta)$ you can look at www.supermath.info/intro_to_complex.pdf where I show how the cosines and sines come from a rotation of the coordinate axes. If you draw the right picture you can understand why the formulas below describe changing the coordinates from (x, y) to $(\bar{x}, \bar{y}$ where the transformed coordinates are rotated by angle θ:*

$$\begin{aligned} \bar{x} &= \cos(\theta)x + \sin(\theta)y \\ \bar{y} &= -\sin(\theta)x + \cos(\theta)y \end{aligned} \qquad \Leftrightarrow \qquad \begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

Theorem 4.1.9.

If $A, B \in \mathbb{R}^{\,n\times n}$ are invertible, $X, Y \in \mathbb{R}^{m\times n}$, $Z, W \in \mathbb{R}^{n\times m}$ and nonzero $c \in \mathbb{R}$ then

1. $(AB)^{-1} = B^{-1}A^{-1}$,
2. $(cA)^{-1} = \frac{1}{c}A^{-1}$,
3. $XA = YA$ implies $X = Y$,
4. $AZ = AW$ implies $Z = W$,

Proof: To prove (1.) simply notice that

$$(AB)B^{-1}A^{-1} = A(BB^{-1})A^{-1} = A(I)A^{-1} = AA^{-1} = I.$$

The proof of (2.) follows from the calculation below,

$$(\tfrac{1}{c}A^{-1})cA = \tfrac{1}{c}cA^{-1}A = A^{-1}A = I.$$

To prove (3.) assume that $XA = YA$ and multiply both sides by A^{-1} on the right to obtain $XAA^{-1} = YAA^{-1}$ which reveals $XI = YI$ or simply $X = Y$. To prove (4.) multiply by A^{-1} on the left. □

Remark 4.1.10.

The proofs just given were all matrix arguments. These contrast the component level proofs needed for 2.3.17. We could give component level proofs for the Theorem above but that is not necessary and those arguments would only obscure the point. I hope you gain your own sense of which type of argument is most appropriate as the course progresses.

We have a simple formula to calculate the inverse of a 2×2 matrix, but sadly no such simple formula exists for bigger matrices. There is a nice method to calculate A^{-1} (if it exists), but we do not have all the theory in place to discuss it at this juncture.

Proposition 4.1.11.

If $A_1, A_2, \dots, A_k \in \mathbb{R}^{\,n\times n}$ are invertible then

$$(A_1A_2\cdots A_k)^{-1} = A_k^{-1}A_{k-1}^{-1}\cdots A_2^{-1}A_1^{-1}$$

Proof: Provided by you in the Problem Set. Your argument will involve induction on the index k. Notice you already have the cases $k = 1, 2$ from the arguments in this section. In particular, $k = 1$ is trivial and $k = 2$ is given by Theorem 4.1.11. □

4.1.1 how to calculate the inverse of a matrix

The problem of calculating an inverse amounts to precisely the problem of simultaneously solving several systems of equations at once. We now put to use the technique disucssed in Section 3.1.

PROBLEM: how should we calculate A^{-1} for a 3×3 matrix ?

Consider that the Proposition 3.1.7 gives us another way to look at the problem,

$$AA^{-1} = I \ \Leftrightarrow \ A[v_1|v_2|v_3] = I_3 = [e_1|e_2|e_3]$$

Where $v_i = col_i(A^{-1})$ and $e_1 = (1,0,0), e_2 = (0,\ 1,\ 0), e_3 = (0,\ 0,\ 1)$. We observe that the problem of finding A^{-1} for a 3×3 matrix amounts to solving three separate systems:

$$Av_1 = e_1, \ Av_2 = e_2, \ Av_3 = e_3$$

when we find the solutions then we can construct $A^{-1} = [v_1|v_2|v_3]$. Think about this, if A^{-1} exists then it is unique thus the solutions v_1, v_2, v_3 are likewise unique. Consequently, by Theorem 1.5.3,

$$rref[A|e_1] = [I|v_1], \ rref[A|e_2] = [I|v_2], \ rref[A|e_3] = [I|v_3].$$

Each of the systems above required the same sequence of elementary row operations to cause $A \mapsto I$. We can just as well do them at the same time in one big matrix calculation:

$$rref[A|e_1|e_2|e_3] = [I|v_1|v_2|v_3]$$

While this discuss was done for $n = 3$ we can just as well do the same for $n > 3$. This provides the proof for the first sentence of the theorem below. Theorem 1.5.3 together with the discussion above proves the second sentence.

Theorem 4.1.12.

$A \in \mathbb{R}^{\ n\times n}$ is invertible iff $rref[A|I] = [I|A^{-1}]$. In contrast, for $A \in \mathbb{R}^{\ n\times n}$, A^{-1} **not** invertible iff $rref(A) \neq I$ iff $rref[A|I] \neq [I|B]$.

This theorem tells us how and when we can find an inverse for a square matrix.

Example 4.1.13. *Recall that in Example 1.2.7 we worked out the details of*

$$rref \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 & 1 & 0 \\ 4 & 4 & 4 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 1/2 & 0 \\ 0 & 0 & 1 & 0 & -1/2 & 1/4 \end{array}\right]$$

Thus,

$$\left[\begin{array}{ccc} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 4 & 4 & 4 \end{array}\right]^{-1} = \left[\begin{array}{ccc} 1 & 0 & 0 \\ -1 & 1/2 & 0 \\ 0 & -1/2 & 1/4 \end{array}\right].$$

Example 4.1.14. *I omit the details of the Gaussian elimination,*

$$rref\left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 1 & 0 & 0 \\ 1 & 0 & -1 & 0 & 1 & 0 \\ 6 & 2 & 3 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2 & -3 & -1 \\ 0 & 1 & 0 & -3 & -3 & -1 \\ 0 & 0 & 1 & -2 & -4 & -1 \end{array}\right]$$

Thus,

$$\begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & 2 & 3 \end{bmatrix}^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}.$$

4.2 matrices with special shapes

In this section we learn about a few special types of matrices.

4.2.1 symmetric and antisymmetric matrices

Definition 4.2.1.

Let $A \in \mathbb{R}^{\,n\times n}$. We say A is **symmetric** iff $A^T = A$. We say A is **antisymmetric** iff $A^T = -A$.

At the level of components, $A^T = A$ gives $A_{ij} = A_{ji}$ for all i, j. Whereas, $A^T = -A$ gives $A_{ij} = -A_{ji}$ for all i, j. I should mention **skew-symmetric** is another word for antisymmetric. In physics, second rank (anti)symmetric tensors correspond to (anti)symmetric matrices. In electromagnetism, the electromagnetic field tensor has components which can be written as an antisymmetric 4×4 matrix. In classical mechanics, a solid's propensity to spin in various directions is described by the intertia tensor which is represented by a symmetric matrix. The energy-momentum tensor from electrodynamics is also represented by a symmetric matrix. Matrices are everywhere if we look for them.

Example 4.2.2. *Some matrices are symmetric:*

$$I, \ O, \ E_{ii}, \ \begin{bmatrix} 1 & 2 \\ 2 & 0 \end{bmatrix}$$

Some matrices are antisymmetric:

$$O, \ \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$$

Only 0 *is both symmetric and antisymmetric. Other matrices are neither symmetric nor antisymmetric:*

$$e_i, \ E_{i,i+1}, \ \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

I assumed $n > 1$ *so that* e_i *is a column vector which is not square.*

Proposition 4.2.3.

Let $A, B \in \mathbb{R}^{\,n\times n}$ and $c \in \mathbb{R}$ then

1. $(A^T)^T = A$
2. $(AB)^T = B^T A^T$ socks-shoes property for transpose of product
3. $(cA)^T = cA^T$
4. $(A+B)^T = A^T + B^T$
5. $(A^T)^{-1} = (A^{-1})^T$.

Proof: Just apply the definition. □

Proposition 4.2.4.

All square matrices are formed by the sum of a symmetric and antisymmetric matrix.

Proof: Let $A \in \mathbb{R}^{\,n\times n}$. Utilizing Proposition 4.2.3 we find

$$\big(\tfrac{1}{2}(A + A^T)\big)^T = \tfrac{1}{2}(A^T + (A^T)^T) = \tfrac{1}{2}(A^T + A) = \tfrac{1}{2}(A + A^T)$$

thus $\frac{1}{2}(A + A^T)$ is a symmetric matrix. Likewise,

$$\big(\tfrac{1}{2}(A - A^T)\big)^T = \tfrac{1}{2}(A^T - (A^T)^T) = \tfrac{1}{2}(A^T - A) = -\tfrac{1}{2}(A - A^T)$$

thus $\frac{1}{2}(A - A^T)$ is an antisymmetric matrix. Finally, note the identity below:

$$A = \tfrac{1}{2}(A + A^T) + \tfrac{1}{2}(A - A^T)$$

The theorem follows. □

The proof that any function on $\mathbb{R}$ is the sum of an even and odd function uses the same trick.

Example 4.2.5. *The proof of the Proposition above shows us how to break up the matrix into its symmetric and antisymmetric pieces:*

$$\begin{aligned} \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} &= \tfrac{1}{2}\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\right) + \tfrac{1}{2}\left(\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} - \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\right) \\ &= \begin{bmatrix} 1 & 5/2 \\ 5/2 & 4 \end{bmatrix} + \begin{bmatrix} 0 & -1/2 \\ 1/2 & 0 \end{bmatrix}. \end{aligned}$$

Example 4.2.6. *What are the symmetric and antisymmetric parts of the standard basis* E_{ij} *in* $\mathbb{R}^{\,n\times n}$*? Here the answer depends on the choice of* i,j*. Note that* $(E_{ij})^T = E_{ji}$ *for all* i,j*.*
Suppose $i=j$ *then* $E_{ij} = E_{ii}$ *is clearly symmetric, thus there is no antisymmetric part.*
If $i \neq j$ *we use the standard trick,*

$$E_{ij} = \tfrac{1}{2}(E_{ij} + E_{ji}) + \tfrac{1}{2}(E_{ij} - E_{ji})$$

where $\frac{1}{2}(E_{ij} + E_{ji})$ *is the symmetric part of* E_{ij} *and* $\frac{1}{2}(E_{ij} - E_{ji})$ *is the antisymmetric part of* E_{ij}*.*

Proposition 4.2.7.

Let $A \in \mathbb{R}^{\,m\times n}$ then A^TA is symmetric.

Proof: Proposition 4.2.3 yields $(A^TA)^T = A^T(A^T)^T = A^TA$. Thus A^TA is symmetric. □

4.2.2 exponent laws for matrices

The power of a matrix is defined in the natural way. Notice we need for A to be square in order for the product AA to be defined.

Definition 4.2.8.

Let $A \in \mathbb{R}^{\,n\times n}$. We define $A^0 = I$, $A^1 = A$ and $A^m = AA^{m-1}$ for all $m \geq 1$. If A is invertible then $A^{-p} = (A^{-1})^p$.

As you would expect, $A^3 = AA^2 = AAA$.

Proposition 4.2.9.

Let $A, B \in \mathbb{R}^{\,n\times n}$ and $p, q \in \mathbb{N} \cup \{0\}$

1. $(A^p)^q = A^{pq}$.
2. $A^pA^q = A^{p+q}$.
3. If A is invertible, $(A^{-1})^{-1} = A$.

Proof: left to reader. □

You should notice that $(AB)^p \neq A^pB^p$ for matrices. Instead,

$$(AB)^2 = ABAB, \qquad (AB)^3 = ABABAB, etc...$$

This means the binomial theorem will not hold for matrices. For example,

$$(A+B)^2 = (A+B)(A+B) = A(A+B) + B(A+B) = AA + AB + BA + BB$$

hence $(A+B)^2 \neq A^2 + 2AB + B^2$ as the matrix product is not generally commutative. If we have A and B commute then $AB = BA$ and we can prove that $(AB)^p = A^pB^p$ and the binomial theorem holds true.

Proposition 4.2.10.

If A is symmetric then A^k is symmetric for all $k \in \mathbb{N}$.

Proof: Suppose $A^T = A$. Proceed inductively. Clearly $k = 1$ holds true since $A^1 = A$. Assume inductively that A^k is symmetric.

$$\begin{aligned} (A^{k+1})^T &= (AA^k)^T && \text{defn. of matrix exponents,} \\ &= (A^k)^T A^T && \text{socks-shoes prop. of transpose,} \\ &= A^k A && \text{using induition hypothesis.} \\ &= A^{k+1} && \text{defn. of matrix exponents,} \end{aligned}$$

thus by proof by mathematical induction A^k is symmetric for all $k \in \mathbb{N}$. □
There are many other fun identities about symmetric and invertible matrices. I'll probably put a few in the Problem Set since they make nice easy proof problems.

4.2.3 diagonal and triangular matrices

Definition 4.2.11.

Let $A \in \mathbb{R}^{m\times n}$. If $A_{ij} = 0$ for all i, j such that $i \neq j$ then A is called a **diagonal** matrix. If A has components $A_{ij} = 0$ for all i, j such that $i \leq j$ then we call A a **upper triangular** matrix. If A has components $A_{ij} = 0$ for all i, j such that $i \geq j$ then we call A a **lower triangular** matrix.

Example 4.2.12. *Let me illustrate a generic example of each case for 3×3 matrices:*

$$\begin{bmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix} \qquad \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ 0 & A_{22} & A_{23} \\ 0 & 0 & A_{33} \end{bmatrix} \qquad \begin{bmatrix} A_{11} & 0 & 0 \\ A_{21} & A_{22} & 0 \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$$

As you can see the diagonal matrix only has nontrivial entries on the diagonal, and the names lower triangular and upper triangular are likewise natural.

If an upper triangular matrix has zeros on the diagonal then it is said to be **strictly upper triangular**. Likewise, if a lower triangular matrix has zeros on the diagonal then it is said to be **strictly lower triangular**. Obviously and matrix can be written as a sum of a diagonal and strictly upper and strictly lower matrix,

$$\begin{aligned} A &= \sum_{i,j} A_{ij} E_{ij} \\ &= \sum_{i} A_{ii} E_{ii} + \sum_{i<j} A_{ij} E_{ij} + \sum_{i>j} A_{ij} E_{ij} \end{aligned}$$

There is an algorithm called LU-factorization which for many matrices A finds a lower triangular matrix L and an upper triangular matrix U such that $A = LU$. We may discuss it at the end of the course. It is one of several factorization schemes which is calculationally advantageous for large systems. There are many many ways to solve a system, but some are faster methods. Algorithmics is the study of which method is optimal.

Proposition 4.2.13.

Let $A, B \in \mathbb{R}^{\ n\times n}$.

1. If A, B are upper diagonal then AB is diagonal.
2. If A, B are upper triangular then AB is upper triangular.
3. If A, B are lower triangular then AB is lower triangular.

Proof of (1.): Suppose A and B are diagonal. It follows there exist a_i, b_j such that $A = \sum_i a_i E_{ii}$ and $B = \sum_j b_j E_{jj}$. Calculate,

$$\begin{aligned} AB &= \sum_i a_i E_{ii} \sum_j b_j E_{jj} \\ &= \sum_i \sum_j a_i b_j E_{ii} E_{jj} \\ &= \sum_i \sum_j a_i b_j \delta_{ij} E_{ij} \\ &= \sum_i a_i b_i E_{ii} \end{aligned}$$

thus the product matrix AB is also diagonal and we find that the diagonal of the product AB is just the product of the corresponding diagonals of A and B.
Proof of (2.): Suppose A and B are upper diagonal. It follows there exist A_{ij}, B_{ij} such that $A = \sum_{i\leq j} A_{ij} E_{ij}$ and $B = \sum_{k\leq l} B_{kl} E_{kl}$. Calculate,

$$\begin{aligned} AB &= \sum_{i\leq j} A_{ij} E_{ij} \sum_{k\leq l} B_{kl} E_{kl} \\ &= \sum_{i\leq j} \sum_{k\leq l} A_{ij} B_{kl} E_{ij} E_{kl} \\ &= \sum_{i\leq j} \sum_{k\leq l} A_{ij} B_{kl} \delta_{jk} E_{il} \\ &= \sum_{i\leq j} \sum_{j\leq l} A_{ij} B_{jl} E_{il} \end{aligned}$$

Notice that every term in the sum above has $i \leq j$ and $j \leq l$ hence $i \leq l$. It follows the product is upper triangular since it is a sum of upper triangular matrices. The proof of (3.) is similar. $\square$.

It is entirely likely that I give a less technical proof for the special case $n = 2$ or 3 in lecture.

4.2.4 block matrices

If you look at most undergraduate linear algbera texts they will not bother to even attempt much of a proof that block-multiplication holds in general. I will foolishly attempt it here. However, I'm going to cheat a little and employ uber-sneaky physics notation. The Einstein summation convention states that if an index is repeated then it is assumed to be summed over it's values. This means that the letters used for particular indices are reserved. If i, j, k are used to denote components of a spatial vector then you cannot use them for a spacetime vector at the same time. A typical notation in physics would be that v^j is a vector in xyz-space whereas v^μ is a vector in $txyz$-spacetime. A spacetime vector could be written as a sum of space components and a time component; $v = v^\mu e_\mu = v^0 e_0 + v^1 e_1 + v^2 e_2 + v^3 e_3 = v^0 e_0 + v^j e_j$. This is not the sort of langauge we use in mathematics. For us notation is usually not reserved. Anyway, cultural commentary aside, if we were to use Einstein-type notation in linear algebra then we would likely omit sums as follows:

$$v = \sum_i v_i e_i \quad \longrightarrow \quad v = v_i e_i$$

$$A = \sum_{ij} A_{ij} E_{ij} \quad \longrightarrow \quad A = A_{ij} E_{ij}$$

We wish to partition a matrices A and B into 4 parts, use indices M, N which split into subindices m, μ and n, ν respectively. In this notation there are 4 different types of pairs possible:

$$A = [A_{MN}] = \left[\begin{array}{c|c} A_{mn} & A_{m\nu} \\ \hline A_{\mu n} & A_{\mu\nu} \end{array}\right] \qquad B = [B_{NJ}] = \left[\begin{array}{c|c} B_{nj} & B_{n\gamma} \\ \hline B_{\mu j} & B_{\mu\gamma} \end{array}\right]$$

Then the sum over M, N breaks into 2 cases,

$$A_{MN} B_{NJ} = A_{Mn} B_{nJ} + A_{M\nu} B_{\nu J}$$

But, then there are 4 different types of M, J pairs,

$$[AB]_{mj} = A_{mN} B_{Nj} = A_{mn} B_{nj} + A_{m\nu} B_{\nu j}$$

$$[AB]_{m\gamma} = A_{mN} B_{N\gamma} = A_{mn} B_{n\gamma} + A_{m\nu} B_{\nu\gamma}$$

$$[AB]_{\mu j} = A_{\mu N} B_{Nj} = A_{\mu n} B_{nj} + A_{\mu\nu} B_{\nu j}$$

$$[AB]_{\mu\gamma} = A_{\mu N} B_{N\gamma} = A_{\mu n} B_{n\gamma} + A_{\mu\nu} B_{\nu\gamma}$$

Let me summarize,

$$\left[\begin{array}{c|c} A_{mn} & A_{m\nu} \\ \hline A_{\mu n} & A_{\mu\nu} \end{array}\right] \left[\begin{array}{c|c} B_{nj} & B_{n\gamma} \\ \hline B_{\mu j} & B_{\mu\gamma} \end{array}\right] = \left[\begin{array}{c|c} [A_{mn}][B_{nj}] + [A_{m\nu}][B_{\nu j}] & [A_{mn}][B_{n\gamma}] + [A_{m\nu}][B_{\nu\gamma}] \\ \hline [A_{\mu n}][B_{nj}] + [A_{\mu\nu}][B_{\nu j}] & [A_{\mu n}][B_{n\gamma}] + [A_{\mu\nu}][B_{\nu\gamma}] \end{array}\right]$$

Let me again summarize, but this time I'll drop the annoying indices:

Theorem 4.2.14. *block multiplication.*

Suppose $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$ such that both A and B are partitioned as follows:

$$A = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right] \qquad \text{and} \qquad B = \left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right]$$

where A_{11} is an $m_1 \times n_1$ block, A_{12} is an $m_1 \times n_2$ block, A_{21} is an $m_2 \times n_1$ block and A_{22} is an $m_2 \times n_2$ block. Likewise, $B_{n_k p_k}$ is an $n_k \times p_k$ block for $k = 1, 2$. We insist that $m_1 + m_2 = m$ and $n_1 + n_2 = n$. If the partitions are compatible as decribed above then we may multiply A and B by multiplying the blocks as if they were scalars and we were computing the product of 2×2 matrices:

$$\left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \end{array}\right]\left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right] = \left[\begin{array}{c|c} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \end{array}\right].$$

To give a careful proof we'd just need to write out many sums and define the partition with care from the outset of the proof. In any event, notice that once you have this partition you can apply it twice to build block-multiplication rules for matrices with more blocks. The basic idea remains the same: you can parse two matrices into matching partitions then the matrix multiplication follows a pattern which is as if the blocks were scalars. However, the blocks are not scalars so the multiplication of the blocks is nonabelian. For example,

$$AB = \left[\begin{array}{c|c} A_{11} & A_{12} \\ \hline A_{21} & A_{22} \\ \hline A_{31} & A_{32} \end{array}\right]\left[\begin{array}{c|c} B_{11} & B_{12} \\ \hline B_{21} & B_{22} \end{array}\right] = \left[\begin{array}{c|c} A_{11}B_{11} + A_{12}B_{21} & A_{11}B_{12} + A_{12}B_{22} \\ \hline A_{21}B_{11} + A_{22}B_{21} & A_{21}B_{12} + A_{22}B_{22} \\ \hline A_{31}B_{11} + A_{32}B_{21} & A_{31}B_{12} + A_{32}B_{22} \end{array}\right].$$

where if the partitions of A and B are compatible it follows that the block-multiplications on the RHS are all well-defined.

Example 4.2.15. *Let* $R(\theta) = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$ *and* $B(\gamma) = \begin{bmatrix} \cosh(\gamma) & \sinh(\gamma) \\ \sinh(\gamma) & \cosh(\gamma) \end{bmatrix}$. *Furthermore construct* 4×4 *matrices* Λ_1 *and* Λ_2 *as follows:*

$$\Lambda_1 = \left[\begin{array}{c|c} B(\gamma_1) & 0 \\ \hline 0 & R(\theta_1) \end{array}\right] \qquad \Lambda_2 = \left[\begin{array}{c|c} B(\gamma_2) & 0 \\ \hline 0 & R(\theta_2) \end{array}\right]$$

Multiply Λ_1 *and* Λ_2 *via block multiplication:*

$$\begin{aligned} \Lambda_1\Lambda_2 &= \left[\begin{array}{c|c} B(\gamma_1) & 0 \\ \hline 0 & R(\theta_1) \end{array}\right]\left[\begin{array}{c|c} B(\gamma_2) & 0 \\ \hline 0 & R(\theta_2) \end{array}\right] \\ &= \left[\begin{array}{c|c} B(\gamma_1)B(\gamma_2) + 0 & 0 + 0 \\ \hline 0 + 0 & 0 + R(\theta_1)R(\theta_2) \end{array}\right] \\ &= \left[\begin{array}{c|c} B(\gamma_1 + \gamma_2) & 0 \\ \hline 0 & R(\theta_1 + \theta_2) \end{array}\right]. \end{aligned}$$

The last calculation is actually a few lines in detail, if you know the adding angles formulas for cosine, sine, cosh and sinh it's easy. If $\theta = 0$ and $\gamma \neq 0$ then Λ would represent a **velocity boost** *on spacetime. Since it mixes time and the first coordinate the velocity is along the x-coordinate. On the other hand, if $\theta \neq 0$ and $\gamma = 0$ then Λ gives a* **rotation** *in the yz spatial coordinates in space time. If both parameters are nonzero then we can say that Λ is a* **Lorentz transformation** *on spacetime. Of course there is more to say here, perhaps we could offer a course in special relativity if enough students were interested in concert.*

Example 4.2.16. Problem: *Suppose M is a square matrix with submatrices $A, B, C, 0$. What conditions should we insist on for $M = \left[\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right]$ to be invertible.*

Solution: *I propose we partition the potential inverse matrix $M^{-1} = \left[\begin{array}{c|c} D & E \\ \hline F & G \end{array}\right]$. We seek to find conditions on A, B, C such that there exist D, E, F, G and $MM^{-1} = I$. Each block of the equation $MM^{-1} = I$ gives us a separate submatrix equation:*

$$MM^{-1} = \left[\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right] \left[\begin{array}{c|c} D & E \\ \hline F & G \end{array}\right] = \left[\begin{array}{c|c} AD + BF & AE + BG \\ \hline 0D + CF & 0E + CG \end{array}\right] = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right]$$

We must solve simultaneously the following:

$$(1.)\ AD + BF = I, \qquad (2.)\ AE + BG = 0, \qquad (3.)\ CF = 0, \qquad (4.)\ CG = I$$

If C^{-1} exists then $G = C^{-1}$ from (4.). Moreover, (3.) then yields $F = C^{-1}0 = 0$. Our problem thus reduces to (1.) and (2.) which after substituting $F = 0$ and $G = C^{-1}$ yield

$$(1.)\ AD = I, \qquad (2.)\ AE + BC^{-1} = 0.$$

Equation (1.) says $D = A^{-1}$. Finally, let's solve (2.) for E,

$$E = -A^{-1}BC^{-1}.$$

Let's summarize the calculation we just worked through. IF A, C are invertible then the matrix $M = \left[\begin{array}{c|c} A & B \\ \hline 0 & C \end{array}\right]$ is invertible with inverse

$$\boxed{M^{-1} = \left[\begin{array}{c|c} A^{-1} & -A^{-1}BC^{-1} \\ \hline 0 & C^{-1} \end{array}\right].}$$

Consider the case that M is a 2×2 matrix and $A, B, C \in \mathbb{R}$. Then the condition of invertibility reduces to the simple conditions $A, C \neq 0$ and $-A^{-1}BC^{-1} = \frac{-B}{AC}$ we find the formula:

$$M^{-1} = \left[\begin{array}{c|c} \frac{1}{A} & \frac{-B}{AC} \\ \hline 0 & \frac{1}{C} \end{array}\right] = \frac{1}{AC}\left[\begin{array}{c|c} C & -B \\ \hline 0 & A \end{array}\right].$$

This is of course the formula for the 2×2 matrix in this special case where $M_{21} = 0$.

Of course the real utility of formulas like those in the last example is that they work for partitions of arbitrary size. If we can find a block of zeros somewhere in the matrix then we may reduce the size of the problem. The time for a computer calculation is largely based on some power of the size of the matrix. For example, if the calculation in question takes n^2 steps then parsing the matrix into 3 nonzero blocks which are $n/2 \times n/2$ would result in something like $[n/2]^2 + [n/2]^2 + [n/2]^2 = \frac{3}{4}n^2$ steps. If the calculation took on order n^3 computer operations (flops) then my toy example of 3 blocks would reduce to something like $[n/2]^3 + [n/2]^3 + [n/2]^3 = \frac{3}{8}n^2$ flops. A savings of more than 60% of computer time. If the calculation was typically order n^4 for an $n \times n$ matrix then the saving is even more dramatic. If the calculation is a determinant then the cofactor formula depends on the factorial of the size of the matrix. Try to compare 10!+10! verses say 20!. Hope your calculator has a big display:

$$10! = 3628800 \quad \Rightarrow \quad 10! + 10! = 7257600 \qquad \text{or} \qquad 20! = 2432902008176640000.$$

Perhaps you can start to appreciate why numerical linear algebra software packages often use algorithms which make use of block matrices to streamline large matrix calculations.

Finally, I would comment that breaking a matrix into blocks is basically the bread and butter of quantum mechanics. One attempts to find a basis of state vectors which makes the Hamiltonian into a block-diagonal matrix. Each block corresponds to a certain set of statevectors sharing a common energy. The goal of representation theory in physics is basically to break down matrices into blocks with nice physical meanings. On the other hand, abstract algebraists also use blocks to rip apart a matrix into it's most basic form. For linear algebraists[1], the so-called Jordan form is full of blocks. Wherever reduction of a linear system into smaller subsystems is of interest there will be blocks.

4.3 LU factorization

In this section we will use elementary matrices which correspond to the forward pass of the Gaussian elimination to factor matrices into a pair of simpler matrices; our goal is to factor a matrix A into a lower triangular matrix L and a upper triangular matrix U; we hope to find $A = LU$. In the abstract the idea for the factorization simply comes from thinking about how we calculate $ref(A)$. To obtain $ref(A)$ one begins with A and then performs row operations until we have reduced the matrix to the form $ref(A)$. Each row operation can be implemented by a corresponding left multiplication by an elementary matrix so symbollically we can summarize the forward pass by the following equation:

$$\boxed{E_k E_{k-1} \cdots E_3 E_2 E_1 A = ref(A)}$$

The matrix $ref(A)$ has pivot positions with a nonzero number $\star$ in each such entry. Moreover, by construction there are no nonzero entries below the pivot positions hence $ref(A)$ is an upper

[1]mostly dead by now sad to say.

triangular matrix. Generically the pattern is something like

$$ref(A) = \begin{bmatrix} \star & * & * & * \\ 0 & 0 & \star & * \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

where again $\star \neq 0$ but $*$'s can be anything. Solve the boxed equation for A,

$$\boxed{A = \underbrace{E_1^{-1}E_2^{-1}E_3^{-1}\cdots E_{k-1}^{-1}E_k^{-1}}_{maybe\ L\ ?}\underbrace{ref(A)}_{U}}$$

The inverse of elementary matrices are easily obtained and the product of those matrices is easily assembled if we just keep track of the row reduction to produce $ref(A)$. Let's see how this works out for a few examples.

Example 4.3.1. *Let me modify the row reduction we studied in Example 1.2.3,*

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 3 & -3 & 0 \\ 2 & -2 & -3 \end{bmatrix} \xrightarrow[\text{(this is } E_1)]{r_2 - 3r_1 \to r_2} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & -3 \\ 2 & -2 & -3 \end{bmatrix} \xrightarrow[\text{(this is } E_2)]{r_3 - 2r_1 \to r_3}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & -5 \end{bmatrix} \xrightarrow[\text{(this is } E_3)]{r_3 - \frac{5}{3}r_2 \to r_3} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix} = U$$

Recall that elementary matrices are obtained by performing the corresponding operations on the identity matrix. We have $U = E_3E_2E_1A$, in particular $E_1 = \{I : r_2 - 3r_1 \to r_2\}$. Observe that $A = E_1^{-1}E_2^{-1}E_3^{-1}U$ and calculate the product $E_1^{-1}E_2^{-1}E_3^{-1}$ as follows: [2]

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow[\text{(this is } E_3^{-1})]{r_3 + \frac{5}{3}r_2 \to r_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{5}{3} & 1 \end{bmatrix} \xrightarrow{r_3 + 2r_1 \to r_3}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & \frac{5}{3} & 1 \end{bmatrix} \xrightarrow{r_2 + 3r_1 \to r_2} \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & \frac{5}{3} & 1 \end{bmatrix} = L$$

At the end of this section I'll return to this example once more and streamline the calculation. I'm trying to explain why the later algorithm works in detail to begin. The reason we are doing this and not just the algorithm at the end of the section is that you still need to think more about elementary

[2] note the E_3^{-1} goes first since $E_1^{-1}E_2^{-1}E_3^{-1} = E_1^{-1}E_2^{-1}E_3^{-1}I$, we have to multiply I on left to interpret elementary matrices as row operations, if I was to multiply on the right then it does column operations instead... anyway, this is how to quickly calculate the product of elementary matrices. If for some reason this is confusing then perhaps you might try writing down the 3×3 matrices for each elementary matrix $E_1^{-1}, E_2^{-1}, E_3^{-1}$ then explicitly multiply these out. I prefer to do a few row operations on the identity matrix instead

matrices and this is a pretty good mathematical laboratory to test things out. We find A is factored into a lower and upper triangular matrix:

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & \frac{5}{3} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}}_{U}.$$

Example 4.3.2. *Find an LU-decomposition of A given below (if possible).*

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & 3 & 4 & 5 \\ 2 & 2 & 2 & 2 \end{bmatrix} \xrightarrow{r_2 - 2r_1 \to r_2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 2 & 2 & 2 & 2 \end{bmatrix} \xrightarrow{r_3 - 2r_1 \to r_3}$$

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U$$

In this example, we have $U = E_2E_1A$ hence $A = E_1^{-1}E_2^{-1}U$ and we can calculate the product $E_1^{-1}E_2^{-1}$ as follows:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_3 + 2r_1 \to r_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \xrightarrow{r_2 + 2r_1 \to r_2} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = L$$

We find A factors as follows:

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{U}$$

Notice that in both of the last examples the L was really obtained by taking the identity matrix and inserting a couple numbers below the diagonal. In both cases those numbers were linked to the row operations performed in the forward pass. Keep this in mind for the end of the section.

If we can reduce A to an upper triangular matrix U using only row additions as in the last two examples then it seems entirely plausible that we will be able to find an LU-decomposition for A. However, in the next example we'll see that row-interchanges spoils the simplicity of the method. Let's see how:

Example 4.3.3. *In Example 1.2.2 we needed to use row interchanges to reduce the matrix. For that reason I chose to study it again here.*

$$A = \begin{bmatrix} 1 & 2 & -3 & 1 \\ 2 & 4 & 0 & 7 \\ -1 & 3 & 2 & 0 \end{bmatrix} \xrightarrow{r_2 - 2r_1 \to r_2} \begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 0 & 6 & 5 \\ -1 & 3 & 2 & 0 \end{bmatrix} \xrightarrow{r_3 + r_1 \to r_3}$$

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 0 & 6 & 5 \\ 0 & 5 & -1 & 1 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 5 & -1 & 1 \\ 0 & 0 & 6 & 5 \end{bmatrix} = U$$

We have $U = E_3E_2E_1A$ hence $A = E_1^{-1}E_2^{-1}E_3^{-1}U$ and we can calculate the product $E_1^{-1}E_2^{-1}E_3^{-1}$ as follows:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \xrightarrow{r_3 - r_1 \to r_3}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \xrightarrow{r_2 + 2r_1 \to r_2} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} = PL$$

I have inserted a "P" in front of the L since the matrix above is not lower triangular. However, if we go one step further and let $r_2 \leftrightarrow r_3$ then we will obtain a lower triangular matrix:

$$PL = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} = L$$

Therefore, we find that $E_1^{-1}E_2^{-1}E_3^{-1} = PL$ where L is as above and $P = E_{2\leftrightarrow 3}$. This means that A has a modified LU-decomposition. Some mathemticians call it a PLU-decomposition,

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{P} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 5 & -1 & 1 \\ 0 & 0 & 6 & 5 \end{bmatrix}}_{U} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}}_{PL} \underbrace{\begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 5 & -1 & 1 \\ 0 & 0 & 6 & 5 \end{bmatrix}}_{U}.$$

Since permutation matrices all satisfy the condition $P^2 = I$ the existence of a PLU-decomposition for A naturally suggests that $PA = LU$. Therefore, even when a LU decomposition is not available we can just flip a few rows to find a LU-decomposable matrix. This is a useful observation because it means that the slick algorithms developed for LU-decompositions apply to all matrices with just a little extra fine print. We'll examine how the LU-decomposition allows efficient solution of the problem $Ax = b$ at the conclusion of this section.

As I have hinted at several times, if you examine the calculation of the LU-decomposition carefully you'll see certain patterns. If no permutations are needed then whenever we make the row operation

$r_j + \lambda r_k \to r_j$ it inevitably places a $-\lambda$ in the jk-position of L. Basically we just need to keep track of the λ-multipliers from each row operation. Let me do our first example in a slick notation that avoids explicit stand-alone computation of L

Example 4.3.4.

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 3 & -3 & 0 \\ 2 & -2 & -3 \end{bmatrix} \xrightarrow{r_2 - 3r_1 \to r_2} \begin{bmatrix} 1 & -1 & 1 \\ (3) & 0 & -3 \\ 2 & -2 & -3 \end{bmatrix} \xrightarrow{r_3 - 2r_1 \to r_3}$$

$$\begin{bmatrix} 1 & -1 & 1 \\ (3) & 0 & -3 \\ (2) & 0 & -5 \end{bmatrix} \xrightarrow{r_3 - \frac{5}{3}r_2 \to r_3} \begin{bmatrix} 1 & -1 & 1 \\ (3) & 0 & -3 \\ (2) & (\frac{5}{3}) & 0 \end{bmatrix} = U$$

The parenthetical entries are deleted to obtain U and they are inserted into the identity matrix to obtain the product $E_3^{-1}E_2^{-1}E_1^{-1}$ as follows:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & \frac{5}{3} & 1 \end{bmatrix} \qquad \text{and} \qquad U = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}.$$

Which is precisely what we found before.

Example 4.3.5. *Returning to our example for which $A = PLU$ let's try the slick notation and see if it still works.*

$$A = \begin{bmatrix} 1 & 2 & -3 & 1 \\ 2 & 4 & 0 & 7 \\ -1 & 3 & 2 & 0 \end{bmatrix} \xrightarrow{r_2 - 2r_1 \to r_2} \begin{bmatrix} 1 & 2 & -3 & 1 \\ (2) & 0 & 6 & 5 \\ -1 & 3 & 2 & 0 \end{bmatrix} \xrightarrow{r_3 + r_1 \to r_3}$$

$$\begin{bmatrix} 1 & 2 & -3 & 1 \\ (2) & 0 & 6 & 5 \\ (-1) & 5 & -1 & 1 \end{bmatrix} \xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 1 & 2 & -3 & 1 \\ (-1) & 5 & -1 & 1 \\ (2) & 0 & 6 & 5 \end{bmatrix} = U$$

We find if we remove the parenthetical entries from U and ajoing them to I then it gives back the matrix L we found previously:

$$U = \begin{bmatrix} 1 & 2 & -3 & 1 \\ 0 & 5 & -1 & 1 \\ 0 & 0 & 6 & 5 \end{bmatrix} \qquad L = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

The matrices above give us the LU-decomposition of PA where P is precisely the permutation we encountered in the calculation of U.

Remark 4.3.6.

I hope these examples are sufficient to exhibit the method. If we insist that L has units on the diagonal then I believe the factorization we have calculated is unique provided the matrix A is invertible. Uniqueness aside the application of the factorization to ladder networks is fascinating. Lay explains how the U-factor corresponds to a series circuit whereas a L-factor corresponds to a shunt circuit. The problem of finding an LU-decomposition for a given transfer matrix amounts to finding the necessary shunt and series circuits which in tandem will produce the desired transfer characteristic. We study the mathematical application of LU-decompositions in this course.

4.3.1 application of LU factorization to equation solving

Suppose we wish to solve $Ax = b$ and we are given an LU-decomposition of A. This means that we wish to solve $LUx = b$. Define $y = Ux$ and note that we then have two separate problems to solve:

$$\boxed{Ax = b, \;\; A = LU \qquad \Leftrightarrow \qquad \begin{array}{l} (1.)\ y = Ux \\ (2.)\ Ly = b \end{array}.}$$

It's easy to solve (2.) and then (1.).

Example 4.3.7. *Solve $Ax = b$ given that*

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 3 & -3 & 0 \\ 2 & -2 & -3 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & \frac{5}{3} & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}}_{U} \qquad \text{and} \qquad b = \begin{bmatrix} 3 \\ 1 \\ \frac{200}{3} \end{bmatrix}.$$

Solve $Ly = b$ by **forward substitution**

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & \frac{5}{3} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ \frac{200}{3} \end{bmatrix} \;\Rightarrow\; \begin{array}{l} y_1 = 3, \\ 9 + y_2 = 1 \;\Rightarrow\; y_2 = -8, \\ 6 - \frac{40}{3} + y_3 = \frac{200}{3} \;\Rightarrow\; y_3 = 0. \end{array}$$

Then solve $Ux = y$ by **back substitution**

$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -8 \\ 0 \end{bmatrix} \;\Rightarrow\; \begin{array}{l} -3x_3 = -8, \;\Rightarrow\; x_3 = 8/3, \\ x_1 + x_2 + 8/3 = 3 \;\Rightarrow\; x_1 = -1/3 - x_2. \end{array}$$

We find that $Ax = b$ has solutions of the form $(-\frac{1}{3} - t, t, \frac{8}{3})$ for $t \in \mathbb{R}$.

Note all the possibilities we encountered in previous work are still possible here. A different choice of b could make $Ax = b$ inconsistent. On the other hand, no choice of b will force a unique solution for the A considered here. In any event, it should be clear enough that forward/back substitution will provide a speedy solution to the problem $Ax = b$.

4.4 applications

Definition 4.4.1.

Let $P \in \mathbb{R}^{\,n\times n}$ with $P_{ij} \geq 0$ for all i, j. If the sum of the entries in any column of P is one then we say P is a stochastic matrix.

Example 4.4.2. Stochastic Matrix: *A medical researcher*[3] *is studying the spread of a virus in 1000 lab. mice. During any given week it's estimated that there is an 80% probability that a mouse will overcome the virus, and during the same week there is an 10% likelyhood a healthy mouse will become infected. Suppose 100 mice are infected to start, (a.) how many sick next week? (b.) how many sick in 2 weeks ? (c.) after many many weeks what is the steady state solution?*

$$I_k = \text{infected mice at beginning of week } k \qquad N_k = \text{noninfected mice at beginning of week } k \qquad P = \begin{bmatrix} 0.2 & 0.1 \\ 0.8 & 0.9 \end{bmatrix}$$

We can study the evolution of the system through successive weeks by multiply the state-vector $X_k = (I_k, N_k)$ *by the probability transition matrix* P *given above. Notice we are given that* $X_1 = (100, 900)$*. Calculate then,*

$$X_2 = \begin{bmatrix} 0.2 & 0.1 \\ 0.8 & 0.9 \end{bmatrix} \begin{bmatrix} 100 \\ 900 \end{bmatrix} = \begin{bmatrix} 110 \\ 890 \end{bmatrix}$$

After one week there are 110 infected mice Continuing to the next week,

$$X_3 = \begin{bmatrix} 0.2 & 0.1 \\ 0.8 & 0.9 \end{bmatrix} \begin{bmatrix} 110 \\ 890 \end{bmatrix} = \begin{bmatrix} 111 \\ 889 \end{bmatrix}$$

After two weeks we have 111 mice infected. What happens as $k \to \infty$*? Generally we have* $X_k = PX_{k-1}$*. Note that as* k *gets large there is little difference between* k *and* $k-1$*, in the limit they both tend to infinity. We define the steady-state solution to be* $X^* = \lim_{k\to\infty} X_k$*. Taking the limit of* $X_k = PX_{k-1}$ *as* $k \to \infty$ *we obtain the requirement* $X^* = PX^*$*. In other words, the steady state solution is found from solving* $(P - I)X^* = 0$*. For the example considered here we find,*

$$(P-I)X^* = \begin{bmatrix} -0.8 & 0.1 \\ 0.8 & -0.1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0 \qquad v = 8u \qquad X^* = \begin{bmatrix} u \\ 8u \end{bmatrix}$$

However, by conservation of mice, $u + v = 1000$ *hence* $9u = 1000$ *and* $u = 111.\bar{1}1$ *thus the steady state can be shown to be* $X^* = (111.\bar{1}1, 888.\bar{8}8)$

[3]this example and most of the other applied examples in these notes are borrowed from my undergraduate linear algebra course taught from Larson's text by Dr. Terry Anderson of Appalachian State University

Example 4.4.3. Diagonal matrices are nice: *Suppose that demand for doorknobs halves every week while the demand for yo-yos it cut to 1/3 of the previous week's demand every week due to an amazingly bad advertising campaign*[4]. *At the beginning there is demand for* 2 *doorknobs and* 5 *yo-yos.*

$$D_k = \text{demand for doorknobs at beginning of week } k \qquad P = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$$
$$Y_k = \text{demand for yo-yos at beginning of week } k$$

We can study the evolution of the system through successive weeks by multiply the state-vector $X_k = (D_k, Y_k)$ *by the transition matrix* P *given above. Notice we are given that* $X_1 = (2, 5)$. *Calculate then,*

$$X_2 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 5/3 \end{bmatrix}$$

Notice that we can actually calculate the k-th state vector as follows:

$$X_k = P^k X_1 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}^k \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2^{-k} & 0 \\ 0 & 3^{-k} \end{bmatrix}^k \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2^{-k+1} \\ 5(3^{-k}) \end{bmatrix}$$

Therefore, assuming this silly model holds for 100 weeks, we can calculate the 100*-the step in the process easily,*

$$X_{100} = P^{100} X_1 = \begin{bmatrix} 2^{-101} \\ 5(3^{-100}) \end{bmatrix}$$

Notice that for this example the analogue of X^* *is the zero vector since as* $k \to \infty$ *we find* X_k *has components which both go to zero.*

Example 4.4.4. Naive encryption: *in Example 4.1.14 we found observed that the matrix* A *has inverse matrix* A^{-1} *where:*

$$A = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 0 & -1 \\ 6 & 2 & 3 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} -2 & -3 & -1 \\ -3 & -3 & -1 \\ -2 & -4 & -1 \end{bmatrix}.$$

We use the alphabet code

$$A = 1,\ B = 2,\ C = 3,\ \ldots,\ Y = 25,\ Z = 26$$

and a space is encoded by 0. *The words are parsed into row vectors of length 3 then we multiply them by* A *on the right;* $[decoded]A = [coded]$. *Suppose we are given the string, already encoded by* A

$$[9, -1, -9], [38, -19, -19], [28, -9, -19], [-80, 25, 41], [-64, 21, 31], [-7, 4, 7].$$

Find the hidden message by undoing the multiplication by A. *Simply multiply by* A^{-1} *on the right,*

$$[9, -1, -9]A^{-1},\ [38, -19, -19]A^{-1},\ [28, -9, -19]A^{-1},$$

[4]insert your own more interesting set of quantities that doubles/halves or triples during some regular interval of time

$$[-80, 25, 41]A^{-1}, \ [-64, 21, 31]A^{-1}, \ [-7, 4, 7]A^{-1}$$

This yields,

$$[19, 19, 0], [9, 19, 0], [3, 1, 14], [3, 5, 12], [12, 5, 4]$$

which reads C*LASS IS CANCELLED* [5].

If you enjoy this feel free to peruse my Math 121 notes, I have additional examples of this naive encryption. I say it's naive since real encryption has much greater sophistication by this time. There are many other applications. A parial list: matrix multiplication of the input by the transfer matrix gives the output signal for linear systems in electrical engineering. There are coincidence matrices, permutation matrices, Leslie matrices, tri-banded matrices, shear transformation matrices, matrices to model an affine transformation of computer graphics... My goal in this section is simply to show you a few simple applications and also to invite you to study more as your interests guide you. One nice source of applications is found at the end of Anton and Rorres' *Elementary Linear Algebra: applications version.* They have twenty interesting applications you can study if you need additional motivation as to the applicability of linear algebra.

Remark 4.4.5.

Matrix multiplication and the composition of linear operators is the heart of the chain rule in multivariate calculus. The derivative of a function $f : \mathbb{R}^n \to \mathbb{R}^m$ at a point $p \in \mathbb{R}^n$ gives the best linear approximation to f in the sense that

$$L_f(p + h) = f(p) + D_pf(h) \approxeq f(p + h)$$

if $h \in \mathbb{R}^n$ is close to the zero vector; the graph of L_f gives the tangent line or plane or hypersurface depending on the values of m, n. The so-called Frechet derivative is D_pf, it is a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$. The simplest case is $f : \mathbb{R} \to \mathbb{R}$ where $D_pf(h) = f'(p)h$ and you should recognize $L_f(p+h) = f(p) + f'(p)h$ as the function whose graph is the tangent line, perhaps $L_f(x) = f(p) + f'(p)(x - p)$ is easier to see but it's the same just set $p + h = x$. Given two functions, say $f : \mathbb{R}^n \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^p$ then it can be shown that $D(g \circ f) = Dg \circ Df$. In turn, the matrix of $D(g \circ f)$ is simply obtain by multiplying the matrices of Dg and Df. The matrix of the Frechet derivative is called the Jacobian matrix. The determinant of the Jacobian matrix plays an important role in changing variables for multiple integrals. It is likely we would cover this discussion in some depth in the Advanced Calculus course, while linear algebra is not a pre-req, it sure would be nice if you had it. Linear is truly foundational for most interesting math.

[5] Larson's pg. 100-102 # 22

4.5 conclusions

The theorem that follows here collects the various ideas we have discussed concerning an $n \times n$ matrix and invertibility and solutions of $Ax = b$.

Theorem 4.5.1.

Let A be a real $n \times n$ matrix then the following are equivalent:

(a.) A is invertible,

(b.) $rref[A|0] = [I|0]$ where $0 \in \mathbb{R}^{n\times 1}$,

(c.) $Ax = 0$ iff $x = 0$,

(d.) A is the product of elementary matrices,

(e.) there exists $B \in \mathbb{R}^{n\times n}$ such that $AB = I$,

(f.) there exists $B \in \mathbb{R}^{n\times n}$ such that $BA = I$,

(g.) $rref[A] = I$,

(h.) $rref[A|b] = [I|x]$ for an $x \in \mathbb{R}^{n\times 1}$,

(i.) $Ax = b$ is consistent for every $b \in \mathbb{R}^{n\times 1}$,

(j.) $Ax = b$ has exactly one solution for every $b \in \mathbb{R}^{n\times 1}$,

(k.) A^T is invertible.

These are in no particular order. If you examine the arguments in this chapter you'll find we've proved most of this theorem. What did I miss? [6]

[6] teaching moment or me trying to get you to do my job, you be the judge.

Chapter 5

determinants

I should warn you there are some difficult calculations in this Chapter. However, the good news is these are primarily to justify the various properties of the determinant. The determinant of a square matrix is simply a number which contains lots of useful information. We will conclude this Chapter with a discussion of what the determinant says about systems of equations. There are a lot of different ways to introduce the determinant, my approach is rooted in my love of index calculations from physics. A pure mathematician would likely take another approach (mine is better). Geometrically, determinants are used to capture the idea of an oriented volume. I illustrate this with several examples before we get too deep into the more esoteric calculations.

5.1 determinants and geometry

The determinant of a square matrix can be defined by the following formulas. I'll give the general formula in the next section, but more often than not the formulas given here are more than enough. Well, this one is just silly:

$$\det a = a.$$

Then the 2×2 case is perhaps more familar,

$$\det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = ad - bc.$$

we've seen this before somewhere. Then the 3×3 formula is:

$$\det \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} = a \cdot \det \begin{pmatrix} e & f \\ h & i \end{pmatrix} - b \cdot \det \begin{pmatrix} d & f \\ g & i \end{pmatrix} + c \cdot \det \begin{pmatrix} d & e \\ g & h \end{pmatrix}$$

and finally the 4×4 determinant is given by

$$\det \begin{pmatrix} a & b & c & d \\ e & f & g & h \\ i & j & k & l \\ m & n & o & p \end{pmatrix} = a \cdot \det \begin{pmatrix} f & g & h \\ j & k & l \\ n & o & p \end{pmatrix} - b \cdot \det \begin{pmatrix} e & g & h \\ i & k & l \\ m & o & p \end{pmatrix} \tag{5.1}$$

$$+ c \cdot \det \begin{pmatrix} e & f & h \\ i & j & l \\ m & n & p \end{pmatrix} - d \cdot \det \begin{pmatrix} e & f & g \\ i & j & k \\ m & n & o \end{pmatrix} \tag{5.2}$$

What do these formulas have to do with geometry?

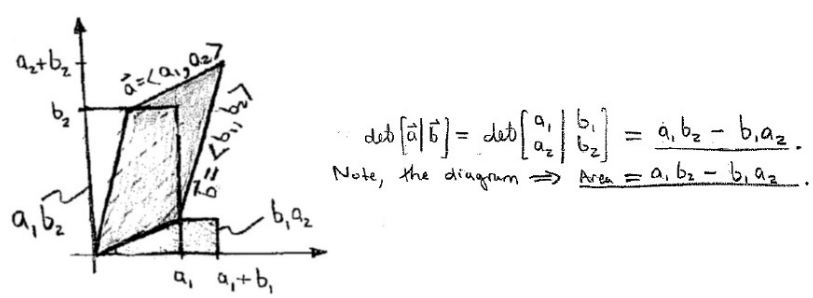
Example 5.1.1. *Consider the vectors $< l, 0 >$ and $< 0, w >$. They make two sides of a rectangle with length l and width w. Notice*

$$det \begin{bmatrix} l & 0 \\ 0 & w \end{bmatrix} = lw.$$

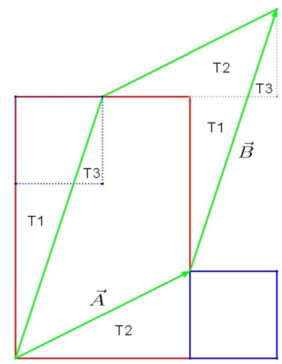
In contrast,

$$det \begin{bmatrix} 0 & w \\ l & 0 \end{bmatrix} = -lw.$$

Interestingly this works for parallellograms with sides $< a, b >$ and $< c, d >$ the area is given by $\pm det \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$.



Maybe you can see it better in the diagram below: the point is that triangles $T1$ and $T2$ match nicely but the $T3$ is included in the red rectangle but is excluded from the green parallelogram. The area of the red rectangle A_1B_2 less the area of the blue square A_2B_1 is precisely the area of the green parallelogram.



If you've taken calculus III the you may have learned that a parallelogram with sides $\vec{A}, \vec{B}$ can be parametrized by $\vec{r}(u,v) = u\vec{A} + v\vec{B}$. We have $\vec{A} = (a,b,0)$ and $\vec{B} = (c,d,0)$ if you view the parallelogram from a three dimensional perspective. Moreover,

$$\vec{A} \times \vec{B} = det \begin{bmatrix} e_1 & e_2 & e_3 \\ a & b & 0 \\ c & d & 0 \end{bmatrix} = (ad - bc)e_3.$$

The sign of $ad - bc$ indicates the **orientation** *of the paralellogram. If the paralellogram lives in the xy-plane then it has an up-ward pointing normal if the determinant is positive whereas it has a downward pointing normal if the determinant is negative.*

Example 5.1.2. *If we look at a three dimensional box with vectors $\vec{A}, \vec{B}, \vec{C}$ pointing along three edges with from a common corner then it can be shown that the volume V is given by the determinant*

$$V = \pm det\left[\vec{A}|\vec{B}|\vec{C}\right]$$

Of course it's easy to see that $V = lwh$ if the sides have length l, width w and height h. However, this formula is more general than that, it also holds if the vectors lie along a paralell piped. Again the sign of the determinant has to do with the **orientation** *of the box. If the determinant is positive then that means that the set of vectors $\{\vec{A}, \vec{B}, \vec{C}\}$ forms a righted-handed set of vectors. In terms of calculus III, $\vec{(C)}$ and $\vec{A} \times \vec{B}$ both point off the same side of the plane containing $\vec{A}$ and $\vec{B}$; the ordering of the vectors is roughly consistent with the right-hand rule. If the determinant of the three vectors is negative then they will be consistent with the (inferior and evil) left-hand rule. I say "roughly" because $\vec{A} \times \vec{B}$ need not be parallel with $\vec{C}$. The sign of the determinant just reveals if $\vec{C}$ is above or below the plane spanned by $\vec{A}, \vec{B}$.*

If you study the geometry of cross and dot products it is not too hard to see that $V = |\vec{A} \cdot (\vec{B} \times \vec{C})|$. This formula is easy to reproduce,

$$det \begin{bmatrix} A_1 & A_2 & A_3 \\ \hline B_1 & B_2 & B_3 \\ \hline C_1 & C_2 & C_3 \end{bmatrix} = A_1(B_2C_3 - B_3C_2) + A_2(B_1C_3 - B_3C_1) + A_3(B_1C_2 - B_2C_1)$$
$$= \vec{A} \cdot (\vec{B} \times \vec{C}).$$

If you'd like to know more about the geometry of cross products then you should take calculus III and read more than a standard calculus text. It is interesting that the determinant gives formulas for cross products and the so-called "triple product" above.

Example 5.1.3. *To calculate the cross-product of $\vec{A}$ and $\vec{B}$ we can use the heuristic rule*

$$\vec{A} \times \vec{B} = det \begin{bmatrix} e_1 & e_2 & e_3 \\ A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \end{bmatrix}$$

technically this is not a real "determinant" because there are vectors in the top row but numbers in the last two rows.

Example 5.1.4. *The infinitesimal area element for polar coordinate is calculated from the Jacobian:*

$$dS = det \begin{bmatrix} r\sin(\theta) & -r\cos(\theta) \\ \cos(\theta) & \sin(\theta) \end{bmatrix} drd\theta = (r\sin^2(\theta) + r\cos^2(\theta))drd\theta = rdrd\theta$$

Example 5.1.5. *The infinitesimal volume element for cylindrical coordinate is calculated from the Jacobian:*

$$dV = det \begin{bmatrix} r\sin(\theta) & -r\cos(\theta) & 0 \\ \cos(\theta) & \sin(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} drd\theta dz = (r\sin^2(\theta) + r\cos^2(\theta))drd\theta dz = rdrd\theta dz$$

Jacobians are needed to change variables in multiple integrals. The Jacobian[1] is a determinant which measures how a tiny volume is rescaled under a change of coordinates. Each row in the matrix making up the Jacobian is a tangent vector which points along the direction in which a coordinate increases when the remaining coordinates are fixed.

5.2 cofactor expansion for the determinant

The precise definition of the determinant is intrinsically combinatorial. A permutation $\sigma : \mathbb{N}_n \rightarrow \mathbb{N}_n$ is a bijection. Every permutation can be written as a product of an even or odd composition of transpositions. The $sgn(\sigma) = 1$ if σ is formed from an even product of transpositions. The $sgn(\sigma) = -1$ if σ is formed from an odd product of transpositions. The sum below is over all possible permutations,

$$\boxed{det(A) = \sum_{\sigma} sgn(\sigma) A_{1\sigma(1)} A_{2\sigma(2)} \cdots A_{n\sigma(n)}}$$

this provides an explicit definition of the determinant. For example, in the $n = 2$ case we have $\sigma_o(x) = x$ or $\sigma_1(1) = 2, \sigma_1(2) = 1$. The sum over all permutations has just two terms in the $n = 2$ case,

$$det(A) = sgn(\sigma_o) A_{1\sigma_o(1)} A_{2\sigma_o(2)} + sgn(\sigma_1) A_{1\sigma_1(1)} A_{2\sigma_1(2)} = A_{11}A_{22} - A_{12}A_{21}$$

[1]see pages 206-208 of Spence Insel and Friedberg or perhaps my advanced calculus notes where I develop differentiation from a linear algebraic viewpoint.

In the notation $A_{11} = a, A_{12} = b, A_{21} = c, A_{22} = d$ the formula above says $det(A) = ad - bc$.

Pure mathematicians tend to prefer the definition above to the one I am preparing below. I would argue mine has the advantage of not summing over functions. My sums are simply over integers. The calculations I make in the proofs in this Chapter may appear difficult to you, but if you gain a little more experience with index calculations I think you would find them accessible. I will not go over them all in lecture. I would recommend you at least read over them.

Definition 5.2.1.

Let $\epsilon_{i_1 i_2 \dots i_n}$ be defined to be the completely antisymmetric symbol in n-indices. We define $\epsilon_{12\dots n} = 1$ then all other values are generated by demanding the interchange of any two indices is antisymmetric. This is also known as the **Levi-Civita** symbol.

We have nice formulas for the determinant with the help of the Levi-Civita symbol, the following is yet another way of stating the definition for $det(A)$,

$$det(A) = \sum_{i_1, i_2, \dots, i_n} \epsilon_{i_1, i_2, \dots, i_n} A_{1i_1} A_{2i_2} \cdots A_{ni_n}$$

Example 5.2.2. *I prefer this definition. I can actually calculate it faster, for example the $n = 3$ case is pretty quick:*

$$\begin{aligned} det(A) &= \epsilon_{123} A_{11} A_{22} A_{33} + \epsilon_{231} A_{12} A_{23} A_{31} + \epsilon_{312} A_{13} A_{21} A_{32} \\ &+ \epsilon_{321} A_{13} A_{22} A_{31} + \epsilon_{213} A_{12} A_{21} A_{33} + \epsilon_{132} A_{11} A_{23} A_{32} \end{aligned}$$

In principle there are 27 terms above but only these 6 are nontrivial because if any index is repeated the ϵ_{ijk} is zero. The only nontrivial terms are $\epsilon_{123} = \epsilon_{231} = \epsilon_{312} = 1$ and $\epsilon_{321} = \epsilon_{213} = \epsilon_{132} = -1$. Thus,

$$\begin{aligned} det(A) &= A_{11} A_{22} A_{33} + A_{12} A_{23} A_{31} + A_{13} A_{21} A_{32} \\ &- A_{13} A_{22} A_{31} - A_{12} A_{21} A_{33} - A_{11} A_{23} A_{32} \end{aligned}$$

This formula is much closer to the trick-formula for calculating the determinant without using minors. See Anton for the "arrow technique".

The formalism above will be used in all my proofs. I take the Levi-Civita definition as the primary definition for the determinant. All other facts flow from that source. The cofactor expansions of the determinant could also be used as a definition.

Definition 5.2.3.

Let $A = [A_{ij}] \in \mathbb{R}^{\,n \times n}$. The minor of A_{ij} is denoted M_{ij} which is defined to be the determinant of the $\mathbb{R}^{(n-1)\times(n-1)}$ matrix formed by deleting the i-th column and the j-th row of A. The (i, j)-th co-factor of A is $C_{ij} = (-1)^{i+j} M_{ij}$.

Theorem 5.2.4.

The determinant of $A \in \mathbb{R}^{\,n\times n}$ can be calculated from a sum of cofactors either along any row or column;

1. $det(A) = A_{i1}C_{i1} + A_{i2}C_{i2} + \cdots + A_{in}C_{in}$ (i-th row expansion)
2. $det(A) = A_{1j}C_{1j} + A_{2j}C_{2j} + \cdots + A_{nj}C_{nj}$ (j-th column expansion)

Proof: I'll attempt to sketch a proof of (2.) directly from the general definition. Let's try to identify A_{1i_1} with A_{1j} then A_{2i_2} with A_{2j} and so forth, keep in mind that j is a fixed but arbitrary index, it is not summed over.

$$det(A) = \sum_{i_1,i_2,\dots,i_n} \epsilon_{i_1,i_2,\dots,i_n} A_{1i_1}A_{2i_2}\cdots A_{ni_n}$$

$$= \sum_{i_2,\dots,i_n} \epsilon_{j,i_2,\dots,i_n} A_{1j}A_{2i_2}\cdots A_{ni_n} + \sum_{i_1\neq j,i_2,\dots,i_n} \epsilon_{i_1,i_2,\dots,i_n} A_{1i_1}A_{2i_2}\cdots A_{ni_n}$$

$$\begin{aligned} &= \sum_{i_2,\dots,i_n} \epsilon_{j,i_2,\dots,i_n} A_{1j}A_{2i_2}\cdots A_{ni_n} + \sum_{i_1\neq j,i_3,\dots,i_n} \epsilon_{i_1,j,\dots,i_n} A_{1i_1}A_{2j}\cdots A_{ni_n} \\ &\quad + \cdots + \sum_{i_1\neq j,i_2\neq j,\dots,i_{n-1}\neq j} \epsilon_{i_1,i_2,\dots,i_{n-1},j} A_{1i_1}\cdots A_{n-1,i_{n-1}}A_{nj} \\ &\quad + \sum_{i_1\neq j,\dots,i_n\neq j} \epsilon_{i_1,\dots,i_n} A_{1i_1}A_{1i_2}\cdots A_{ni_n} \end{aligned}$$

Consider the summand. If all the indices $i_1, i_2, \dots i_n \neq j$ then there must be at least one repeated index in each list of such indices. Consequently the last sum vanishes since $\epsilon_{i_1,\dots,i_n}$ is zero if any two indices are repeated. We can pull out A_{1j} from the first sum, then A_{2j} from the second sum, and so forth until we eventually pull out A_{nj} out of the last sum.

$$\begin{aligned} det(A) = A_{1j}\Big(\sum_{i_2,\dots,i_n} \epsilon_{j,i_2,\dots,i_n} A_{2i_2}\cdots A_{ni_n}\Big) + A_{2j}\Big(\sum_{i_1\neq j,\dots,i_n} \epsilon_{i_1,j,\dots,i_n} A_{1i_1}\cdots A_{ni_n}\Big) + \cdots \\ + A_{nj}\Big(\sum_{i_1\neq j,i_2\neq j,\dots,j\neq i_{n-1}} \epsilon_{i_1,i_2,\dots,j} A_{1i_1}A_{2i_2}\cdots A_{n-1,i_{n-1}}\Big) \end{aligned}$$

The terms appear different, but in fact there is a hidden symmetry. If any index in the summations above takes the value j then the Levi-Civita symbol with have two j's and hence those terms are zero. Consequently we can just as well take all the sums over all values **except** j. In other words, each sum is a completely antisymmetric sum of products of $n-1$ terms taken from all columns except j. For example, the first term has an antisymmetrized sum of a product of $n-1$ terms not including column j or row 1.Reordering the indices in the Levi-Civita symbol generates a sign of

$(-1)^{1+j}$ thus the first term is simply $A_{1j}C_{1j}$. Likewise the next summand is $A_{2j}C_{2j}$ and so forth until we reach the last term which is $A_{nj}C_{nj}$. In other words,

$$\boxed{det(A) = A_{1j}C_{1j} + A_{2j}C_{2j} + \cdots + A_{nj}C_{nj}}$$

The proof of (1.) is probably similar. We will soon learn that $det(A^T) = det(A)$ thus (2.) $\implies$ (1.). since the j-th row of A^T is the j-th columns of A.

All that remains is to show why $det(A) = det(A^T)$. Recall $(A^T)_{ij} = A_{ji}$ for all i, j, thus

$$\begin{aligned} det(A^T) &= \sum_{i_1,i_2,\dots,i_n} \epsilon_{i_1,i_2,\dots,i_n} (A^T)_{1i_1}(A^T)_{2i_2}\cdots(A^T)_{ni_n} \\ &= \sum_{i_1,i_2,\dots,i_n} \epsilon_{i_1,i_2,\dots,i_n} A_{i_1 1}A_{i_2 2}\cdots A_{i_n n} \\ &= \sum_{i_1,i_2,\dots,i_n} \epsilon_{i_1,i_2,\dots,i_n} A_{1i_1}A_{2i_2}\cdots A_{ni_n} = det(A) \end{aligned}$$

to make the last step one need only see that both sums contain all the same terms just written in a different order. Let me illustrate explicitly how this works in the $n = 3$ case,

$$\begin{aligned} det(A^T) = \; &\epsilon_{123}A_{11}A_{22}A_{33} + \epsilon_{231}A_{21}A_{32}A_{13} + \epsilon_{312}A_{31}A_{12}A_{23} \\ &+ \epsilon_{321}A_{31}A_{22}A_{13} + \epsilon_{213}A_{21}A_{12}A_{33} + \epsilon_{132}A_{11}A_{32}A_{23} \end{aligned}$$

The I write the entries so the column indices go $1, 2, 3$

$$\begin{aligned} det(A^T) = \; &\epsilon_{123}A_{11}A_{22}A_{33} + \epsilon_{231}A_{13}A_{21}A_{32} + \epsilon_{312}A_{12}A_{23}A_{31} \\ &+ \epsilon_{321}A_{13}A_{22}A_{31} + \epsilon_{213}A_{12}A_{21}A_{33} + \epsilon_{132}A_{11}A_{23}A_{32} \end{aligned}$$

But, the indices of the Levi-Civita symbol are not in the right order yet. Fortunately, we have identities such as $\epsilon_{231} = \epsilon_{312}$ which allow us to reorder the indices without introducing any new signs,

$$\begin{aligned} det(A^T) = \; &\epsilon_{123}A_{11}A_{22}A_{33} + \epsilon_{312}A_{13}A_{21}A_{32} + \epsilon_{231}A_{12}A_{23}A_{31} \\ &+ \epsilon_{321}A_{13}A_{22}A_{31} + \epsilon_{213}A_{12}A_{21}A_{33} + \epsilon_{132}A_{11}A_{23}A_{32} \end{aligned}$$

But, these are precisely the terms in $det(A)$ just written in a different order (see Example 5.2.2). Thus $det(A^T) = det(A)$. I leave the details of how to reorder the order n sum to the reader. □

Remark 5.2.5.

The best way to prove things about determinants is likely the wedge product formalism. For a $n \times n$ matrix the $det(A)$ is defined implicitly by the formula $col_1(A) \wedge col_2(A) \wedge \cdots \wedge col_n(A) = det(A)e_1 \wedge e_2 \wedge \cdots \wedge e_n$. One nice place to read more about these things from a purely linear-algebraic perspective is the text *Abstract Linear Algebra* by Morton L. Curtis.

Remark 5.2.6.

Lay's text circumnavigates many of the difficulties I face in this chapter by using the co-factor definition as the definition of the determinant. One place you can also find a serious treatment of determinants is in *Linear Algebra* by Insel, Spence and Friedberg where you'll find the proof of the co-factor expansion is somewhat involved. However, the heart of the proof involves multilinearity. Multilinearity is practically manifest with our Levi-Civita definition. Anywho, a better definition for the determinant is as follows: **the determinant is the alternating, n-multilinear, real valued map such that** $det(I) = 1$. It can be shown this uniquely defines the determinant. All these other things like permutations and the Levi-Civita symbol are just notation.

Example 5.2.7. *I suppose it's about time for an example. Let*

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

I usually calculate by expanding across the top row out of habit,

$$\begin{aligned} det(A) &= 1det\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix} - 2det\begin{bmatrix} 4 & 6 \\ 7 & 9 \end{bmatrix} + 3det\begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix} \\ &= 1(45-48) - 2(36-42) + 3(32-35) \\ &= -3 + 12 - 9 \\ &= 0. \end{aligned}$$

Now, we could also calculate by expanding along the middle row,

$$\begin{aligned} det(A) &= -4det\begin{bmatrix} 2 & 3 \\ 8 & 9 \end{bmatrix} + 5det\begin{bmatrix} 1 & 3 \\ 7 & 9 \end{bmatrix} - 6det\begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix} \\ &= -4(18-24) + 5(9-21) - 6(8-14) \\ &= 24 - 60 + 36 \\ &= 0. \end{aligned}$$

Many other choices are possible, for example expan along the right column,

$$\begin{aligned} det(A) &= 3det\begin{bmatrix} 4 & 5 \\ 7 & 8 \end{bmatrix} - 6det\begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix} + 9det\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \\ &= 3(32-35) - 6(8-14) + 9(5-8) \\ &= -9 + 36 - 27 \\ &= 0. \end{aligned}$$

which is best? Certain matrices might have a row or column of zeros, then it's easiest to expand along that row or column.

Example 5.2.8. *Let's look at an example where we can exploit the co-factor expansion to greatly reduce the difficulty of the calculation. Let*

$$A = \begin{bmatrix} 1 & 2 & 3 & 0 & 4 \\ 0 & 0 & 5 & 0 & 0 \\ 6 & 7 & 8 & 0 & 0 \\ 0 & 9 & 3 & 4 & 0 \\ -1 & -2 & -3 & 0 & 1 \end{bmatrix}$$

Begin by expanding down the 4-th column,

$$det(A) = (-1)^{4+4}M_{44} = 4det \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 5 & 0 \\ 6 & 7 & 8 & 0 \\ -1 & -2 & -3 & 1 \end{bmatrix}$$

Next expand along the 2-row of the remaining determinant,

$$det(A) = (4)(5(-1)^{2+3}M_{23}) = -20det \begin{bmatrix} 1 & 2 & 4 \\ 6 & 7 & 0 \\ -1 & -2 & 1 \end{bmatrix}$$

Finish with the trick for 3×3 determinants, it helps me to write out

$$\left[\begin{array}{ccc|cc} 1 & 2 & 4 & 1 & 2 \\ 6 & 7 & 0 & 6 & 7 \\ -1 & -2 & 1 & -1 & -2 \end{array}\right]$$

then calculate the products of the three down diagonals and the three upward diagonals. Subtract the up-diagonals from the down-diagonals.

$$det(A) = -20(7 + 0 - 48 - (-28) - (0) - (12)) = -20(-25) = 500.$$

5.3 properties of determinants

We're finally getting towards the good part.

Proposition 5.3.1.

Let $A \in \mathbb{R}^{\,n\times n}$,

1. $det(A^T) = det(A)$,
2. If there exists j such that $row_j(A) = 0$ then $det(A) = 0$,
3. If there exists j such that $col_j(A) = 0$ then $det(A) = 0$,
4. $det[A_1|A_2|\cdots|aA_k + bB_k|\cdots A_n] = adet[A_1|\cdots|A_k|\cdots|A_n] + bdet[A_1|\cdots|B_k|\cdots|A_n]$,
5. $det(kA) = k^n det(A)$
6. if $B = \{A : r_k \leftrightarrow r_j\}$ then $det(B) = -det(A)$,
7. if $B = \{A : r_k + ar_j \to r_k\}$ then $det(B) = det(A)$,
8. if $row_i(A) = krow_j(A)$ for $i \neq j$ then $det(A) = 0$

where I mean to denote $r_k \leftrightarrow r_j$ as the row interchange and $r_k + ar_j \to r_k$ as a column addition and I assume $k < j$.

Proof: we already proved (1.) in the proof of the cofactor expansion Theorem 5.2.4. The proof of (2.) and (3.) follows immediately from the cofactor expansion if we expand along the zero row or column. The proof of (4.) is not hard given our Levi-Civita defintion, let

$$C = [A_1|A_2|\cdots|aA_k + bB_k|\cdots|A_n]$$

Calculate from the definition,

$$\begin{aligned}
det(C) &= \sum_{i_1,i_2,\dots,i_n} \epsilon_{i_1,i_2,\dots,i_n} C_{1i_1}\cdots C_{ki_k}\cdots C_{ni_n} \\
&= \sum_{i_1,i_2,\dots,i_n} \epsilon_{i_1,i_2,\dots,i_n} A_{1i_1}\cdots (aA_{ki_k} + bB_{ki_k})\cdots A_{ni_n} \\
&= a\Big(\sum_{i_1,i_2,\dots,i_n} \epsilon_{i_1,i_2,\dots,i_n} A_{1i_1}\cdots A_{ki_k}\cdots A_{ni_n}\Big) \\
&\qquad + b\Big(\sum_{i_1,i_2,\dots,i_n} \epsilon_{i_1,i_2,\dots,i_n} A_{1i_1}\cdots B_{ki_k}\cdots A_{ni_n}\Big) \\
&= a\,det[A_1|A_2|\cdots|A_k|\cdots|A_n] + b\,det[A_1|A_2|\cdots|B_k|\cdots|A_n].
\end{aligned}$$

by the way,the property above is called multilinearity. The proof of (5.) is similar,

$$\begin{aligned} det(kA) &= \sum_{i_1,i_2,\dots,i_n} \epsilon_{i_1,i_2,\dots,i_n} kA_{1i_1} kA_{2i_2} \cdots kA_{ni_n} \\ &= k^n \sum_{i_1,i_2,\dots,i_n} \epsilon_{i_1,i_2,\dots,i_n} A_{1i_1} A_{2i_2} \cdots A_{ni_n} \\ &= k^n \, det(A) \end{aligned}$$

Let B be as in (6.), this means that $col_k(B) = col_j(A)$ and vice-versa,

$$\begin{aligned} det(B) &= \sum_{i_1,i_2,\dots,i_n} \epsilon_{i_1,\dots,i_k,\dots,i_j,\dots,i_n} A_{1i_1} \cdots A_{ji_k} \cdots A_{ki_j} \cdots A_{ni_n} \\ &= \sum_{i_1,i_2,\dots,i_n} -\epsilon_{i_1,\dots,i_j,\dots,i_k,\dots,i_n} A_{1i_1} \cdots A_{ji_k} \cdots A_{ki_j} \cdots A_{ni_n} \\ &= -det(A) \end{aligned}$$

where the minus sign came from interchanging the indices i_j and i_k.

To prove (7.) let us define B as in the Proposition: let $row_k(B) = row_k(A) + arow_j(A)$ and $row_i(B) = row_i(A)$ for $i \neq k$. This means that $B_{kl} = A_{kl} + aA_{jl}$ and $B_{il} = A_{il}$ for each l. Consequently,

$$\begin{aligned} det(B) &= \sum_{i_1,i_2,\dots,i_n} \epsilon_{i_1,\dots,i_k,\dots,i_n} A_{1i_1} \cdots (A_{ki_k} + aA_{ji_k}) \cdots A_{ni_n} \\ &= \sum_{i_1,i_2,\dots,i_n} \epsilon_{i_1,\dots,i_n} A_{1i_1} \cdots A_{ki_k} \cdots A_{ni_n} \\ &\quad + a\Bigg(\sum_{i_1,i_2,\dots,i_n} \epsilon_{i_1,\dots,i_j,\dots,i_k,\dots,i_n} A_{1i_1} \cdots A_{j,i_j} \cdots A_{ji_k} \cdots A_{ni_n} \Bigg) \\ &= \sum_{i_1,i_2,\dots,i_n} \epsilon_{i_1,\dots,i_n} A_{1i_1} \cdots A_{ki_k} \cdots A_{ni_n} \\ &= det(A). \end{aligned}$$

The term in parenthesis vanishes because it has the sum of an antisymmetric tensor in i_j, i_k against a symmetric tensor in i_j, i_k. Here is the pattern, suppose $S_{ij} = S_{ji}$ and $T_{ij} = -T_{ji}$ for all i, j then consider

$$\begin{aligned} \sum_i \sum_j S_{ij} T_{ij} &= \sum_j \sum_i S_{ji} T_{ji} && \text{switched indices} \\ &= \sum_j \sum_i -S_{ij} T_{ij} && \text{used sym. and antisym.} \\ &= -\sum_i \sum_j S_{ij} T_{ij} && \text{interchanged sums.} \end{aligned}$$

thus we have $\sum S_{ij}T_{ij} = -\sum S_{ij}T_{ij}$ which indicates the sum is zero. We can use the same argument on the pair of indices i_j, i_k in the expression since $A_{ji_j}A_{ji_k}$ is symmetric in i_j, i_k whereas the Levi-Civita symbol is antisymmetric in i_j, i_k.

We get (8.) as an easy consequence of (2.) and (7.), just subtract one row from the other so that we get a row of zeros. $\square$

Proposition 5.3.2.

The determinant of a diagonal matrix is the product of the diagonal entries.

Proof: Use multilinearity on each row,

$$det \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} = d_1\, det \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} = \cdots = d_1 d_2 \cdots d_n det \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

Thus $det(D) = d_1 d_2 \cdots d_n$ as claimed. $\square$

Proposition 5.3.3.

Let L be a lower triangular square matric and U be an upper triangular square matrix.

1. $det(L) = L_{11}L_{22}\cdots L_{nn}$
2. $det(U) = U_{11}U_{22}\cdots U_{nn}$

Proof: I'll illustrate the proof of (2.) for the 3×3 case. We use the co-factor expansion across the first column of the matrix to begin,

$$det \begin{bmatrix} U_{11} & U_{12} & U_{13} \\ 0 & U_{22} & U_{23} \\ 0 & 0 & U_{33} \end{bmatrix} = A_{11} det \begin{bmatrix} U_{22} & U_{23} \\ 0 & U_{33} \end{bmatrix} = U_{11}U_{22}U_{33}$$

The proof of the $n \times n$ case is essentially the same. For (1.) use the co-factor expansion across the top row of L, to get $det(L) = L_{11}C_{11}$. Not the submatrix for calculating C_{11} is again has a row of zeros across the top. We calculate $C_{11} = L_{22}C_{22}$. This continues all the way down the diagonal. We find $det(L) = L_{11}L_{22}\cdots L_{nn}$. $\square$

Proposition 5.3.4.

Let $A \in \mathbb{R}^{n \times n}$ and $k \neq 0 \in \mathbb{R}$,

1. $det(E_{r_i \leftrightarrow r_j}) = -1$,
2. $det(E_{kr_i \rightarrow r_i}) = k$,
3. $det(E_{r_i + br_j \rightarrow r_i}) = 1$,
4. for any square matrix B and elementary matrix E, $det(EB) = det(E)det(B)$
5. if $E_1, E_2, \dots, E_k$ are elementary then $det(E_1 E_2 \cdots E_k) = det(E_1)det(E_2) \cdots det(E_k)$

Proof: Proposition 5.6.2 shows us that $det(I) = 1$ since $I^{-1} = I$ (there are many easier ways to show that). Note then that $E_{r_i \leftrightarrow r_j}$ is a row-swap of the identity matrix thus by Proposition 5.3.1 we find $det(E_{r_i \leftrightarrow r_j}) = -1$. To prove (2.) we use multilinearity from Proposition 5.3.1. For (3.) we use multilinearity again to show that:

$$det(E_{r_i + br_j \rightarrow r_i}) = det(I) + bdet(E_{ij})$$

Again $det(I) = 1$ and since the unit matrix E_{ij} has a row of zeros we know by Proposition 5.3.1 $det(E_{ij}) = 0$.

To prove (5.) we use Proposition 5.3.1 multiple times in the arguments below. Let $B \in \mathbb{R}^{n \times n}$ and suppose E is an elementary matrix. If E is multiplication of a row by k then $det(E) = k$ from (2.). Also EB is the matrix B with some row multiplied by k. Use multilinearity to see that $det(EB) = kdet(B)$. Thus $det(EB) = det(E)det(B)$. If E is a row interchange then EB is B with a row swap thus $det(EB) = -det(B)$ and $det(E) = -1$ thus we again find $det(EB) = det(E)det(B)$. Finally, if E is a row addition then EB is B with a row addition and $det(EB) = det(B)$ and $det(E) = 1$ hence $det(EB) = det(E)det(B)$. Notice that (6.) follows by repeated application of (5.). $\square$

Proposition 5.3.5.

A square matrix A is invertible iff $det(A) \neq 0$.

Proof: recall there exist elementary matrices $E_1, E_2, \dots, E_k$ such that $rref(A) = E_1 E_2 \cdots E_k A$. Thus $det(rref(A)) = det(E_1)det(E_2) \cdots det(E_k)det(A)$. Either $det(rref(A)) = 0$ and $det(A) = 0$ or they are both nonzero.

Suppose A is invertible. Then $Ax = 0$ has a unique solution and thus $rref(A) = I$ hence $det(rref(A)) = 1 \neq 0$ implying $det(A) \neq 0$.

Conversely, suppose $det(A) \neq 0$, then $det(rref(A)) \neq 0$. But this means that $rref(A)$ does not have a row of zeros. It follows $rref(A) = I$. Therefore $A^{-1} = E_1 E_2 \cdots E_k$. $\square$

Proposition 5.3.6.

If $A, B \in \mathbb{R}^{\,n\times n}$ then $det(AB) = det(A)det(B)$.

Proof: If either A or B is not invertible then the reduced row echelon form of the nonivertible matrix will have a row of zeros hence $det(A)det(B) = 0$. Without loss of generality, assume A is not invertible. Note $rref(A) = E_1E_2\cdots E_kA$ hence $E_3{}^{-1}E_2{}^{-1}E_1{}^{-1}rref(A)B = AB$. Notice that $rref(A)B$ will have at least one row of zeros since $rref(A)$ has a row of zeros. Thus $det(E_3{}^{-1}E_2{}^{-1}E_1{}^{-1}rref(A)B) = det(E_3{}^{-1}E_2{}^{-1}E_1{}^{-1})det(rref(A)B) = 0$.

Suppose that both A and B are invertible. Then there exist elementary matrices such that $A = E_1\cdots E_p$ and $B = E_{p+1}\cdots E_{p+q}$ thus

$$\begin{aligned} det(AB) &= det(E_1\cdots E_pE_{p+1}\cdots E_{p+q}) \\ &= det(E_1\cdots E_p)det(E_{p+1}\cdots E_{p+q}) \\ &= det(A)det(B). \end{aligned}$$

We made repeated use of (6.) in Proposition 5.3.4. □

Proposition 5.3.7.

If $A \in \mathbb{R}^{\,n\times n}$ is invertible then $det(A^{-1}) = \frac{1}{det(A)}$.

Proof: If A is invertible then there exists $A^{-1} \in \mathbb{R}^{\,n\times n}$ such that $AA^{-1} = I$. Apply Proposition 5.3.6 to see that

$$det(AA^{-1}) = det(A)det(A^{-1}) = det(I) \;\Rightarrow\; det(A)det(A^{-1}) = 1.$$

Thus, $det(A^{-1}) = 1/det(A)$ □

Many of the properties we used to prove $det(AB) = det(A)det(B)$ are easy to derive if you were simply given the assumption $det(AB) = det(A)det(B)$. When you look at what went into the proof of Proposition 5.3.6 it's not surprising that $det(AB) = det(A)det(B)$ is a powerful formula to know.

Proposition 5.3.8.

If A is block-diagonal with square blocks $A_1, A_2, \dots, A_k$ then

$$det(A) = det(A_1)det(A_2)\cdots det(A_k).$$

Proof: for a 2×2 matrix this is clearly true since a block diagonal matrix is simply a diagonal matrix. In the 3×3 nondiagonal case we have a 2×2 block A_1 paired with a single diagonal entry A_2. Simply apply the cofactor expansion on the row of the diagonal entry to find that $det(A) = A_2det(A_1) = det(A_2)det(A_1)$. For a 4×4 we have more cases but similar arguments apply. I leave the general proof to the reader. □

Example 5.3.9. *If* $M = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right]$ *is a block matrix where* A, B *are square blocks then* $det(M) = det(A)det(B)$.

5.4 examples and select applications of determinants

In the preceding section I worked pretty hard to prove a number of useful properties for determinants. I show how to use them in this section.

Example 5.4.1. *Notice that row* 2 *is twice row* 1*,*

$$det\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 7 & 8 & 9 \end{bmatrix} = 0.$$

Example 5.4.2. *To calculate this one we make a single column swap to get a diagonal matrix. The determinant of a diagonal matrix is the product of the diagonals, thus:*

$$det\begin{bmatrix} 0 & 6 & 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = -det\begin{bmatrix} 6 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = 48.$$

Example 5.4.3. *I choose the the column/row for the co-factor expansion to make life easy each time:*

$$\begin{aligned} det\begin{bmatrix} 0 & 1 & 0 & 2 \\ 13 & 71 & 5 & \pi \\ 0 & 3 & 0 & 4 \\ -2 & e & 0 & G \end{bmatrix} &= -5det\begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 4 \\ -2 & e & G \end{bmatrix} \\ &= -5(-2)det\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \\ &= 10(4-6) \\ &= -20. \end{aligned}$$

Example 5.4.4. *Find the values of* λ *such that the matrix* $A - \lambda I$ *is singular given that*

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}$$

The matrix $A - \lambda I$ is singular iff $det(A - \lambda I) = 0$,

$$\begin{aligned} det(A-\lambda I) &= det\begin{bmatrix} 1-\lambda & 0 & 2 & 3 \\ 1 & -\lambda & 0 & 0 \\ 0 & 0 & 2-\lambda & 0 \\ 0 & 0 & 0 & 3-\lambda \end{bmatrix} \\ &= (3-\lambda)det\begin{bmatrix} 1-\lambda & 0 & 2 \\ 1 & \lambda & 0 \\ 0 & 0 & 2-\lambda \end{bmatrix} \\ &= (3-\lambda)(2-\lambda)det\begin{bmatrix} 1-\lambda & 0 \\ 1 & \lambda \end{bmatrix} \\ &= (3-\lambda)(2-\lambda)(1-\lambda)(-\lambda) \\ &= \lambda(\lambda-1)(\lambda-2)(\lambda-3) \end{aligned}$$

Thus we need $\lambda = 0, 1, 2$ or 3 in order that $A - \lambda I$ be a noninvertible matrix. These values are called the **eigenvalues** *of A. We will have much more to say about that later.*

Example 5.4.5. *Suppose we are given the LU-factorization of a particular matrix (borrowed from the text by Spence, Insel and Friedberg see Example 2 on pg. 154-155.)*

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 3 & -1 & 7 \\ 2 & -4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} = LU$$

The LU-factorization is pretty easy to find, we may discuss that at the end of the course. It is an important topic if you delve into serious numerical work where you need to write your own code and so forth. Anyhow, notice that L, U are triangular so we can calculate the determinant very easily,

$$det(A) = det(L)det(U) = 1 \cdot 1 \cdot 1 \cdot 1 \cdot 2 \cdot 2 = 4.$$

From a numerical perspective, the LU-factorization is a superior method for calculating $det(A)$ as compared to the co-factor expansion. It has much better "convergence" properties. For a very large matrix the technique of calculation could result in a great reduction in computing time. See the chapter on factoring for further discussion of the LU-decomposition.

Example 5.4.6. *Recall that the columns in A are linearly independent iff $Ax = 0$ has only the $x = 0$ solution. We also found that the existence of A^{-1} was equivalent to that claim in the case A was square since $Ax = 0$ implies $A^{-1}Ax = A^{-1}0 = 0$ hence $x = 0$. Clearly then the columns of a square matrix A are linearly independent iff A^{-1} exists. Suppose A^{-1} exists then $AA^{-1} = I$ thus $det(AA^{-1}) = det(A)det(A^{-1} = det(I) = 1$ hence $det(A) \neq 0$. Conversely, the adjoint formula for the inverse is well-defined if $det(A) \neq 0$. To summarize: for $A \in \mathbb{R}^{\ n\times n}$*

$$\boxed{\textit{columns of } A \textit{ are linearly independent} \quad \Leftrightarrow \quad det(A) \neq 0.}$$

Observe that this criteria is only useful if we wish to examine the linear independence of preciely n-vectors in $\mathbb{R}^n$. For example, $(1,1,1),(1,0,1),(2,1,2) \in \mathbb{R}^3$ have

$$det \left[\begin{array}{c|c|c} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 1 & 1 & 2 \end{array} \right] = 0.$$

Therefore, $\{(1,1,1),(1,0,1),(2,1,2)\}$ form a linearly dependent set of vectors.

5.5 Cramer's Rule

The numerical methods crowd seem to think this is a loathsome brute. It is an incredibly clumsy way to calculate the solution of a system of equations $Ax = b$. Moreover, Cramer's rule fails in the case $det(A) = 0$ so it's not nearly as general as our other methods. However, it does help calculate the variation of parameters formulas in differential equations so it is still of theoretical interest at a minimum. Students sometimes like it because it gives you a *formula* to find the solution. Students sometimes incorrectly jump to the conclusion that a formula is easier than say a *method*. It is certainly wrong here, the method of Gaussian elimination beats Cramer's rule by just about every objective criteria in so far as concrete numerical examples are concerned.

Proposition 5.5.1.

If $Ax = b$ is a linear system of equations with $x = [x_1 \ x_2 \ \cdots \ x_n]^T$ and $A \in \mathbb{R}^{\ n \times n}$ such that $det(A) \neq 0$ then we find solutions

$$x_1 = \frac{det(A_1)}{det(A)}, \quad x_2 = \frac{det(A_2)}{det(A)}, \quad \dots, \quad x_n = \frac{det(A_n)}{det(A)}$$

where we define A_k to be the $n \times n$ matrix obtained by replacing the k-th column of A by the inhomogeneous term b.

Proof: Since $det(A) \neq 0$ we know that $Ax = b$ has a unique solution. Suppose $x_j = \frac{det(A_j)}{det(A)}$ where $A_j = [col_1(A)|\cdots|col_{j-1}(A)|b|col_{j+1}(A)|\cdots|col_n(A)]$. We seek to show $x = [x_j]$ is a solution to $Ax = b$. Notice that the n-vector equations

$$Ae_1 = col_1(A), \dots, Ae_{j-1} = col_{j-1}(A), Ae_{j+1} = col_{j+1}(A), \dots, Ae_n = col_n(A), Ax = b$$

can be summarized as a single matrix equation:

$$A[e_1|\dots|e_{j-1}|x|e_{j+1}|\cdots|e_n] = \underbrace{[col_1(A)|\cdots|col_{j-1}(A)|b|col_{j+1}(A)|\cdots|col_n(A)]}_{\text{this is precisely } A_j} = A_j$$

Notice that if we expand on the j-th column it's obvious that

$$det[e_1|\dots|e_{j-1}|x|e_{j+1}|\cdots|e_n] = x_j$$

Returning to our matrix equation, take the determinant of both sides and use that the product of the determinants is the determinant of the product to obtain:

$$det(A)x_j = det(A_j)$$

Since $det(A) \neq 0$ it follows that $x_j = \frac{det(A_j)}{det(A)}$ for all j. □

This is the proof that is given in Lay's text. The construction of the matrix equation is not really an obvious step in my estimation. Whoever came up with this proof originally realized that he would need to use the determinant product identity to overcome the subtlety in the proof. Once you realize that then it's natural to look for that matrix equation. This is a clever proof[2]

Example 5.5.2. *Solve $Ax = b$ given that*

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 8 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 5 \end{bmatrix}$$

where $x = [x_1\ x_2]^T$. Apply Cramer's rule, note $det(A) = 2$,

$$x_1 = \frac{1}{2} det \begin{bmatrix} 1 & 3 \\ 5 & 8 \end{bmatrix} = \frac{1}{2}(8 - 15) = \frac{-7}{2}.$$

and,

$$x_2 = \frac{1}{2} det \begin{bmatrix} 1 & 1 \\ 2 & 5 \end{bmatrix} = \frac{1}{2}(5 - 2) = \frac{3}{2}.$$

The original system of equations would be $x_1 + 3x_2 = 1$ and $2x_1 + 8x_2 = 5$. As a quick check we can substitute in our answers $x_1 = -7/2$ and $x_2 = 3/2$ and see if they work.

I conclude this section with examples from other courses. In particular, the subsections that follow show several particularly beautiful examples of how linear algebra is used to do calculus.

5.5.1 constrained partial differentiation, an application of Cramer's rule*

Suppose[3] $x + y + z + w = 3$ and $x^2 - 2xyz + w^3 = 5$. **Calculate partial derivatives of z and w with respect to the independent variables x, y.** Solution: we begin by calculation of the differentials of both equations:

$$\begin{aligned} &dx + dy + dz + dw = 0 \\ &(2x - 2yz)dx - 2xzdy - 2xydz + 3w^2dw = 0 \end{aligned}$$

We can solve for (dz, dw). In this calculation we can treat the differentials as formal variables.

$$\begin{aligned} &dz + dw = -dx - dy \\ &-2xydz + 3w^2dw = -(2x - 2yz)dx + 2xzdy \end{aligned}$$

[2] as seen from my humble vantage point naturally

[3] This discussion is take from my Advanced Calculus 2011 notes.

I find matrix notation is often helpful,

$$\left[\begin{array}{cc} 1 & 1 \\ -2xy & 3w^2 \end{array}\right]\left[\begin{array}{c} dz \\ dw \end{array}\right] = \left[\begin{array}{c} -dx-dy \\ -(2x-2yz)dx+2xzdy \end{array}\right]$$

Use Cramer's rule, multiplication by inverse, substitution, adding/subtracting equations etc... whatever technique of solving linear equations you prefer. Our goal is to solve for dz and dw in terms of dx and dy. I'll use Cramer's rule this time:

$$dz = \frac{det\left[\begin{array}{c|c} -dx-dy & 1 \\ -(2x-2yz)dx+2xzdy & 3w^2 \end{array}\right]}{det\left[\begin{array}{cc} 1 & 1 \\ -2xy & 3w^2 \end{array}\right]} = \frac{3w^2(-dx-dy)+(2x-2yz)dx-2xzdy}{3w^2+2xy}$$

Collecting terms,

$$dz = \left(\frac{-3w^2+2x-2yz}{3w^2+2xy}\right)dx + \left(\frac{-3w^2-2xz}{3w^2+2xy}\right)dy$$

From the expression above we can read various implicit derivatives,

$$\boxed{\left(\frac{\partial z}{\partial x}\right)_y = \frac{-3w^2+2x-2yz}{3w^2+2xy} \qquad \& \qquad \left(\frac{\partial z}{\partial y}\right)_x = \frac{-3w^2-2xz}{3w^2+2xy}}$$

The notation above indicates that z is understood to be a function of independent variables x, y. $\left(\frac{\partial z}{\partial x}\right)_y$ means we take the derivative of z with respect to x while holding y fixed. The appearance of the dependent variable w can be removed by using the equations $G(x,y,z,w) = (3,5)$. Similar ambiguities exist for implicit differentiation in calculus I. Apply Cramer's rule once more to solve for dw:

$$dw = \frac{det\left[\begin{array}{c|c} 1 & -dx-dy \\ -2xy & -(2x-2yz)dx+2xzdy \end{array}\right]}{det\left[\begin{array}{cc} 1 & 1 \\ -2xy & 3w^2 \end{array}\right]} = \frac{-(2x-2yz)dx+2xzdy-2xy(dx+dy)}{3w^2+2xy}$$

Collecting terms,

$$dw = \left(\frac{-2x+2yz-2xy}{3w^2+2xy}\right)dx + \left(\frac{2xzdy-2xydy}{3w^2+2xy}\right)dy$$

We can read the following from the differential above:

$$\boxed{\left(\frac{\partial w}{\partial x}\right)_y = \frac{-2x+2yz-2xy}{3w^2+2xy} \qquad \& \qquad \left(\frac{\partial w}{\partial y}\right)_x = \frac{2xzdy-2xydy}{3w^2+2xy}}$$

If you'd like to see another nontrivial application of Cramer's Rule, I would encourage you to read my Differential Equations notes where I use Cramer's Rule to derive the general method of variation of parameters.

5.6 adjoint matrix

Definition 5.6.1.

Let $A \in \mathbb{R}^{\,n\times n}$ the the matrix of cofactors is called the **adjoint** of A. It is denoted $adj(A)$ and is defined by and $adj(A)_{ij} = C_{ij}^T$ where C_{ij} is the (i,j)-th cofactor.

I'll keep it simple here, lets look at the 2×2 case:

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

has cofactors $C_{11} = (-1)^{1+1}det(d) = d$, $C_{12} = (-1)^{1+2}det(c) = -c$, $C_{21} = (-1)^{2+1}det(b) = -b$ and $C_{22} = (-1)^{2+2}det(a) = a$. Collecting these results,

$$adj(A) = \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

This is interesting. Recall we found a formula for the inverse of A (if it exists). The formula was

$$A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Notice that $det(A) = ad - bc$ thus in the 2×2 case the relation between the inverse and the adjoint is rather simple:

$$\boxed{A^{-1} = \frac{1}{det(A)}adj(A)^T}$$

It turns out this is true for larger matrices as well:

Proposition 5.6.2.

If A is invertible then $A^{-1} = \frac{1}{det(A)}adj(A)^T$.

Proof: To find the inverse of A we need only apply Cramer's rule to solve the equations implicit within $AA^{-1} = I$. Let $A^{-1} = [v_1|v_2|\cdots|v_n]$ we need to solve

$$Av_1 = e_1, \qquad Av_2 = e_2, \qquad \dots \qquad Av_n = e_n$$

Cramer's rule gives us $(v_1)_j = \frac{C_{1j}}{det(A)}$ where $C_{1j} = (-1)^{1+j}M_{ij}$ is the cofactor formed from deleting the first row and j-th column. More generally we may apply Cramer's rule to deduce the j-component of the i-th column in the inverse $(v_i)_j = \frac{C_{ij}}{det(A)}$. Therefore, $col_i(A^{-1})_j = (A^{-1})_{ji} = \frac{C_{ij}}{det(A)}$. By definition $adj(A) = [C_{ij}]$ hence $adj(A)^T_{ij} = C_{ji}$. It follows that $A^{-1} = \frac{1}{det(A)}adj(A)^T$. $\square$

Example 5.6.3. *Let's calculate the general formula for the inverse of a 3×3 matrix (assume it exists). Let*

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Calculate the cofactors,

$$C_{11} = det \left[\begin{smallmatrix} e & f \\ h & i \end{smallmatrix}\right] = ei - fh,$$

$$C_{12} = -det \left[\begin{smallmatrix} d & f \\ g & i \end{smallmatrix}\right] = fg - di,$$

$$C_{13} = det \left[\begin{smallmatrix} d & e \\ g & h \end{smallmatrix}\right] = dh - eg,$$

$$C_{21} = -det \left[\begin{smallmatrix} b & c \\ h & i \end{smallmatrix}\right] = ch - bi,$$

$$C_{22} = det \left[\begin{smallmatrix} a & c \\ g & i \end{smallmatrix}\right] = ai - cg,$$

$$C_{23} = -det \left[\begin{smallmatrix} a & b \\ g & h \end{smallmatrix}\right] = bg - ah,$$

$$C_{31} = det \left[\begin{smallmatrix} b & c \\ e & f \end{smallmatrix}\right] = bf - ce,$$

$$C_{32} = -det \left[\begin{smallmatrix} a & c \\ d & f \end{smallmatrix}\right] = cd - af,$$

$$C_{33} = det \left[\begin{smallmatrix} a & b \\ d & e \end{smallmatrix}\right] = ae - bd.$$

Hence the adjoint is

$$adj(A) = \left[\begin{array}{c|c|c} ei - fh & fg - di & dh - eg \\ \hline ch - bi & ai - cg & bg - ah \\ \hline bf - ce & cd - af & ae - bd \end{array}\right]$$

Thus, using the $A^{-1} = det(A)adj(A)^T$

$$\boxed{\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}^{-1} = \frac{1}{aei + bfg + cdh - gec - hfa - idb} \left[\begin{array}{c|c|c} ei - fh & ch - bi & bf - ce \\ \hline fg - di & ai - cg & cd - af \\ \hline dh - eg & bg - ah & ae - bd \end{array}\right]}$$

You should notice that are previous method for finding A^{-1} is far superior to this method. It required much less calculation. Let's check my formula in the case $A = 3I$, this means $a = e = i = 3$ and the others are zero.

$$I^{-1} = \frac{1}{27} \left[\begin{array}{c|c|c} 9 & 0 & 0 \\ \hline 0 & 9 & 0 \\ \hline 0 & 0 & 9 \end{array}\right] = \frac{1}{3} I$$

This checks, $(3I)(\frac{1}{3}I) = \frac{3}{3}II = I$. I do not recommend that you memorize this formula to calculate inverses for 3×3 matrices.

5.7 applications

The determinant is a convenient mnemonic to create expressions which are antisymmetric. The key property is that if we switch a row or column it creates a minus sign. This means that if any two rows are repeated then the determinant is zero. Notice this is why the cross product of two vectors is naturally phrased in terms of a determinant. The antisymmetry of the determinant insures the formula for the cross-product will have the desired antisymmetry. In this section we examine a few more applications for the determinant.

Example 5.7.1. *The Pauli's exclusion principle in quantum mechanics states that the wave function of a system of fermions is antisymmetric. Given N-electron wavefunctions $\chi_1, \chi_2, \dots, \chi_N$ the following is known as the* **Slater Determinant**

$$\Psi(\vec{r}_1, \vec{r}_2, \dots, \vec{r}_N) = det \begin{bmatrix} \chi_1(\vec{r}_1) & \chi_2(\vec{r}_1) & \cdots & \chi_N(\vec{r}_1) \\ \chi_1(\vec{r}_2) & \chi_2(\vec{r}_2) & \cdots & \chi_N(\vec{r}_2) \\ \vdots & \vdots & \cdots & \vdots \\ \chi_1(\vec{r}_N) & \chi_2(\vec{r}_N) & \cdots & \chi_N(\vec{r}_N) \end{bmatrix}$$

Notice that $\Psi(\vec{r}_1, \vec{r}_1, \dots, \vec{r}_N) = 0$ and generally if any two of the position vectors $\vec{r}_i = \vec{r}_j$ then the total wavefunction $\Psi = 0$. In quantum mechanics the wavefunction's modulus squared gives the probability density of finding the system in a particular circumstance. In this example, the fact that any repeated entry gives zero means that no two electrons can share the same position. This is characteristic of particles with half-integer **spin***, such particles are called* **fermions***. In contrast,* **bosons** *are particles with integer spin and they can occupy the same space. For example, light is made of photons which have spin 1 and in a laser one finds many waves of light traveling in the same space.*

Example 5.7.2. *This is an example of a Vandermonde determinant. Note the following curious formula:*

$$det \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x & y \end{bmatrix} = 0$$

Let's reduce this by row-operations[4]

$$\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x & y \end{bmatrix} \xrightarrow[r_3 - r_1 \to r_3]{r_2 - r_1 \to r_2} \begin{bmatrix} 1 & x_1 & y_1 \\ 0 & x_2 - x_1 & y_2 - y_1 \\ 0 & x - x_1 & y - y_1 \end{bmatrix}$$

Notice that the row operations above could be implemented by multiply on the left by $E_{r_2 - r_1 \to r_2}$ and $E_{r_3 - r_1 \to r_3}$. These are invertible matrices and thus $det(E_{r_2 - r_1 \to r_2}) = k_1$ and $det(E_{r_3 - r_1 \to r_3}) = k_2$

[4] of course we could calculate it straight from the co-factor expansion, I merely wish to illustrate how we can use row operations to simplify a determinant

for some pair of nonzero constants k_1, k_2. If X is the given matrix and Y is the reduced matrix above then $Y = E_{r_3 - r_1 \to r_3} E_{r_2 - r_1 \to r_2} X$ thus,

$$0 = det \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x & y \end{bmatrix} = k_1 k_2 det \begin{bmatrix} 1 & x_1 & y_1 \\ 0 & x_2 - x_1 & y_2 - y_1 \\ 0 & x - x_1 & y - y_1 \end{bmatrix}$$
$$= k_1 k_2 \big[(x_2 - x_1)(y - y_1) - (y_2 - y_1)(x - x_1)\big]$$

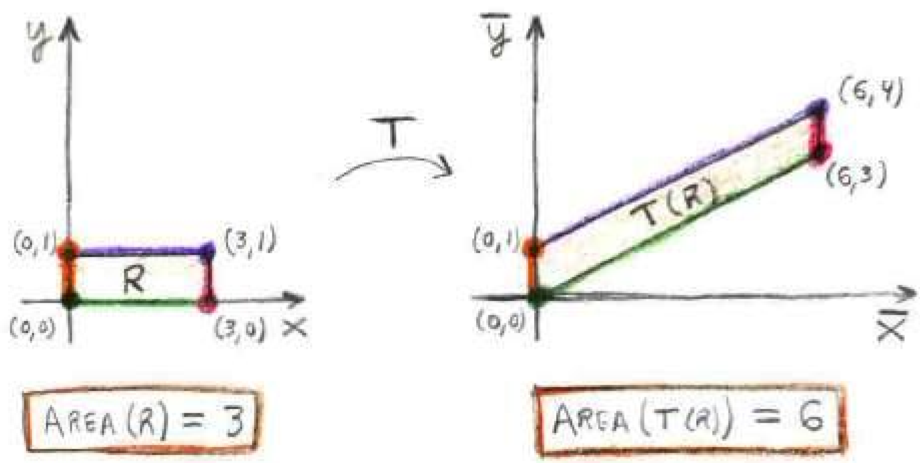
Divide by $k_1 k_2$ and rearrange to find:

$$(x_2 - x_1)(y - y_1) = (y_2 - y_1)(x - x_1) \qquad \Rightarrow \qquad \boxed{y = y_1 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x_2 - x_1)}$$

The boxed equation is the famous two-point formula for a line.

Example 5.7.3. *Let us consider a linear transformation $T((x, y)) = (2x, x + y)$. Furthermore, let's see how a rectangle R with corners $(0, 0), (3, 0), (3, 1), (0, 1)$. Since this linear transformation is invertible (I invite you to prove that) it follows that the image of a line is again a line. Therefore, if we find the image of the corners under the mapping T then we can just connect the dots in the image to see what $T(R)$ resembles. Our goal here is to see what a linear transformation does to a rectangle.*

$$T((0, 0)) = (0, 0)$$
$$T((3, 0)) = (6, 3)$$
$$T((3, 1)) = (6, 4)$$
$$T((0, 1)) = (0, 1).$$



As you can see from the picture we have a paralellogram with base 6 and height 1 thus $Area(T(R)) = 6$. In constrast, $Area(R) = 3$. You can calculate that $det(T) = 2$. Curious, $Area(T(R)) = det(T)Area(R)$. I wonder if this holds in general? [5]

[5] ok, actually I don't wonder, I just make it a homework problem sometimes.

5.8 conclusions

We continue Theorem 4.5.1 from the previous chapter.

Theorem 5.8.1.

Let A be a real $n \times n$ matrix then the following are equivalent:

(a.) A is invertible,

(b.) $rref[A|0] = [I|0]$ where $0 \in \mathbb{R}^n$,

(c.) $Ax = 0$ iff $x = 0$,

(d.) A is the product of elementary matrices,

(e.) there exists $B \in \mathbb{R}^{\ n\times n}$ such that $AB = I$,

(f.) there exists $B \in \mathbb{R}^{\ n\times n}$ such that $BA = I$,

(g.) $rref[A] = I$,

(h.) $rref[A|b] = [I|x]$ for an $x \in \mathbb{R}^n$,

(i.) $Ax = b$ is consistent for every $b \in \mathbb{R}^n$,

(j.) $Ax = b$ has exactly one solution for every $b \in \mathbb{R}^n$,

(k.) A^T is invertible,

(l.) $det(A) \neq 0$,

(m.) Cramer's rule yields solution of $Ax = b$ for every $b \in \mathbb{R}^n$.

It's a small addition, however the determinant is a nice tool for small systems since it's pretty easy to calculate. Also, Cramer's rule is nice for small systems since it just gives us the solution. This is all a very special case, in general we could have an inconsistent system or infinitely many solutions.

Theorem 5.8.2.

Let A be a real $n \times n$ matrix then the following are equivalent:

(a.) A is not invertible,

(b.) $Ax = 0$ has at least one nontrivial solution.,

(c.) there exists $b \in \mathbb{R}^n$ such that $Ax = b$ is inconsistent,

(d.) $det(A) = 0$,

It turns out this theorem is also useful. We shall see it is fundamental to the theory of eigenvectors.

Chapter 6

linear transformations and coordinates

Linear transformations are simply functions[1] from one vector space to another which preserve both vector addition and scalar multiplication. In short, if V and W are vector spaces and $L : V \to W$ is a function with (**1**.) $L(x+y) = L(x) + L(y)$ & (**2**.) $L(cx) = cL(x)$ for all $x, y \in V$ and numbers c. Such a function is both additive(1) and homogeneous(2). Perhaps you identify that many of the operations from calculus share these patterns: limits, definite integrals, d/dx etc... In this work, we mainly study the case $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$ in which there is a simple correspondance between linear transformations and $m \times n$ matrices. In particular, the Fundamental Theorem of Linear Algebra claims that a mapping $L : \mathbb{R}^n \to \mathbb{R}^m$ is linear iff there exists $A \in \mathbb{R}^{m \times n}$ such that $L(x) = Ax$ for all $x \in \mathbb{R}^n$. This matrix A is important and has special notation; $A = [L]$ the standard matrix. We'll see how the structure of A is reflected in the properties of L_A. This gives us a chance to once more apply our work from earlier chapters on LI and spanning.

The geometry of linear tranformations is interesting, I study how the unit square is mapped by a variety of linear transformations. Anton and Rorres have a nice and much more comprehensive list of possible linear mappings of the plane. Please take a few minutes to appreciate those pictures.

We introduce the concept of coordinates. We show how to find coordinate vectors as well as how to calculate the matrix with respect to a nonstandard basis. We see how the matrix of a linear transformation changes undergoes a similarity transformation as we exchange one basis for another.

[1] Let me briefly some standard terminology about functions. Recall that a function $f : A \to B$ is an single-valued assignment of elements in A to elements in B. We say that $dom(f) = A$ and $codomain(f) = B$. Furthermore, recall that the range of the function is the set of all outputs: $range(f) = f(A)$. If $f(A) = B$ then we say that f is a **surjection** or equivalently f is **onto**. If $f(x) = f(y)$ implies $x = y$ for all $x, y \in A$ then we say that f is **injective** or equivalently f is **1-1**. Don't dismay if the wording in this chapter is too much to follow in places. Just make a note and ask. There are many unproved assertions in this chapter. If you percieve a gap, it's likely there. It's also likely that is something that is shown in Math 321.

6.1 theory of linear transformations part one

Definition 6.1.1.

Let V, W be vector spaces. If a mapping $L : V \to W$ satisfies

1. $L(x + y) = L(x) + L(y)$ for all $x, y \in V$,
2. $L(cx) = cL(x)$ for all $x \in V$ and $c \in \mathbb{R}$

then we say L is a linear transformation. If $L : V \to V$ is linear then say L is a linear transformation on V.

Many simple examples are given in the next section. You can look ahead to gain a better sense, but first I develop a little theory.

Proposition 6.1.2.

Let $L : V \to W$ be a linear transformation on vector spaces V and W,

1. $L(0) = 0$
2. $L(c_1v_1 + c_2v_2 + \cdots c_nv_n) = c_1L(v_1) + c_2L(v_2) + \cdots + c_nL(v_n)$ for all $v_i \in V$ and $c_i \in \mathbb{R}$.

Proof: to prove of (1.) let $x \in \mathbb{R}^n$ and notice that $x - x = 0$ thus

$$L(0) = L(x - x) = L(x) + L(-1x) = L(x) - L(x) = 0.$$

To prove (2.) we use induction on n. Notice the proposition is true for n=1,2 by definition of linear transformation. Assume inductively $L(c_1v_1+c_2v_2+\cdots c_nv_n) = c_1L(v_1)+c_2L(v_2)+\cdots+c_nL(v_n)$ for all $v_i \in V$ and $c_i \in \mathbb{R}$ where $i = 1, 2, \dots, n$. Let $v_1, v_2, \dots, v_n, v_{n+1} \in V$ and $c_1, c_2, \dots c_n, c_{n+1} \in \mathbb{R}$ and consider, $L(c_1v_1 + c_2v_2 + \cdots c_nv_n + c_{n+1}v_{n+1}) =$

$$\begin{aligned} &= L(c_1v_1 + c_2v_2 + \cdots c_nv_n) + c_{n+1}L(v_{n+1}) && \text{by linearity of } L \\ &= c_1L(v_1) + c_2L(v_2) + \cdots + c_nL(v_n) + c_{n+1}L(v_{n+1}) && \text{by the induction hypothesis.} \end{aligned}$$

Hence the proposition is true for $n+1$ and we conclude by the principle of mathematical induction that (2.) is true for all $n \in \mathbb{N}$. $\square$

Item (2.) simply says that the linear transformations map spans to spans. Thus:

Proposition 6.1.3.

Let V, W be vector spaces and $\{v_1, v_2, \dots, v_k\} \subset V$ if $L : V \to W$ is linear then

$$L(span\{v_1, v_2, \dots, v_k\}) = span\{L(v_1), L(v_2), \dots, L(v_k)\}.$$

This is a very nice result. Why? Think about $V = \mathbb{R}^n$ and $W = \mathbb{R}^m$ for $m, n \geq k$. We find L maps k-dimensional planes spanned by $\{v_1, v_2, \dots, v_k\}$ map to a new j-dimensional plane spanned by $\{L(v_1), L(v_2), \dots, L(v_k)\}$ where $j \leq k$. We have to allow for the possibility that $j < k$ as is illustrated explicitly in the examples of the next section. Then in Section 6.3 we turn to the question of how to anticipate this loss of dimension. We'll study how and why a linear transformation maintains the dimension of the objects it transforms.

There is also a similar proposition for line-segments.

Proposition 6.1.4.

Let $\mathcal{L} = \{p + tv \mid t \in [0,1], p, v \in \mathbb{R}^n \text{ with } v \neq 0\}$ define a line segment from p to $p + v$ in $\mathbb{R}^n$. If $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation then $T(\mathcal{L})$ is a either a line-segment from $T(p)$ to $T(p + v)$ or a point.

Proof: suppose T and $\mathcal{L}$ are as in the proposition. Let $y \in T(\mathcal{L})$ then by definition there exists $x \in \mathcal{L}$ such that $T(x) = y$. But this implies there exists $t \in [0,1]$ such that $x = p + tv$ so $T(p + tv) = y$. Notice that

$$y = T(p + tv) = T(p) + T(tv) = T(p) + tT(v).$$

which implies $y \in \{T(p) + sT(v) \mid s \in [0,1]\} = \mathcal{L}_2$. Therefore, $T(\mathcal{L}) \subseteq \mathcal{L}_2$. Conversely, suppose $z \in \mathcal{L}_2$ then $z = T(p) + sT(v)$ for some $s \in [0,1]$ but this yields by linearity of T that $z = T(p + sv)$ hence $z \in T(\mathcal{L})$. Since we have that $T(\mathcal{L}) \subseteq \mathcal{L}_2$ and $\mathcal{L}_2 \subseteq T(\mathcal{L})$ it follows that $T(\mathcal{L}) = \mathcal{L}_2$. Note that $\mathcal{L}_2$ is a line-segment provided that $T(v) \neq 0$, however if $T(v) = 0$ then $\mathcal{L}_2 = \{T(p)\}$ and the proposition follows. $\square$

We can say something a bit more general without much more work. However, this suffices for our current endeavor. Ask me if you are interested in how to generalize the proposition to a higher-dimensional object.

Proposition 6.1.5.

If $A \in \mathbb{R}^{m \times n}$ and $L : \mathbb{R}^n \to \mathbb{R}^m$ is defined by $L(x) = Ax$ for each $x \in \mathbb{R}^n$ then L is a linear transformation.

Proof: Let $A \in \mathbb{R}^{m \times n}$ and define $L : \mathbb{R}^n \to \mathbb{R}^m$ by $L(x) = Ax$ for each $x \in \mathbb{R}^n$. Let $x, y \in \mathbb{R}^n$ and $c \in \mathbb{R}$,

$$L(x + y) = A(x + y) = Ax + Ay = L(x) + L(y)$$

and

$$L(cx) = A(cx) = cAx = cL(x)$$

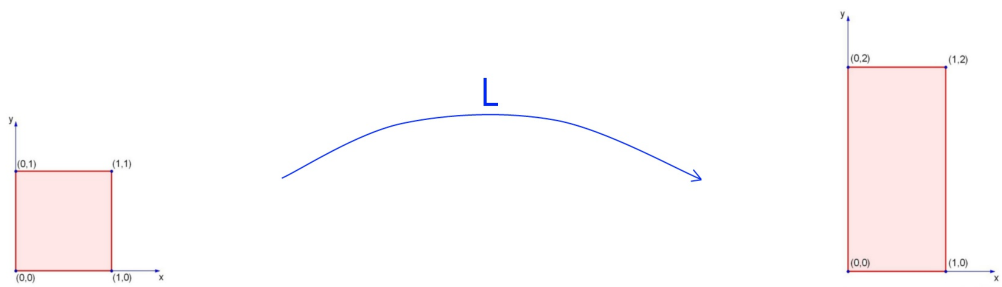
thus L is a linear transformation. $\square$

Obviously this gives us a nice way to construct examples. It's so simple. If you can write the formula for a mapping as a matrix multiplication then that **proves** the mapping is linear. That is what the proposition above gives us. Please make a note in your mind on this point.

6.2 examples of linear transformations on euclidean spaces

My choice of mapping the unit square has no particular signficance in the examples below. I merely wanted to keep it simple and draw your eye to the distinction between the examples. In each example we'll map the four corners of the square to see where the transformation takes the unit-square. Those corners are simply $(0,0),(1,0),(1,1),(0,1)$ as we traverse the square in a counter-clockwise direction.

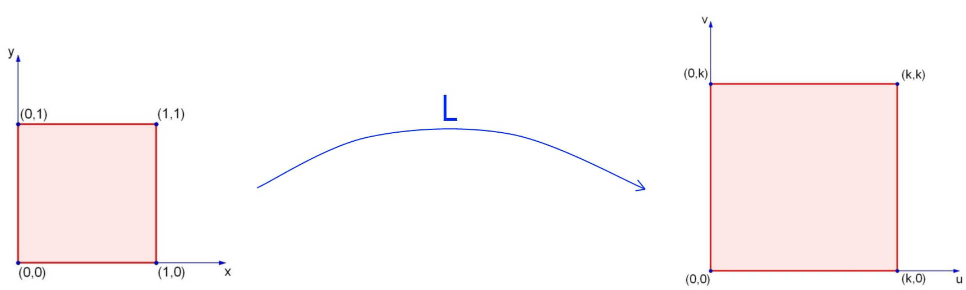
Example 6.2.1. *Let $L(x,y)=(x,2y)$. This is a mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$. This mapping has stretches the vertical direction.*



Example 6.2.2. *Let $A=\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}$ for some $k>0$. Define $L(v)=Av$ for all $v\in\mathbb{R}^2$. In particular this means,*

$$L(x,y)=A(x,y)=\begin{bmatrix} k & 0 \\ 0 & k \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} kx \\ ky \end{bmatrix}.$$

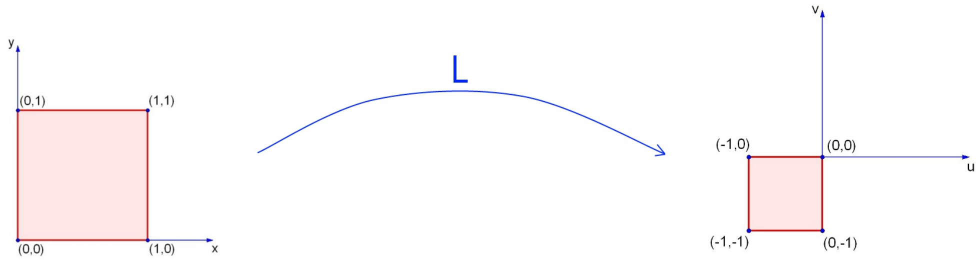
We find $L(0,0)=(0,0),\ L(1,0)=(k,0),\ L(1,1)=(k,k),\ L(0,1)=(0,k)$. This mapping is called a **dilation**.



Example 6.2.3. *Let $A=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$. Define $L(v)=Av$ for all $v\in\mathbb{R}^2$. In particular this means,*

$$L(x,y)=A(x,y)=\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}=\begin{bmatrix} -x \\ -y \end{bmatrix}.$$

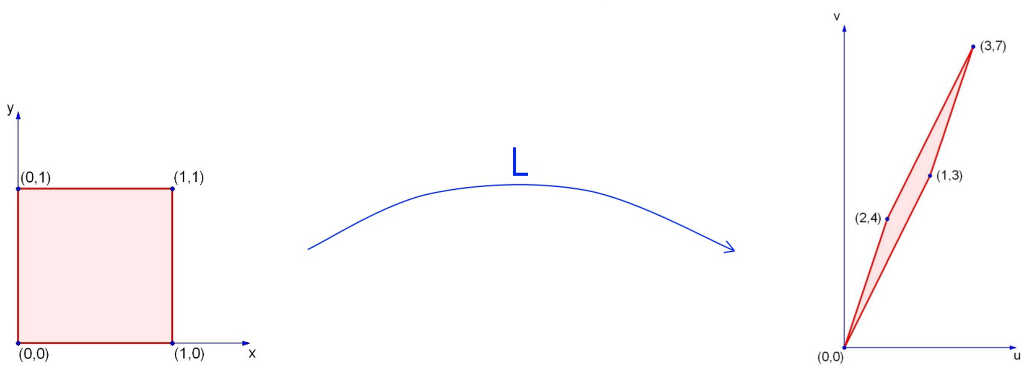
We find $L(0,0)=(0,0),\ L(1,0)=(-1,0),\ L(1,1)=(-1,-1),\ L(0,1)=(0,-1)$. This mapping is called an **inversion**.



Example 6.2.4. *Let* $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$*. Define* $L(v) = Av$ *for all* $v \in \mathbb{R}^2$*. In particular this means,*

$$L(x, y) = A(x, y) = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + 2y \\ 3x + 4y \end{bmatrix}.$$

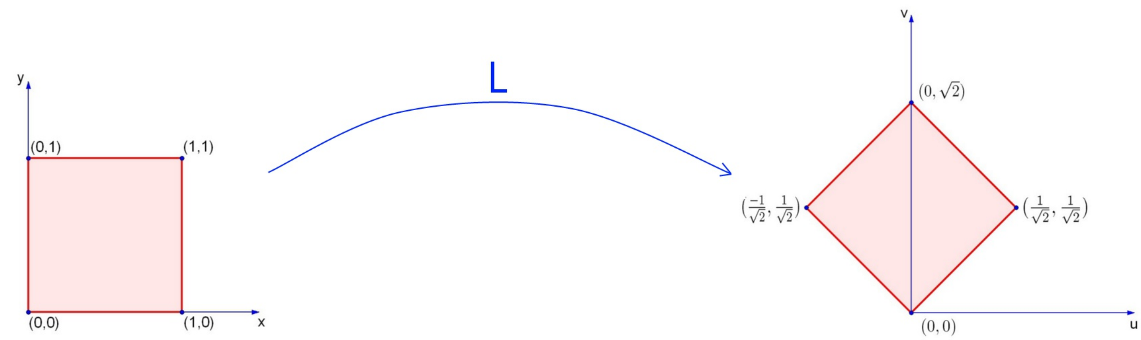
We find $L(0, 0) = (0, 0)$, $L(1, 0) = (1, 3)$, $L(1, 1) = (3, 7)$, $L(0, 1) = (2, 4)$*. This mapping shall remain nameless, it is doubtless a combination of the other named mappings.*



Example 6.2.5. *Let* $A = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$*. Define* $L(v) = Av$ *for all* $v \in \mathbb{R}^2$*. In particular this means,*

$$L(x, y) = A(x, y) = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} x - y \\ x + y \end{bmatrix}.$$

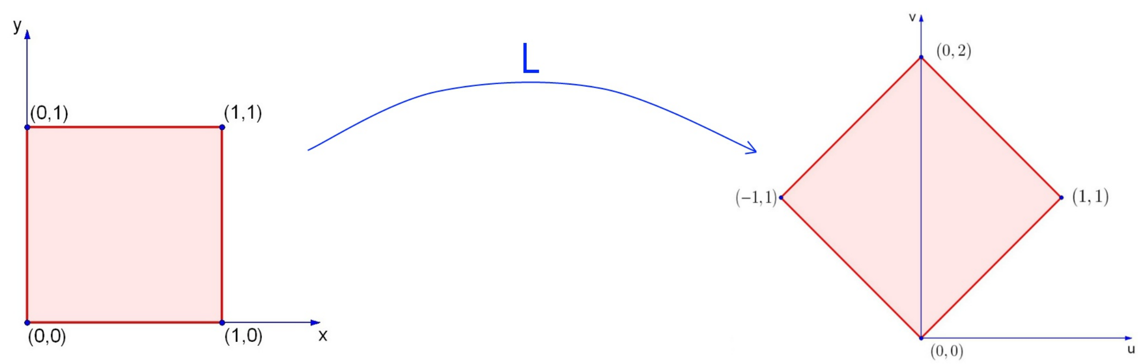
We find $L(0, 0) = (0, 0)$, $L(1, 0) = \frac{1}{\sqrt{2}}(1, 1)$, $L(1, 1) = \frac{1}{\sqrt{2}}(0, 2)$, $L(0, 1) = \frac{1}{\sqrt{2}}(-1, 1)$*. This mapping is a rotation by* $\pi/4$ *radians.*



Example 6.2.6. *Let* $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$. *Define* $L(v) = Av$ *for all* $v \in \mathbb{R}^2$. *In particular this means,*

$$L(x, y) = A(x, y) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x - y \\ x + y \end{bmatrix}.$$

We find $L(0, 0) = (0, 0)$, $L(1, 0) = (1, 1)$, $L(1, 1) = (0, 2)$, $L(0, 1) = (-1, 1)$. *This mapping is a rotation followed by a dilation by* $k = \sqrt{2}$.

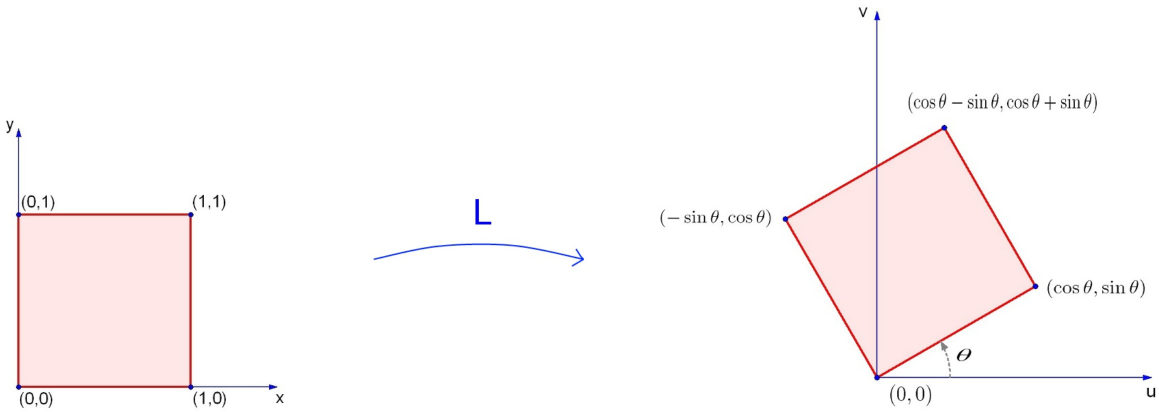


We will come back to discuss rotations a few more times this semester, you'll see they give us interesting and difficult questions later this semester. Also, if you so choose there are a few bonus applied problems on computer graphics which are built from an understanding of the mathematics in the next example.

Example 6.2.7. *Let* $A = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$. *Define* $L(v) = Av$ *for all* $v \in \mathbb{R}^2$. *In particular this means,*

$$L(x, y) = A(x, y) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos(\theta) - y\sin(\theta) \\ x\sin(\theta) + y\cos(\theta) \end{bmatrix}.$$

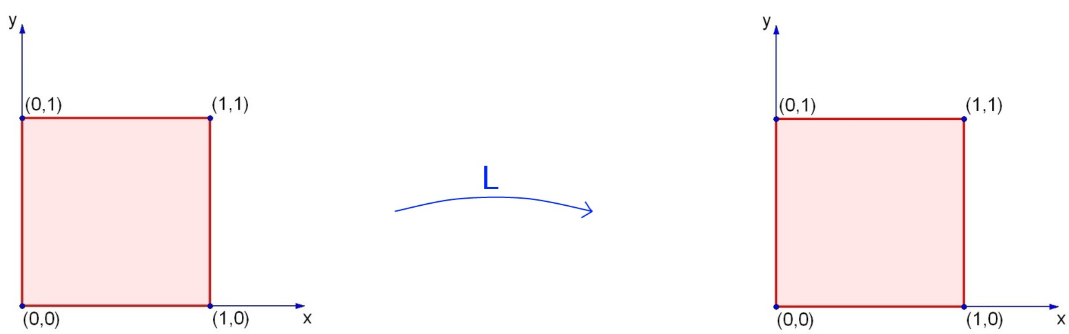
We find $L(0,0) = (0,0),\ L(1,0) = (\cos(\theta), \sin(\theta)),\ L(1,1) = (\cos(\theta)-\sin(\theta), \cos(\theta)+\sin(\theta))\ L(0,1) = (\sin(\theta), \cos(\theta))$. *This mapping is a rotation by* θ *in the counter-clockwise direction. Of course you could have derived the matrix* A *from the picture below.*



Example 6.2.8. *Let* $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. *Define* $L(v) = Av$ *for all* $v \in \mathbb{R}^2$. *In particular this means,*

$$L(x,y) = A(x,y) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

We find $L(0,0) = (0,0),\ L(1,0) = (1,0),\ L(1,1) = (1,1),\ L(0,1) = (0,1)$. *This mapping is a rotation by zero radians, or you could say it is a dilation by a factor of 1, ... usually we call this the identity mapping because the image is identical to the preimage.*



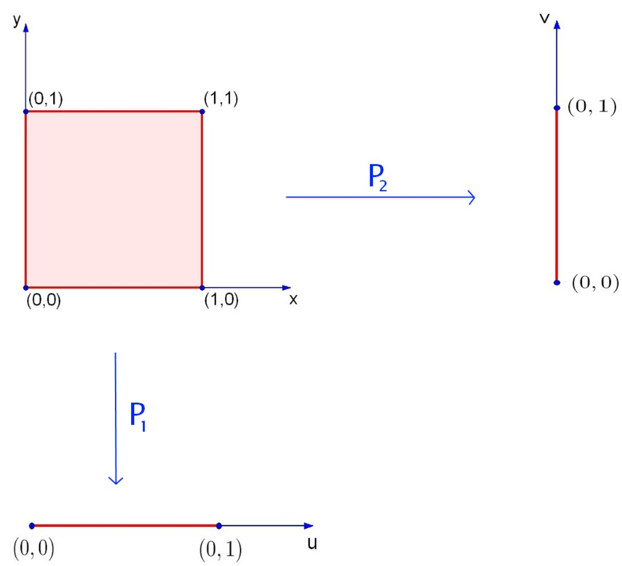
Example 6.2.9. *Let* $A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. *Define* $P_1(v) = A_1 v$ *for all* $v \in \mathbb{R}^2$. *In particular this means,*

$$P_1(x, y) = A_1(x, y) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix}.$$

We find $P_1(0, 0) = (0, 0),\ P_1(1, 0) = (1, 0),\ P_1(1, 1) = (1, 0),\ P_1(0, 1) = (0, 0)$. *This mapping is a* **projection** *onto the first coordinate.*
Let $A_2 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. *Define* $L(v) = A_2 v$ *for all* $v \in \mathbb{R}^2$. *In particular this means,*

$$P_2(x, y) = A_2(x, y) = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ y \end{bmatrix}.$$

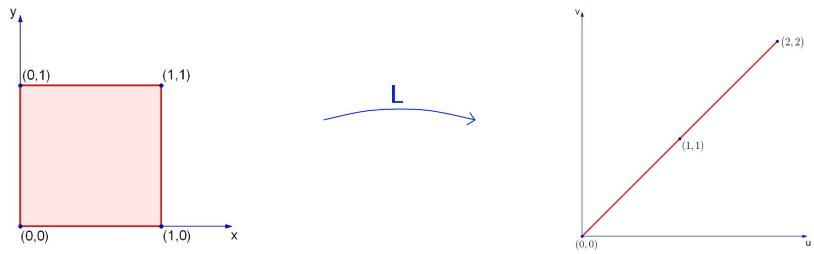
We find $P_2(0, 0) = (0, 0),\ P_2(1, 0) = (0, 0),\ P_2(1, 1) = (0, 1),\ P_2(0, 1) = (0, 1)$. *This mapping is* **projection** *onto the second coordinate.*
We can picture both of these mappings at once:



Example 6.2.10. *Let* $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. *Define* $L(v) = Av$ *for all* $v \in \mathbb{R}^2$. *In particular this means,*

$$L(x, y) = A(x, y) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x + y \\ x + y \end{bmatrix}.$$

We find $L(0, 0) = (0, 0),\ L(1, 0) = (1, 1),\ L(1, 1) = (2, 2),\ L(0, 1) = (1, 1)$. *This mapping is not a projection, but it does collapse the square to a line-segment.*



A projection has to have the property that if it is applied twice then you obtain the same image as if you applied it only once. If you apply the transformation to the image then you'll obtain a line-segment from $(0,0)$ to $(4,4)$. While it is true the transformation "projects" the plane to a line it is not technically a "projection".

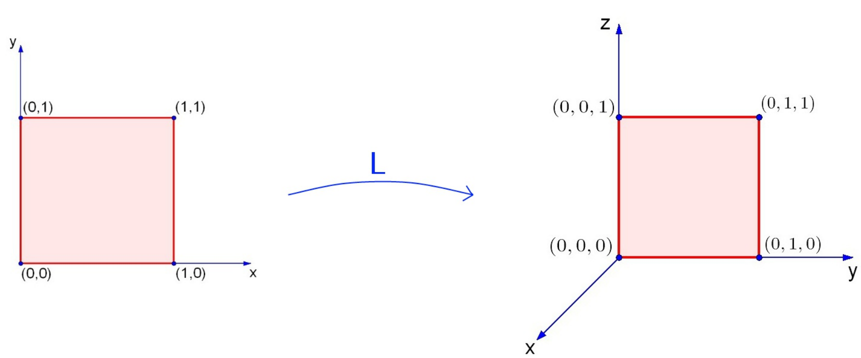
Remark 6.2.11.

The examples here have focused on linear transformations from $\mathbb{R}^2$ to $\mathbb{R}^2$. It turns out that higher dimensional mappings can largely be understood in terms of the geometric operations we've seen in this section.

Example 6.2.12. *Let* $A = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$. *Define* $L(v) = Av$ *for all* $v \in \mathbb{R}^2$. *In particular this means,*

$$L(x,y) = A(x,y) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ x \\ y \end{bmatrix}.$$

We find $L(0,0) = (0,0,0),\ L(1,0) = (0,1,0),\ L(1,1) = (0,1,1),\ L(0,1) = (0,0,1)$. *This mapping moves the* xy*-plane to the* yz*-plane. In particular, the horizontal unit square gets mapped to vertical unit square;* $L([0,1]\times[0,1]) = \{0\}\times[0,1]\times[0,1]$. *This mapping certainly is not surjective because no point with* $x \neq 0$ *is covered in the range.*



Example 6.2.13. *Let* $A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$. *Define* $L(v) = Av$ *for all* $v \in \mathbb{R}^3$. *In particular this means,*

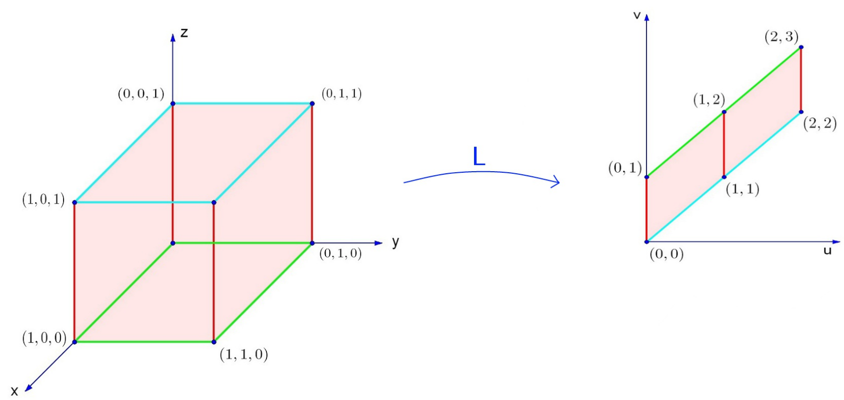
$$L(x, y, z) = A(x, y, z) = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x+y \\ x+y+z \end{bmatrix}.$$

Let's study how L *maps the unit cube. We have* $2^3 = 8$ *corners on the unit cube,*

$$L(0, 0, 0) = (0, 0),\ L(1, 0, 0) = (1, 1),\ L(1, 1, 0) = (2, 2),\ L(0, 1, 0) = (1, 1)$$

$$L(0, 0, 1) = (0, 1),\ L(1, 0, 1) = (1, 2),\ L(1, 1, 1) = (2, 3),\ L(0, 1, 1) = (1, 2).$$

This mapping squished the unit cube to a shape in the plane which contains the points $(0, 0)$, $(0, 1)$, $(1, 1)$, $(1, 2)$, $(2, 2)$, $(2, 3)$. *Face by face analysis of the mapping reveals the image is a parallelogram. This mapping is certainly not injective since two different points get mapped to the same point. In particular, I have color-coded the mapping of top and base faces as they map to line segments. The vertical faces map to one of the two parallelograms that comprise the image.*

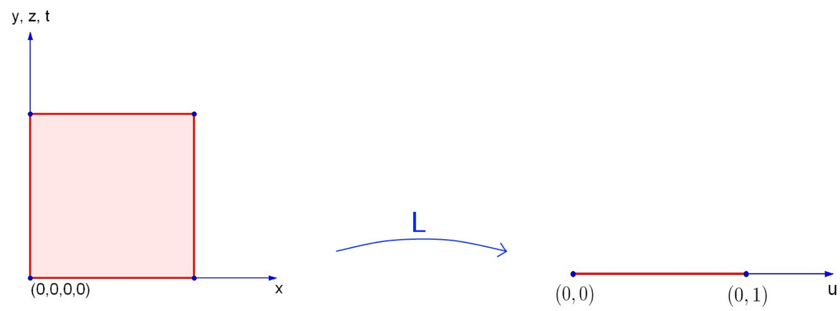


I have used terms like "vertical" or "horizontal" in the standard manner we associate such terms with three dimensional geometry. Visualization and terminology for higher-dimensional examples is not as obvious. However, with a little imagination we can still draw pictures to capture important aspects of mappings.

Example 6.2.14. *Let* $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. *Define* $L(v) = Av$ *for all* $v \in \mathbb{R}^4$. *In particular this means,*

$$L(x, y, z, t) = A(x, y, z, t) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} x \\ x \end{bmatrix}.$$

Let's study how L *maps the unit hypercube* $[0, 1]^4 \subset \mathbb{R}^4$. *We have* $2^4 = 16$ *corners on the unit hypercube, note* $L(1, a, b, c) = (1, 1)$ *whereas* $L(0, a, b, c) = (0, 0)$ *for all* $a, b, c \in [0, 1]$. *Therefore,* *the unit hypercube is squished to a line-segment from* $(0,0)$ *to* $(1,1)$. *This mapping is neither surjective nor injective. In the picture below the vertical axis represents the* y, z, t*-directions.*



Example 6.2.15. *Suppose* $f(t,s) = (\sqrt{t}, s^2 + t)$ *note that* $f(1,1) = (1,2)$ *and* $f(4,4) = (2,20)$. *Note that* $(4,4) = 4(1,1)$ *thus we should see* $f(4,4) = f(4(1,1)) = 4f(1,1)$ *but that fails to be true so* f *is not a linear transformation.*

Example 6.2.16. *Let* $L : \mathbb{R}^n \to \mathbb{R}^m$ *be defined by* $L(x) = 0$ *for all* $x \in V$. *This is a linear transformation known as the* **trivial transformation**

$$L(x+y) = 0 = 0 + 0 = L(x) + L(y) \quad \text{and} \quad L(cx) = 0 = c0 = cL(x)$$

for all $c \in \mathbb{R}$ *and* $x, y \in \mathbb{R}^n$.

Example 6.2.17. *The identity function on a* $\mathbb{R}^n$ *is also a linear transformation. Let* $Id : \mathbb{R}^n \to \mathbb{R}^n$ *satisfy* $L(x) = x$ *for each* $x \in \mathbb{R}^n$. *Observe that*

$$Id(x + cy) = x + cy = x + c(y) = Id(x) + cId(y)$$

for all$x, y \in \mathbb{R}^n$ *and* $c \in \mathbb{R}$.

Example 6.2.18. *Let* $L(x,y) = x^2 + y^2$ *define a mapping from* $\mathbb{R}^2$ *to* $\mathbb{R}$. *This is not a linear transformation since*

$$L(c(x,y)) = L(cx, cy) = (cx)^2 + (cy)^2 = c^2(x^2 + y^2) = c^2 L(x,y).$$

We say L *is a nonlinear transformation.*

Obviously we have not even begun to appreciate the wealth of possibilities that exist for linear mappings. Clearly different types of matrices will decribe different types of geometric transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$. On the other hand, square matrices describe mappings from $\mathbb{R}^n$ to $\mathbb{R}^n$ and these can be thought of as coordinate transformations. A square matrix may give us a way to define new coordinates on $\mathbb{R}^n$. We will return to the concept of linear transformations a number of times in this course. Hopefully you already appreciate that linear algebra is not just about solving equations. It always comes back to that, but there is more here to ponder.

6.3 theory of linear transformations part two

If you are pondering what I am pondering then you probably would like to know if all linear mappings from $\mathbb{R}^n$ to $\mathbb{R}^m$ can be reduced to matrix multiplication? We saw that if a map is defined as a matrix multiplication then it will be linear. A natural question to ask: is the converse true? Given a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$ can we write it as multiplication by a matrix ?

Theorem 6.3.1. *fundamental theorem of linear algebra.*

$L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation if and only if there exists $A \in \mathbb{R}^{m\times n}$ such that $L(x) = Ax$ for all $x \in \mathbb{R}^n$.

Proof: ($\Leftarrow$) Assume there exists $A \in \mathbb{R}^{m\times n}$ such that $L(x) = Ax$ for all $x \in \mathbb{R}^n$. As we argued before,

$$L(x + cy) = A(x + cy) = Ax + cAy = L(x) + cL(y)$$

for all $x, y \in \mathbb{R}^n$ and $c \in \mathbb{R}$ hence L is a linear transformation.

($\Rightarrow$) Assume $L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation. Let e_i denote the standard basis in $\mathbb{R}^n$ and let f_j denote the standard basis in $\mathbb{R}^m$. If $x \in \mathbb{R}^n$ then there exist constants x_i such that $x = x_1e_1 + x_2e_2 + \cdots + x_ne_n$ and

$$\begin{aligned} L(x) &= L(x_1e_1 + x_2e_2 + \cdots + x_ne_n) \\ &= x_1L(e_1) + x_2L(e_2) + \cdots + x_nL(e_n) \end{aligned}$$

where we made use of Proposition 6.1.2. Notice $L(e_i) \in \mathbb{R}^m$ thus there exist constants, say A_{ij}, such that

$$L(e_i) = A_{1i}f_1 + A_{2i}f_2 + \cdots + A_{mi}f_m$$

for each $i = 1, 2, \dots, n$. Let's put it all together,

$$L(x) = \sum_{i=1}^{n} x_iL(e_i) = \sum_{i=1}^{n} x_i \sum_{j=1}^{m} A_{ji}f_j = \sum_{i=1}^{n}\sum_{j=1}^{m} A_{ji}x_if_j = Ax.$$

Notice that $A_{ji} = L(e_i)_j$ for $1 \le i \le n$ and $1 \le j \le m$ hence $A \in \mathbb{R}^{m\times n}$ by its construction. $\square$

The fundamental theorem of algebra allows us to make the following definition.

Definition 6.3.2.

Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation, the matrix $A \in \mathbb{R}^{m\times n}$ such that $L(x) = Ax$ for all $x \in \mathbb{R}^n$ is called the **standard matrix** of L. We denote this by $[L] = A$ or more compactly, $[L_A] = A$, we say that L_A is the linear transformation induced by A. Moreover, the components of the matrix A are found from $A_{ji} = (L(e_i)))_j$.

Example 6.3.3. *Given that $L((x,y,z)) = (x+2y, 3y+4z, 5x+6z)$ for $(x,y,z) \in \mathbb{R}^3$ find the the standard matrix of L. We wish to find a 3×3 matrix such that $L(v) = Av$ for all $v = (x,y,z) \in \mathbb{R}^3$. Write $L(v)$ then collect terms with each coordinate in the domain,*

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} x+2y \\ 3y+4z \\ 5x+6z \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix} + y\begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 4 \\ 6 \end{bmatrix}$$

It's not hard to see that,

$$L\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 5 & 0 & 6 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \Rightarrow \quad A = [L] = \begin{bmatrix} 1 & 2 & 0 \\ 0 & 3 & 4 \\ 5 & 0 & 6 \end{bmatrix}$$

Notice that the columns in A are just as you'd expect from the proof of theorem 6.3.1. $[L] = [L(e_1)|L(e_2)|L(e_3)]$. In future examples I will exploit this observation to save writing.

Example 6.3.4. *Suppose that $L((t,x,y,z)) = (t+x+y+z, z-x, 0, 3t-z)$, find $[L]$.*

$$\begin{array}{l} L(e_1) = L((1,0,0,0)) = (1,0,0,3) \\ L(e_2) = L((0,1,0,0)) = (1,-1,0,0) \\ L(e_3) = L((0,0,1,0)) = (1,0,0,0) \\ L(e_4) = L((0,0,0,1)) = (1,1,0,-1) \end{array} \quad \Rightarrow \quad [L] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 3 & 0 & 0 & -1 \end{bmatrix}.$$

I invite the reader to check my answer here and see that $L(v) = [L]v$ for all $v \in \mathbb{R}^4$ as claimed.

Very well, let's return to the concepts of injective and surjectivity of linear mappings. How do our theorems of LI and spanning inform us about the behaviour of linear transformations? The following pair of theorems summarize the situtation nicely.

Theorem 6.3.5. *linear map is injective iff only zero maps to zero.*

> $L : \mathbb{R}^n \to \mathbb{R}^m$ is an injective linear transformation iff the only solution to the equation $L(x) = 0$ is $x = 0$.

Proof: this is a biconditional statement. I'll prove the converse direction to begin.
($\Leftarrow$) Suppose $L(x) = 0$ iff $x = 0$ to begin. Let $a, b \in \mathbb{R}^n$ and suppose $L(a) = L(b)$. By linearity we have $L(a-b) = L(a) - L(b) = 0$ hence $a - b = 0$ therefore $a = b$ and we find L is injective.
($\Rightarrow$) Suppose L is injective. Suppose $L(x) = 0$. Note $L(0) = 0$ by linearity of L but then by 1-1 property we have $L(x) = L(0)$ implies $x = 0$ hence the unique solution of $L(x) = 0$ is the zero solution. $\square$

Notice this is certainly not true for most functions. For example, $f(x) = x^2$ has $f(0) = 0$ iff $x = 0$. However, f is not one-to-one. The property of linearity gives a very special structure to a mapping. For example, if the values of a linear mapping are given on a basis then there is a unique extension of the formula from that basis to the whole space. Notice[2] that the standard matrix only needs to be given the values of L on the standard basis. That data suffices to fix the values of L everywhere.

Theorem 6.3.6. *injectivity and surjectivity of linear map characterized by its matrix*

$L : \mathbb{R}^n \to \mathbb{R}^m$ is a linear transformation with standard matrix [L] then

1. L is 1-1 iff the columns of $[L]$ are linearly independent,
2. L is onto $\mathbb{R}^m$ iff the columns of $[L]$ span $\mathbb{R}^m$.

Proof: To prove (1.) use Theorem 6.3.5:

$$L \text{ is 1-1} \;\Leftrightarrow\; \Big\{ L(x) = 0 \;\Leftrightarrow\; x = 0 \Big\} \;\Leftrightarrow\; \Big\{ [L]x = 0 \;\Leftrightarrow\; x = 0. \Big\}$$

and the last equation simply states that if a linear combination of columns of L is zero then the coefficients of that linear equation are zero so (1.) follows.

To prove (2.) recall that Theorem 3.3.9 stated that if $A \in \mathbb{R}^{m \times n}$, $v \in \mathbb{R}^n$ then $Av = b$ is consistent for all $b \in \mathbb{R}^m$ iff the columns of A span $\mathbb{R}^m$. To say L is onto $\mathbb{R}^m$ means that for each $b \in \mathbb{R}^m$ there exists $v \in \mathbb{R}^n$ such that $L(v) = b$. But, this is equivalent to saying that $[L]v = b$ is consistent for each $b \in \mathbb{R}^m$ so (2.) follows. $\square$

Example 6.3.7.

1. *You can verify that the linear mappings in Examples 6.2.2, 6.2.3, 6.2.4, 6.2.5, 6.2.6, 6.2.7 and 6.2.8 wer both 1-1 and onto. You can see the columns of the tranformation matrices were both LI and spanned $\mathbb{R}^2$ in each of these examples.*

2. *In constrast, Examples 6.2.9 and 6.2.10 were neither 1-1 nor onto. Moreover, the columns of transformation's matrix were linearly dependent in each of these cases and they did not span $\mathbb{R}^2$. Instead the span of the columns covered only a particular line in the range.*

3. *In Example 6.2.12 the mapping is injective and the columns of A were indeed linearly indpendent. However, the columns do not span $\mathbb{R}^3$ and as expected the mapping is not onto $\mathbb{R}^3$.*

4. *In Example 6.2.13 the mapping is not 1-1 and the columns are obviously linearly dependent. On the other hand, the columns of A do span $\mathbb{R}^2$ and the mapping is onto.*

5. *In Example 6.2.14 the mapping is neither 1-1 nor onto and the columns of the matrix are neither linearly independent nor do they span $\mathbb{R}^2$.*

[2] a similar argument can be made for arbitrary bases of euclidean space or for linear tranformations on finite-dimensional abstract vector spaces.

Inverting a linear transformation is nicely connected to the problem of inverting a matrix.

Theorem 6.3.8. *invertibility of a linear transformation*

> If $L_A : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation induced from $A \in \mathbb{R}^{\ n\times n}$ then L_A is invertible iff A is invertible. Moreover, if A^{-1} exists then $(L_A)^{-1} = L_{A^{-1}}$.

Proof: I'll prove the simple direction. Suppose A^{-1} exists and consider the linear transformation induced by A^{-1}. Note that $L_A(L_{A^{-1}}(x)) = AA^{-1}x = Ix = x$ and $L_{A^{-1}}(L_A(x)) = A^{-1}Ax = Ix = x$ for each $x \in \mathbb{R}^n$ thus L_A is invertible and $(L_A)^{-1} = L_{A^{-1}}$. The proof of the other direction is similar and rests upon the identity $L_A \circ L_B = L_{AB}$ which is shown in the optional Section 6.5. $\square$

6.4 coordinates

Prelude: Think about Cartesian coordinates, if you define positive x, y as the East and North directions and z as the vertical direction with the origin at Sonic on Wards Road then your friend Trogdor defined x, z as the East and North directions and y as the vertical direction with the origin at Sonic on Wards Road then when you compared equations they would not match. To remedy this trouble you and Trogdor should use different notation. We could relable Trogdor coordinates with bars; say $\bar{x}, \bar{y}, \bar{z}$ for the East, vertical, and North directions. Then, to compare equations about villagers you could use the simple equations:

$$\bar{x} = x, \qquad \bar{y} = z, \qquad \bar{z} = y$$

to converse with Trogdor. This is the problem with coordinates for real world problems; there are many choices. Therefore, we should like to develop techniques to change from one choice to another.

If you have a little imagination you might already have wondered why Trogdor chose the same origin as you. Or, why Trogdor didn't use an origin on a train passing at constant velocity. Or worse yet, Trogdor could have used a rotating coordinate system. Translating between moving coordinate systems is more complicated. In fact, it involves a mixture of physics and linear algebra. It is a topic you cover in the Junior level classical mechanics course. I have some notes posted on such things if you wish to read. I will be more boring here, I just include this digression to alert you to the limitations of what we're doing here. Our coordinate change maintains the origin and is fixed in time and space, the coordinates change in a linear way.

6.4.1 the coordinate map for a vector

Given a vector space and a basis[3] we obtain a coordinate system. Given a basis β, for each vector in $v \in V$ we associate a unique[4] column vector $[v]_\beta \in \mathbb{R}$.

Definition 6.4.1.

Suppose $\beta = \{f_1, f_2, \dots, f_n\}$ is a basis for V. If $v \in V$ has

$$v = v_1 f_1 + v_2 f_2 + \cdots + v_n f_n$$

then $[v]_\beta = (v_1 \ v_2 \ \cdots \ v_n)^T \in \mathbb{R}^n$ is called the **coordinate vector** of v with respect to β. The **coordinate map** is $\Phi_\beta : V \to \mathbb{R}^n$ defined by $\Phi_\beta(v) = [v]_\beta$.

Example 6.4.2. *Let* $v = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ *find the coordinates of* v *relative to* β_1, β_2 *and* β_3 *where* $\beta_1 = \{e_1, e_2\}$ *and* $\beta_2 = \{e_1, e_1 + e_2\}$ *and* $\beta_3 = \{e_2, e_1 + e_2\}$*. We'll begin with the standard basis, (I hope you could see this without writing it)*

$$v = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 1e_1 + 3e_2$$

thus $[v]_{\beta_1} = [1 \ 3]^T$*. Find coordinates relative to the other two bases is not quite as obvious. Begin with* β_2*. We wish to find* x, y *such that*

$$v = xe_1 + y(e_1 + e_2)$$

we can just use brute-force,

$$v = e_1 + 3e_2 = xe_1 + y(e_1 + e_2) = (x + y)e_1 + ye_2$$

using linear independence of the standard basis we find $1 = x + y$ *and* $y = 3$ *thus* $x = 1 - 3 = -2$ *and we see* $v = -2e_1 + 3(e_1 + e_2)$ *so* $[v]_{\beta_2} = (-2, 3)$*. This is interesting, the same vector can have different coordinate vectors relative to distinct bases. Finally, let's find coordinates relative to* β_3*. I'll try to be more clever this time: we wish to find* x, y *such that*

$$v = xe_2 + y(e_1 + e_2) \quad \Leftrightarrow \quad \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

---

[3]Technically, the each basis considered in the course is an "ordered basis". This means the set of vectors that forms the basis has an ordering to it. This is more structure than just a plain set since basic set theory does not distinguish $\{1, 2\}$ from $\{2, 1\}$. I should always say "we have an ordered basis" but I will not (and most people do not) say that in this course. Let it be understood that when we list the vectors in a basis they are listed in order and we cannot change that order without changing the basis. For example $v = (1, 2, 3)$ has coordinate vector $[v]_{B_1} = (1, 2, 3)$ with respect to $B_1 = \{e_1, e_2, e_3\}$. On the other hand, if $B_2 = \{e_2, e_1, e_3\}$ then the coordinate vector of v with respect to B_2 is $[v]_{B_2} = (2, 1, 3)$.

[4]see Math 321 notes for proof

We can solve this via the augemented coefficient matrix

$$rref\left[\begin{array}{cc|c} 0 & 1 & 1 \\ 1 & 1 & 3 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 2 \\ 0 & 1 & 1 \end{array}\right] \quad \Leftrightarrow \quad x = 2, \ y = 1.$$

Thus, $[v]_{\beta_3} = (2,1)$. Notice this is precisely the rightmost column in the rref matrix. Perhaps my approach for β_3 is a little like squashing a fly with with a dumptruck. However, once we get to an example with 4-component vectors you may find the matric technique useful.

Example 6.4.3. *Given that $\beta = \{b_1, b_2, b_3, b_4\} = \{e_1 + e_2, e_2 + e_3, e_3 + e_4, e_4\}$ is a basis for $\mathbb{R}^4$ find coordinates for $v = (1,2,3,4) \in \mathbb{R}^4$. Given the discussion in the preceding example it is clear we can find coordinates (x_1, x_2, x_3, x_4) such that $v = \sum_i x_i b_i$ by calculating $rref[b_1|b_2|b_3|b_4|v]$ the rightmost column will be $[v]_\beta$.*

$$rref\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 2 \\ 0 & 1 & 1 & 0 & 3 \\ 0 & 0 & 1 & 1 & 4 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 2 \end{array}\right] \quad \Rightarrow \quad \boxed{[v]_\beta = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}}$$

The proposition below gives us another way to calculate coordinates.

Proposition 6.4.4.

If $\mathbb{R}^n$ has basis $\beta = \{f_1, f_2, \dots, f_n\}$ and we denote $[\beta] = [f_1|f_2|\cdots|f_n]$ then

$$[v]_\beta = [\beta]^{-1}v.$$

Proof: Let $v \in \mathbb{R}^n$ and suppose $v = w_1 f_1 + w_2 f_2 + \cdots + w_n f_n \implies v = [f_i][v]_\beta$ where $[w_i] = [v]_\beta$. Since $\{f_1, f_2, \dots, f_n\}$ are LI, $[\beta] = [f_i]$ is invertible. Thus we find $[v]_\beta = [f_i]^{-1}v$. $\square$

Notice that when $\beta = \{e_1, e_2, \dots, e_n\}$ then $[\beta] = I$ hence $[v]_\beta = v$.

Example 6.4.5. *Work Example 6.4.3 once more using the proposition above. You can calculate the inverse by our usual algorithm:*

$$[\beta]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix} \quad \Rightarrow \quad [v]_\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}$$

This is what we found before. Here's the real beauty, suppose $x = (x_1, x_2, x_3, x_4)$. Calculate,

$$[x]_\beta = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 \\ -x_1 + x_2 \\ x_1 - x_2 + x_3 \\ -x_1 + x_2 - x_3 + x_4 \end{bmatrix}.$$

If we denote the coordinates with respect to the $\beta = \{b_1, b_2, b_3, b_4\}$ *of an arbitrary point by* $(\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4)$ *then the Cartesian coordinates of the point are related by the coordinate change formulas below:*

$$\bar{x}_1 = x_1, \quad \bar{x}_2 = -x_1 + x_2, \quad \bar{x}_3 = x_1 - x_2 + x_3, \quad \bar{x}_4 = -x_1 + x_2 - x_3 + x_4.$$

We can either reach the point by a linear combination of the standard basis or the β *basis.*

$$x = x_1e_1 + x_2e_2 + x_3e_3 + x_4e_4 = \bar{x}_1b_1 + \bar{x}_2b_2 + \bar{x}_3b_3 + \bar{x}_4b_4$$

6.4.2 matrix of linear transformation

This definition is best explained by the diagram later on this page.

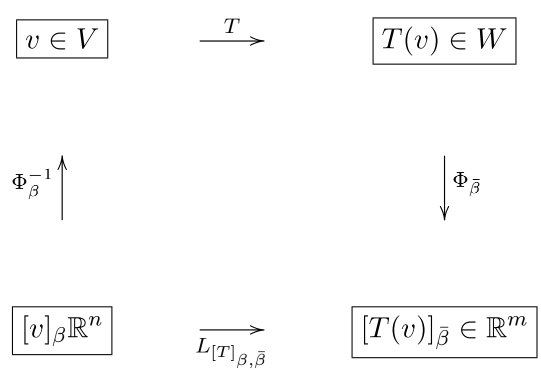
Definition 6.4.6.

Let vector spaces V and W. Let $T : V \to W$ be linear transformation bewteen and suppose $\Phi_\beta : V \to \mathbb{R}^n$ and $\Phi_{\bar{\beta}} : W \to \mathbb{R}^m$ are coordinate mappings with respect to the bases $\beta, \bar{\beta}$ for V, W respective. We define the matrix of T with respect to $\beta, \bar{\beta}$ to be $[T]_{\beta,\bar{\beta}} \in \mathbb{R}^{m\times n}$ to be the standard matrix of the operator $\Phi_{\beta}^{-1} \circ T \circ \Phi_{\bar{\beta}}$.

This means $L_{[T]_{\beta,\bar{\beta}}} = \Phi_{\beta}^{-1} \circ T \circ \Phi_{\bar{\beta}}$. Or, you may find it easier to calculate with $\Phi_{\bar{\beta}} \circ T = L_{[T]_{\beta,\bar{\beta}}} \circ \Phi_\beta$ which suggests we calculate the matrix $[T]_{\beta,\bar{\beta}}$ by inspecting the equation:

$$[T(v)]_{\bar{\beta}} = [T]_{\beta,\bar{\beta}}[v]_\beta.$$

I've denoted where $v, T(v), ...$ are in the diagram below[5] x by the $m \times n$ matrix $[T]_{\beta,\bar{\beta}}$.



Example 6.4.7. *Consider* $V = W = \mathbb{R}^{\,2\times 2}$ *and define* $T(A) = A^T$*. Furthermore, use the usual matrix-unit basis* $\beta = \{E_{11}, E_{12}, E_{21}, E_{22}\}$*. Let* $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ *thus* $T(A) = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$*. We want to*

[5] remember the notation $L_{[T]_{\beta,\bar{\beta}}}$ indicates the operation of left multiplication by the matrix $[T]_{\beta,\bar{\beta}}$; that is $L_{[T]_{\beta,\bar{\beta}}}(x) = [T]_{\beta,\bar{\beta}}x$ for all x.

find a matrix $[T]_{\beta,\beta}$ *such that* $[T]_{\beta,\beta}[A]_\beta = [T(A)]_\beta$

$$[T]_{\beta,\beta}\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} a \\ c \\ b \\ d \end{bmatrix} \quad \Rightarrow \quad [T]_{\beta,\beta} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Just for fun: we know that $(A^T)^T = A$. *What does this suggest about the matrix above? Hint: square the matrix see what happens.*

The method I used in the example above can be replaced by the method suggested by the proposition to follow. You can use whatever makes more sense. If you understand the diagram then you don't really need to memorize.

Proposition 6.4.8.

Given the data in the preceding definition,

$$col_i([T]_{\beta,\bar{\beta}}) = \Phi_{\bar{\beta}}(T(\Phi_\beta^{-1}(e_i))).$$

Proof: Apply Theorem 2.5.12. □

Let us examine see how this is fleshed out in a concrete example.

Example 6.4.9. *Let* $\beta = \{1, x, x^2\}$ *be the basis for* P_2 *and consider the derivative mapping* $D : P_2 \to P_2$. *Find the matrix of* D *assuming that* P_2 *has coordinates with respect to* β *on both copies of* P_2. *Define and observe* $\Phi_\beta(x^n) = e_{n+1}$ *whereas* $\Phi_\beta^{-1}(e_n) = x^{n-1}$ *for* $n = 0, 1, 2$. *Recall* $D(ax^2 + bx + c) = 2ax + bx$.

$$col_1([D]_{\beta,\beta}) = \Phi_\beta(D(\Phi_\beta^{-1}(e_1))) = \Phi_\beta(D(1)) = \Phi_\beta(0) = 0$$
$$col_2([D]_{\beta,\beta}) = \Phi_\beta(D(\Phi_\beta^{-1}(e_2))) = \Phi_\beta(D(x)) = \Phi_\beta(1) = e_1$$
$$col_3([D]_{\beta,\beta}) = \Phi_\beta(D(\Phi_\beta^{-1}(e_3))) = \Phi_\beta(D(x^2)) = \Phi_\beta(2x) = 2e_2$$

Therefore we find $[D]_{\beta,\beta} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. *Calculate* D^3. *Is this surprising?*

Example 6.4.10. *Let* $\beta = \{(1, -1), (1, 1)\}$. *Define* T *by* $T((x, y)) = (2x + 4y,\ 6y)$. *Find the matrix of* T *with respect to* β. *Observe that*

$$[\beta]^{-1} = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} x \\ y \end{bmatrix}_\beta = \frac{1}{2}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(x-y) \\ \frac{1}{2}(x+y) \end{bmatrix}$$

If $v = (x, y)$ *then* $[v]_\beta = \frac{1}{2}(x - y, x + y)$. *Consequently,*

$$[T(v)]_\beta = [\ (2x + 4y,\ 6y)\ ]_\beta = \frac{1}{2}(2x + 4y - 6y, 2x + 4y + 6y) = (x - y, x + 5y).$$

We seek a matrix $[T]_{\beta,\beta} = \left[\begin{array}{cc} a & b \\ c & d \end{array}\right]$ *such that* $[T]_{\beta,\beta}[v]_\beta = [T(v)]_\beta$*. Consider*

$$\left[\begin{array}{cc} a & b \\ c & d \end{array}\right]\left[\begin{array}{c} \frac{1}{2}(x-y) \\ \frac{1}{2}(x+y) \end{array}\right] = \left[\begin{array}{c} x-y \\ x+5y \end{array}\right]$$

If $x = 1$ *and* $y = 1$ *then* $b = 0$ *and* $d = 6$*. On the other hand, if we set* $x = 1$ *and* $y = -1$ *then we obtain* $a = 2$ *and* $c = -4$*. We find* $[T]_{\beta,\beta} = \left[\begin{array}{cc} 2 & 0 \\ -4 & 6 \end{array}\right]$*. In this case, there is a better way. See Example 6.4.13*

6.4.3 coordinate change

We narrow the focus to euclidean space $\mathbb{R}^n$ in this section. We consider the case where the domain and range are the same dimension since this is the case of most interest.

Proposition 6.4.11. *Coordinate change for vectors and linear transformations on* $\mathbb{R}^n$*.*

Let $\mathbb{R}^n$ have bases $\beta = \{f_1, f_2, \dots, f_n\}$ and $\bar{\beta} = \{\bar{f}_1, \bar{f}_2, \dots, \bar{f}_n\}$ such that $[\beta]P = [\bar{\beta}]$ where I denoted $[\beta] = [f_1|f_2|\cdots|f_n]$ and $[\bar{\beta}] = [\bar{f}_1|\bar{f}_2|\cdots|\bar{f}_n]$. If $v = \sum_i v_i f_i$ and $v = \sum_j \bar{v}_j \bar{f}_j$ we denote $[v]_\beta = [v_i]$ and $[v]_{\bar{\beta}} = [\bar{v}_j]$ and the coordinate vectors of v are related by

$$[v]_{\bar{\beta}} = P^{-1}[v]_\beta$$

Moreover, if $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear operator then

$$[T]_{\bar{\beta},\bar{\beta}} = P^{-1}[T]_{\beta,\beta}P.$$

Proof: Given the data above, note we can write $\sum_i v_i f_i = [\beta][v]_\beta$ and $\sum_j \bar{v}_j \bar{f}_j = [\bar{\beta}][v]_{\bar{\beta}}$ (we can do this since we are in $\mathbb{R}^n$)

$$v = [\beta][v]_\beta = [\beta]PP^{-1}[v]_\beta = [\bar{\beta}]P^{-1}[v]_\beta$$

However, we also have $v = [\bar{\beta}][v]_{\bar{\beta}}$. But $[\bar{\beta}]$ is an invertible matrix thus $[\bar{\beta}][v]_{\bar{\beta}} = [\bar{\beta}]P^{-1}[v]_\beta$ implies $[v]_{\bar{\beta}} = P^{-1}[v]_\beta$.

We defined $[T]_{\bar{\beta},\bar{\beta}}$ implicitly through the equation $T = \Phi_{\bar{\beta}}^{-1} \circ L_{[T]_{\bar{\beta},\bar{\beta}}} \circ \Phi_{\bar{\beta}}$. In this special case the coordinate maps and their inverses are matrix multiplication as described by Proposition 6.4.4 and we calculate

$$T = L_{\bar{\beta}} \circ L_{[T]_{\bar{\beta},\bar{\beta}}} \circ L_{\bar{\beta}^{-1}}$$

But the matrix of a composite of linear transformations is the product the matrices of those transformations, thus

$$T = L_{[\bar{\beta}][T]_{\bar{\beta},\bar{\beta}}[\bar{\beta}]^{-1}}$$

Therefore, the standard matrix of T is $[T] = [\bar{\beta}][T]_{\bar{\beta},\bar{\beta}}[\bar{\beta}]^{-1}$. By the same argument we find $[T] = [\beta][T]_{\beta,\beta}[\beta]^{-1}$. Thus,

$$[T] = [\bar{\beta}][T]_{\bar{\beta},\bar{\beta}}[\bar{\beta}]^{-1} = [\beta][T]_{\beta,\beta}[\beta]^{-1} \quad \Rightarrow \quad [T]_{\bar{\beta},\bar{\beta}} = [\bar{\beta}]^{-1}[\beta][T]_{\beta,\beta}[\beta]^{-1}[\bar{\beta}]$$

However, we defined P to be the matrix which satisfies $[\beta]P = [\bar{\beta}]$ thus $P = [\beta]^{-1}[\bar{\beta}]$ and $P^{-1} = [\bar{\beta}]^{-1}[\beta]$. $\square$.

We should pause to consider a special case. If $\bar{\beta}$ is the standard basis then $[\bar{\beta}] = I$ and the change of basis matrix P satisfies $[\beta]P = I$ hence $P = \beta^{-1}$. It follows that

$$[T] = (\beta^{-1})^{-1}[T]_{\beta,\beta}\beta^{-1} \quad \Rightarrow \quad [T]_{\beta,\beta} = [\beta]^{-1}[T][\beta]$$

The Corollary below follows:

Corollary 6.4.12. *how to calculate the matrix w.r.t. nonstandard basis directly*

If $T : \mathbb{R}^n \to \mathbb{R}^n$ has standard matrix $[T]$ and β is a basis for $\mathbb{R}^n$ then $[T]_{\beta,\beta} = [\beta]^{-1}[T][\beta]$.

We worked Example 6.4.10 in the previous subsection without the insight above. You can compare and see which method you think is best.

Example 6.4.13. *Let $\beta = \{(1,-1),(1,1)\}$. Define T by $T((x,y)) = (2x+4y,\ 6y)$. Note $[T] = \left[\begin{array}{cc} 2 & 4 \\ 0 & 6 \end{array}\right]$. The inverse is found by the usual 2×2 formula,*

$$[\beta]^{-1} = \left[\begin{array}{cc} 1 & 1 \\ -1 & 1 \end{array}\right]^{-1} = \frac{1}{2}\left[\begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array}\right]$$

Calculate,

$$\begin{aligned}
[T]_{\beta,\beta} = [\beta]^{-1}[T][\beta] &= \frac{1}{2}\left[\begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array}\right]\left[\begin{array}{cc} 2 & 4 \\ 0 & 6 \end{array}\right]\left[\begin{array}{cc} 1 & 1 \\ -1 & 1 \end{array}\right] \\
&= \frac{1}{2}\left[\begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array}\right]\left[\begin{array}{cc} -2 & 6 \\ -6 & 6 \end{array}\right] \\
&= \frac{1}{2}\left[\begin{array}{cc} 4 & 0 \\ -8 & 12 \end{array}\right] \\
&= \left[\begin{array}{cc} 2 & 0 \\ -4 & 6 \end{array}\right].
\end{aligned}$$

Example 6.4.14. *Let $\beta = \{(1,1),(1,-1)\}$ and $\gamma = \{(1,0),(1,1)\}$ be bases for $\mathbb{R}^2$. Find $[v]_\beta$ and $[v]_\gamma$ if $v = (2,4)$. Let me frame the problem, we wish to solve:*

$$v = [\beta][v]_\beta \qquad \textit{and} \qquad v = [\gamma][v]_\gamma$$

where I'm using the basis in brackets to denote the matrix formed by concatenating the basis into a single matrix,

$$[\beta] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \textit{and} \qquad [\gamma] = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

This is the 2×2 case so we can calculate the inverse from our handy-dandy formula:

$$[\beta]^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \qquad \textit{and} \qquad [\gamma]^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

Then multiplication by inverse yields $[v]_\beta = [\beta]^{-1}v$ and $[v]_\gamma = [\gamma]^{-1}v$ thus:

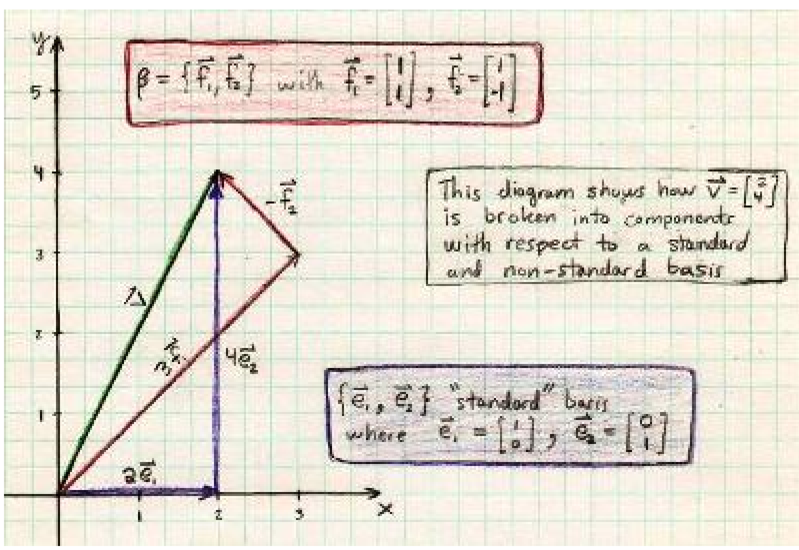
$$[v]_\beta = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ -1 \end{bmatrix} \qquad \textit{and} \qquad [v]_\gamma = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 4 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix}$$

Let's verify the relation of $[v]_\gamma$ and $[v]_\beta$ relative to the change of basis matrix we denoted by P in the prop; we hope to find $[v]_\gamma = P^{-1}[v]_\beta$ (note γ is playing the role of $\bar{\beta}$ in the statement of the prop.)

$$[\beta]P = [\gamma] \quad \Rightarrow \quad P^{-1} = [\gamma]^{-1}[\beta] = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}$$

Consider then (as a check on our calculations and also the proposition)

$$P^{-1}[v]_\beta = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \end{bmatrix} = [v]_\gamma \quad \checkmark$$



Now that we've seen an example, let's find $[v]_\beta$ for an arbitrary $v = (x, y)$,

$$[v]_\beta = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{1}{2}(x+y) \\ \frac{1}{2}(x-y) \end{bmatrix}$$

If we denote $[v]_\beta = (\bar{x}, \bar{y})$ *then we can understand the calculation above as the relation between the barred and standard coordinates:*

$$\bar{x} = \tfrac{1}{2}(x+y) \qquad \bar{y} = \tfrac{1}{2}(x-y)$$

Conversely, we can solve these for x, y *to find the inverse transformations:*

$$x = \bar{x} + \bar{y} \qquad y = \bar{x} - \bar{y}.$$

Similar calculations are possible with respect to the γ*-basis.*

Example 6.4.15. *Let* $\bar{\beta} = \{(1,0,1),(0,1,1),(4,3,1)\}$. *Furthermore, define a linear transformation* $T : \mathbb{R}^3 \to \mathbb{R}^3$ *by the rule* $T((x,y,z)) = (2x-2y+2z,\ x-z,\ 2x-3y+2z)$. *Find the matrix of* T *with respect to the basis* β. *Note first that the standard basis is read from the rule:*

$$T\left(\begin{bmatrix} x \\ y \\ z \end{bmatrix}\right) = \begin{bmatrix} 2x-2y+2z \\ x-z \\ 2x-3y+2z \end{bmatrix} = \begin{bmatrix} 2 & -2 & 2 \\ 1 & 0 & -1 \\ 2 & -3 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Consider then (omitting the details of calculating P^{-1}*) and applying Corollary 6.4.12:*

$$\begin{aligned} P^{-1}[T]P &= \begin{bmatrix} 1/3 & -2/3 & 2/3 \\ -1/2 & 1/2 & 1/2 \\ 1/6 & 1/6 & -1/6 \end{bmatrix}\begin{bmatrix} 2 & -2 & 2 \\ 1 & 0 & -1 \\ 2 & -3 & 2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & 3 \\ 1 & 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1/3 & -2/3 & 2/3 \\ -1/2 & 1/2 & 1/2 \\ 1/6 & 1/6 & -1/6 \end{bmatrix}\begin{bmatrix} 4 & 0 & 4 \\ 0 & -1 & 3 \\ 4 & -1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Therefore, in the $\bar{\beta}$*-coordinates the linear operator* T *takes on a particularly simple form:*

$$\left[T\left(\begin{bmatrix} \bar{x} \\ \bar{y} \\ \bar{z} \end{bmatrix}\right)\right]_{\bar{\beta}} = \begin{bmatrix} 4\bar{x} \\ -\bar{y} \\ \bar{z} \end{bmatrix}$$

In other words, if $\bar{\beta} = \{f_1, f_2, f_3\}$ *then*

$$T((\bar{x}, \bar{y}, \bar{z})) = 4\bar{x}f_1 - \bar{y}f_2 + \bar{z}f_3$$

This linear transformation acts in a special way in the f_1, f_2 *and* f_3 *directions. The basis we considered here is actually what is known as a an* eigenbasis *for* T.

If two matrices are related as $[T]$ and $[T]_{\beta,\beta}$ are related above then the matrices are similar

Definition 6.4.16.

Let $A, B \in \mathbb{R}^{n\times n}$ we say A and B are **similar matrices** and write $A \sim B$ if there exists an invertible matrix P such that $B = P^{-1}AP$. We say that B is obtained from A by a **similarity transformation**.

Past this chapter, much of what we do is in one way or another aimed towards finding the right similarity transformation to simplify a given problem.

6.5 new linear transformations from old*

This section is not required reading. This optional section shows how the set of linear transformations on a pair of vector spaces V, W is itself a vector space $L(V, W)$ because we can add and scalar multiply linear transformations. We then turn to the problem of composition of linear operators on $\mathbb{R}^n$ where we show that the definition of matrix multiplication arises naturally from the formula for the composition of linear mappings.

Definition 6.5.1.

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^n \to \mathbb{R}^m$ are linear transformations then we define $T+S, T-S$ and cT for any $c \in \mathbb{R}$ by the rules

$$(T+S)(x) = T(x)+S(x). \quad (T-S)(x) = T(x)-S(x), \quad (cT)(x) = cT(x)$$

for all $x \in \mathbb{R}^n$.

The following does say something new. Notice I'm talking about adding the transformations themselves not the points in the domain or range.

Proposition 6.5.2.

The sum, difference or scalar multiple of a linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ are once more a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^m$.

Proof: I'll be greedy and prove all three at once:

$$\begin{aligned}
(T+cS)(x+by) &= T(x+by)+(cS)(x+by) && \text{defn. of sum of transformations}\\
&= T(x+by)+cS(x+by) && \text{defn. of scalar mult. of transformations}\\
&= T(x)+bT(y)+c[S(x)+bS(y)] && \text{linearity of } S \text{ and } T\\
&= T(x)+cS(x)+b[T(y)+cS(y)] && \text{vector algebra props.}\\
&= (T+cS)(x)+b(T+cS)(y) && \text{again, defn. of sum and scal. mult. of trans.}
\end{aligned}$$

Let $c=1$ and $b=1$ to see $T+S$ is additive. Let $c=1$ and $x=0$ to see $T+S$ is homogeneous. Let $c=-1$ and $b=1$ to see $T-S$ is additive. Let $c=-1$ and $x=0$ to see $T-S$ is homogeneous. Finally, let $T=0$ to see cS is additive ($b=1$) and homogeneous ($x=0$). $\square$

Proposition 6.5.3.

Suppose $T : \mathbb{R}^n \to \mathbb{R}^m$ and $S : \mathbb{R}^n \to \mathbb{R}^m$ are linear transformations then

$$(1.)\ [T+S] = [T]+[S], \quad (2.)\ [T-S] = [T]-[S], \quad (3.)\ [cT] = c[T].$$

In words, the standard matrix of the sum, difference or scalar multiple of linear transformations is the sum, difference or scalar multiple of the standard matrices of the respsective linear transformations.

Proof: Note $(T+cS)(e_j) = T(e_j)+cS(e_j)$ hence $((T+cS)(e_j))_i = (T(e_j))_i + c(S(e_j))_i$ for all i, j hence $[T+cS] = [T]+c[S]$. $\square$

Example 6.5.4. *Suppose $T(x,y) = (x+y, x-y)$ and $S(x,y) = (2x, 3y)$. It's easy to see that*

$$[T] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \text{ and } [S] = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix} \quad \Rightarrow \quad [T+S] = [T]+[S] = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

Therefore, $(T+S)(x,y) = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3x+y \\ x+2y \end{bmatrix} = (3x+y, x+2y)$. *Naturally this is the same formula that we would obtain through direct addition of the formulas of T and S.*

6.5.1 motivation of matrix multiplication

The definition of matrix multiplication is natural for a variety of reasons. Let's think about composing two linear transformations. This will lead us to see why our definition of matrix multiplication is natural.

Example 6.5.5. *Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be defined by*

$$T((x,y)) = (x+y, 2x-y)$$

for all $(x,y) \in \mathbb{R}^2$. Also let $S : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by

$$S((x,y)) = (x, x, 3x+4y).$$

for all $(x,y) \in \mathbb{R}^2$. We calculate the composite as follows:

$$\begin{aligned} (S \circ T)((x,y)) &= S(T((x,y))) \\ &= S((x+y, 2x-y)) \\ &= (x+y, x+y, 3(x+y)+4(2x-y)) \\ &= (x+y, x+y, 11x-y) \end{aligned}$$

Notice we can write the formula above as a matrix multiplication,

$$(S \circ T)((x,y)) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 11 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} \quad \Rightarrow \quad [S \circ T] = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 11 & -1 \end{bmatrix}.$$

Notice that the standard matrices of S and T are:

$$[S] = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 3 & 4 \end{bmatrix} \qquad [T] = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$$

It's easy to see that $[S \circ T] = [S][T]$ (as we should expect since these are linear operators)

Proposition 6.5.6.

$L_1 : \mathbb{R}^m \to \mathbb{R}^n$ and $L_2 : \mathbb{R}^n \to \mathbb{R}^p$ are linear transformations then $L_2 \circ L_1 : \mathbb{R}^m \to \mathbb{R}^p$ is a linear transformation with matrix $[L_2 \circ L_1]$ such that

$$[L_2 \circ L_1]_{ij} = \sum_{k=1}^{n} [L_2]_{ik}[L_1]_{kj}$$

for all $i = 1, 2, \dots p$ and $j = 1, 2 \dots, m$.

Proof: Let $x, y \in V_1$ and $c \in \mathbb{R}$,

$$\begin{aligned} (L_2 \circ L_1)(x + cy) &= L_2(L_1(x + cy)) && \text{defn. of composite} \\ &= L_2(L_1(x) + cL_1(y)) && L_1 \text{ is linear trans.} \\ &= L_2(L_1(x)) + cL_2(L_1(y)) && L_2 \text{ is linear trans.} \\ &= (L_2 \circ L_1)(x) + c(L_2 \circ L_1)(y) && \text{defn. of composite} \end{aligned}$$

thus $L_2 \circ L_1$ is a linear transformation. To find the matrix of the composite we need only calculate its action on the standard basis: by definition, $[L_2 \circ L_1]_{ij} = ((L_2 \circ L_1)(e_j))_i$, observe

$$\begin{aligned} (L_2 \circ L_1)(e_j) &= L_2(L_1(e_j)) \\ &= L_2([L_1]e_j) \\ &= L_2(\sum_k [L_1]_{kj} e_k) \\ &= \sum_k [L_1]_{kj} L_2(e_k) \\ &= \sum_k [L_1]_{kj} [L_2] e_k \\ &= \sum_k [L_1]_{kj} \sum_i [L_2]_{ik} e_i \\ &= \sum_k \sum_i [L_2]_{ik} [L_1]_{kj} e_i \\ &= \sum_i \left[\sum_k [L_2]_{ik} [L_1]_{kj} \right] e_i. \end{aligned}$$

Therefore, $[L_2 \circ L_1]_{ij} = \sum_k [L_2]_{jk}[L_1]_{ki}$ and Item (2.) follows. $\square$

In other words, we defined matrix multiplication such that the matrix of a composite is simply the product of the composed transformation's matrices. Originally the definition of matrix multiplication was given to help unravel subsititutions. Anton mentions that Eisenstein, a student of Gauss, is credited with finding[6] the matrix product.

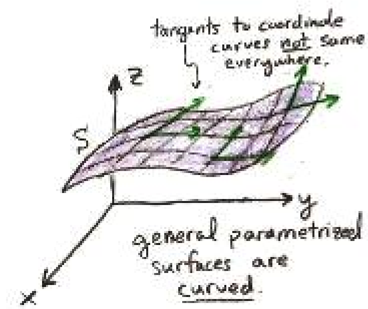
6.6 applications

Geometry is conveniently described by parametrizations. The number of parameters needed to map out some object is the dimension of the object. For example, the rule $t \mapsto \vec{r}(t)$ describes a curve in $\mathbb{R}^n$. Of course we have the most experience in the cases $\vec{r} = \langle x, y \rangle$ or $\vec{r} = \langle x, y, z \rangle$, these give so-called *planar curves* or *space curves* respectively. Generally, a mapping from $\gamma : \mathbb{R} \to S$ where S is some space[7] is called a *path*. The point set $\gamma(S)$ can be identified as a sort of copy of $\mathbb{R}$ which resides in S.

Next, we can consider mappings from $\mathbb{R}^2$ to some space S. In the case $S = \mathbb{R}^3$ we use $X(u, v) = \langle x(u, v), y(u, v), z(u, v) \rangle$ to parametrize a surface. For example,

$$X(\phi, \theta) = \langle \cos(\theta)\sin(\phi), \sin(\theta)\sin(\phi), \cos(\phi) \rangle$$

parametrizes a sphere if we insist that the angles $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$. We call ϕ and θ coordinates on the sphere, however, these are not coordinates in the technical sense later defined in this course. These are so-called *curvelinear coordinates*. Generally a surface in some space is sort-of a copy of $\mathbb{R}^2$ (well, to be more precise it resembles some subset of $\mathbb{R}^2$).



Past the case of a surface we can talk about volumes which are parametrized by three parameters. A volume would have to be embedded into some space which had at least 3 dimensions. For the same reason we can only place a surface in a space with at least 2 dimensions. Perhaps you'd be interested to learn that in relativity theory one considers the world-volume that a particle traces out through spacetime, this is a hyper-volume, it's a 4-dimensional subset of 4-dimensional spacetime.

[6] or creating

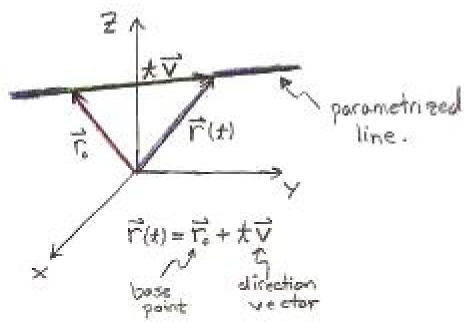
[7] here S could be a set of matrices or functions or an abstract manifold... the concept of a path is very general

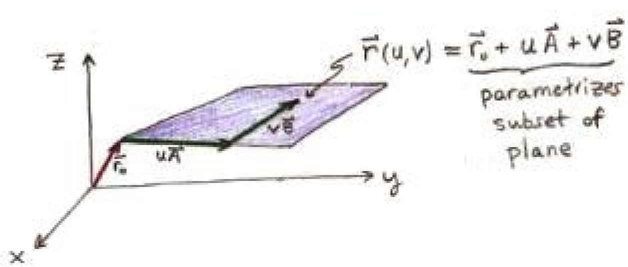
Let me be a little more technical, if the space we consider is to be a k-dimensional parametric subspace of S then that means there exists an **invertible** mapping $X : U \subseteq \mathbb{R}^k \to S \subseteq \mathbb{R}^n$. It turns out that for $S = \mathbb{R}^n$ where $n \geq k$ the condition that X be invertible means that the derivative $D_pX : T_pU \to T_{X(p)}S$ must be an invertible linear mapping at each point p in the parameter space U. This in turn means that the tangent-vectors to the coordinate curves must come together to form a linearly independent set. Linear independence is key.

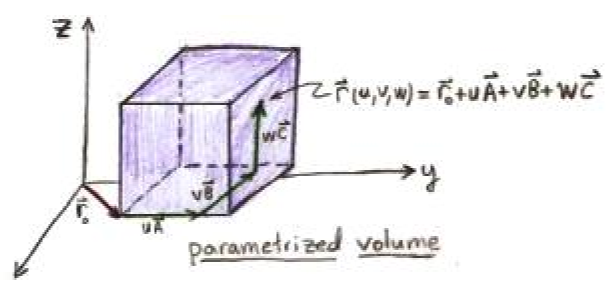
Curvy surfaces and volumes and parametrizations that describe them analytically involve a fair amount of theory which I have only begun to sketch here. However, if we limit our discussion to **affine subspaces** of $\mathbb{R}^n$ we can be explicit. Let me go ahead and write the general form for a line, surface, volume etc... in terms of linearly indpendent vectors $\vec{A}, \vec{B}, \vec{C}, \ldots$

$$\vec{r}(u) = \vec{r}_o + u\vec{A}$$
$$X(u, v) = \vec{r}_o + u\vec{A} + v\vec{B}$$
$$X(u, v, w) = \vec{r}_o + u\vec{A} + v\vec{B} + w\vec{C}$$

I hope you you get the idea.







In each case the parameters give an invertible map only if the vectors are linearly independent. If there was some linear dependence then the dimension of the subspace would collapse. For example,

$$X(u, v) = \langle 1, 1, 1\rangle + u\langle 1, 0, 1\rangle + v\langle 2, 0, 2\rangle$$

appears to give a plane, but upon further inspection you'll notice

$$X(u, v) = \langle 1 + u + 2v, 1, 1 + u + 2v\rangle = \langle 1, 1, 1\rangle + (u + 2v)\langle 1, 0, 1\rangle$$

which reveals this is just a line with direction-vector $\langle 1, 0, 1\rangle$ and parameter $u + 2v$.

Chapter 7

eigenvalues and eigenvectors

The terms eigenvalue and vector originate from the German school of mathematics which was very influential in the early 20-th century. Heisenberg's formulation of quantum mechanics gave new importance to linear algebra and in particular the algebraic structure of matrices. In finite dimensional quantum systems the symmetries of the system were realized by linear operators. These operators acted on states of the system which formed a complex vector space called Hilbert Space. [1]

Operators representing momentum, energy, spin or angular momentum operate on a physical system represented by a sum of eigenfunctions. The eigenvalues then account for possible value which could be measured in an experiment. Generally, quantum mechanics involves complex vector spaces and infinite dimensional vector spaces however many of the mathematical difficulties are already present in our study of linear algebra. For example, one important question is how does one pick a set of states which diagonalize an operator? Moreover, if one operator is diagonalized by a particular basis then can a second operator be diagonalized simultaneously? Linear algebra, and in particular eigenvectors help give an answer for these questions. [2]

Beyond, or perhaps I should say before, quantum mechanics eigenvectors have great application in classical mechanics, optics, population growth, systems of differential equations, chaos theory, difference equations and much much more. They are a fundmental tool which allow us to pick apart a matrix into its very core. Diagonalization of matrices almost always allow us to see the nature of a system more clearly.

However, not all matrices are diagonalizable. It turns out that any matrix is similar to a Jordan Block matrix. Moreover, the similarity transformation is accomplished via a matrix formed from concatenating generalized eigenvectors. When there are enough ordinary eigenvectors then the

[1]Hilbert Spaces and infinite dimensional linear algebra are typically discussed in graduate linear algebra and/or the graduate course in functional analysis, we focus on the basics in this course.

[2]in addition to linear algebra one should also study group theory. In particular, matrix Lie groups and their representation theory explains most of what we call "chemistry". Magic numbers, electronic numbers, etc... all of these are eigenvalues which label the states of the so-called Casimir operators

Jordan Form of the matrix is actually a diagonal matrix. The general theory for Jordan Forms, in particular the proof of the existence of a Jordan Basis, is rather involved. I will forego typical worries about existence and just show you a few examples. I feel this is important because the Jordan Form actually does present itself in applications.

In my Chapter 9 (which I'm covering lightly this semester) we explore how eigenvectors and the Jordan form are connected to solutions of systems of differential equations. The double root solution for constant coefficient 2nd order ODEs actually has the Jordan form hiding in the details. The matrix exponential allows for elegant solutions of any system of differential equations. My approach is similar to that given in the text on DEqns by Nagel, Saff and Snider (the text for math 334). However, I should mention that if you wish to understand generalized eigenvectors and Jordan forms in the abstract then you should really engage in a serious study of modules. If you build a vector space over a ring instead of a field then you get a module. Many of the same theorems hold, if you are interested I would be happy to point you to some sources to begin reading. I would be a good topic for an independent study to follow this course.

Finally, there is the case of complex eigenvalues and complex eigenvectors. These have use in the real case. A general princple for linear systems is that if a complex system has a solution then the corresponding real system will inherit two solutions from the real and imaginary parts of the complex solution. Complex eigenvalues abound in applications. For example, rotation matrices have complex eigenvalues. We'll find that complex eigenvectors are useful and not much more trouble than the real case. The diagonalization provided from complex eigenvectors provides a factorization of the matrix into complex matrices. We examine how to convert such factorizations in terms of rotations.

7.1 why eigenvectors?

In this section I attempt to motivate why eigenvectors are natural to study for both mathematical and physical reasons. In fact, you probably could write a book just on this question.

7.1.1 quantum mechanics

Physically measureable quantities are described by operators and states in quantum mechanics[3]. The operators are linear operators and the states are usually taken to be the eigenvectors with respect to a physical quantity of interest. For example:

$$\hat{p}|p>= p|p> \qquad \hat{J}^2|j>= j(j+1)|j> \qquad \hat{H}|E>= E|E>$$

In the above the eigenvalues were $p, j(j+1)$ and E. Physically, p is the momentum, $j(j+1)$ is the value of the square of the magnitude of the total angular momentum and E is the energy. The exact mathematical formulation of the eigenstates of momentum, energy and angular momentum is

[3] you can skip this if you're not a physics major, but maybe you're interested despite the lack of direct relevance to your major. Maybe your interested in an education not a degree. I believe this is possible so I write these words

in general a difficult problem and well-beyond the scope of the mathematics we cover this semester. You have to study Hilbert space which is an infinite-dimensional vector space with rather special properties. In any event, if the physical system has nice boundary conditions then the quantum mechanics gives mathematics which is within the reach of undergraduate linear algebra. For example, one of the very interesting aspects of quantum mechanics is that we can only measure a certain pairs of operators simultaneously. Such operators have eigenstates which are simultaneously eigenstates of both operators at once. The careful study of how to label states with respect to the energy operator (called the Hamiltonian) and some other commuting symmetry operator (like isospin or angular momentum etc...) gives rise to what we call Chemistry. In other words, Chemistry is largely the tabulation of the specific interworkings of eigenstates as the correlate to the energy, momentum and spin operators (this is a small part of what is known as *representation theory* in modern mathematics). I may ask a question about simultaneous diagonalization. This is a hard topic compared to most we study.

7.1.2 stochastic matrices

Definition 7.1.1.

Let $P \in \mathbb{R}^{\,n\times n}$ with $P_{ij} \geq 0$ for all i, j. If the sum of the entries in any column of P is one then we say P is a stochastic matrix.

Example 7.1.2. Stochastic Matrix: *A medical researcher*[4] *is studying the spread of a virus in 1000 lab. mice. During any given week it's estimated that there is an 80% probability that a mouse will overcome the virus, and during the same week there is an 10% likelyhood a healthy mouse will become infected. Suppose 100 mice are infected to start, (a.) how many sick next week? (b.) how many sick in 2 weeks ? (c.) after many many weeks what is the steady state solution?*

$$I_k = \text{infected mice at beginning of week } k \qquad P = \begin{bmatrix} 0.2 & 0.1 \\ 0.8 & 0.9 \end{bmatrix}$$
$$N_k = \text{noninfected mice at beginning of week } k$$

We can study the evolution of the system through successive weeks by multiply the state-vector $X_k = [I_k, N_k]$ *by the probability transition matrix* P *given above. Notice we are given that* $X_1 = (100, 900)$. *Calculate then,*

$$X_2 = \begin{bmatrix} 0.2 & 0.1 \\ 0.8 & 0.9 \end{bmatrix} \begin{bmatrix} 100 \\ 900 \end{bmatrix} = \begin{bmatrix} 110 \\ 890 \end{bmatrix}$$

After one week there are 110 infected mice Continuing to the next week,

$$X_3 = \begin{bmatrix} 0.2 & 0.1 \\ 0.8 & 0.9 \end{bmatrix} \begin{bmatrix} 110 \\ 890 \end{bmatrix} = \begin{bmatrix} 111 \\ 889 \end{bmatrix}$$

After two weeks we have 111 mice infected. What happens as $k \to \infty$*? Generally we have* $X_k = PX_{k-1}$*. Note that as* k *gets large there is little difference between* k *and* $k-1$*, in the limit they*

[4] this example and most of the other applied examples in these notes are borrowed from my undergraduate linear algebra course taught from Larson's text by Dr. Terry Anderson of Appalachian State University

both tend to infinity. We define the steady-state solution to be $X^* = \lim_{k\to\infty} X_k$. *Taking the limit of* $X_k = PX_{k-1}$ *as* $k \to \infty$ *we obtain the requirement* $X^* = PX^*$. *In other words, the steady state solution is found from solving* $(P - I)X^* = 0$. *For the example considered here we find,*

$$(P-I)X^* = \begin{bmatrix} -0.8 & 0.1 \\ 0.8 & -0.1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = 0 \qquad v = 8u \qquad X^* = \begin{bmatrix} u \\ 8u \end{bmatrix}$$

However, by conservation of mice, $u + v = 1000$ *hence* $9u = 1000$ *and* $u = 111.\bar{1}$ *thus the steady state can be shown to be* $X^* = [111.\bar{1}, 888.\bar{8}]$

Example 7.1.3. Diagonal matrices are nice: *Suppose that demand for doorknobs halves every week while the demand for yo-yos it cut to 1/3 of the previous week's demand every week due to an amazingly bad advertising campaign*[5]. *At the beginning there is demand for* 2 *doorknobs and* 5 *yo-yos.*

$$D_k = \text{demand for doorknobs at beginning of week } k \qquad P = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}$$
$$Y_k = \text{demand for yo-yos at beginning of week } k$$

We can study the evolution of the system through successive weeks by multiply the state-vector $X_k = (D_k, Y_k)$ *by the transition matrix* P *given above. Notice we are given that* $X_1 = (2, 5)$. *Calculate then,*

$$X_2 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 5/3 \end{bmatrix}$$

Notice that we can actually calculate the k-th state vector as follows:

$$X_k = P^k X_1 = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix}^k \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2^{-k} & 0 \\ 0 & 3^{-k} \end{bmatrix}^k \begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2^{-k+1} \\ 5(3^{-k}) \end{bmatrix}$$

Therefore, assuming this silly model holds for 100 weeks, we can calculate the 100*-the step in the process easily,*

$$X_{100} = P^{100} X_1 = \begin{bmatrix} 2^{-101} \\ 5(3^{-100}) \end{bmatrix}$$

Notice that for this example the analogue of X^* *is the zero vector since as* $k \to \infty$ *we find* X_k *has components which both go to zero.*

For some systems we'll find a special state we called the "steady-state" for the system. If the system was attracted to some particular final state as $t \to \infty$ then that state satisfied $PX^* = X^*$. We will learn in this chapter to identify this makes X^* is an eigenvector of P with eigenvalue 1.

[5] insert your own more interesting set of quantities that doubles/halves or triples during some regular interval of time

7.1.3 motion of points under linear transformations

Remark 7.1.4.

What follows here is just intended to show you how you might stumble into the concept of an eigenvector even if you didn't set out to find it. The calculations we study here are not what we aim to ultimately disect in this chapter. This is purely a mathematical experiment to show how eigenvectors arise naturally through repeated matrix multiplication on a given point. Physically speaking the last two subsections were way more interesting.

I'll focus on two dimensions to begin for the sake of illustration. Let's take a matrix A and a point x_o and study what happens as we multiply by the matrix. We'll denote $x_1 = Ax_o$ and generally $x_{k+1} = Ax_k$. It is customary to call x_k the "k-th state of the system". As we multiply the k-th state by A we generate the $k+1$-th state.[6]

Example 7.1.5. *Let* $A = \left[\begin{smallmatrix} 3 & 0 \\ 8 & -1 \end{smallmatrix}\right]$ *and let* $x_o = \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$. *Calculate,*

$$\begin{aligned}
x_1 &= \left[\begin{smallmatrix} 3 & 0 \\ 8 & -1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 3 \\ 6 \end{smallmatrix}\right] \\
x_2 &= \left[\begin{smallmatrix} 3 & 0 \\ 8 & -1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 3 \\ 6 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 9 \\ 18 \end{smallmatrix}\right] \\
x_3 &= \left[\begin{smallmatrix} 3 & 0 \\ 8 & -1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 9 \\ 18 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 27 \\ 54 \end{smallmatrix}\right] \\
x_4 &= \left[\begin{smallmatrix} 3 & 0 \\ 8 & -1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 27 \\ 54 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 81 \\ 162 \end{smallmatrix}\right]
\end{aligned}$$

Each time we multiply by A we scale the vector by a factor of three. If you want to look at x_o as a point in the plane the matrix A pushes the point x_k to the point $x_{k+1} = 3x_k$. Its not hard to see that $x_k = 3^k x_o$. What if we took some other point, say $y_o = \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ then what will A do?

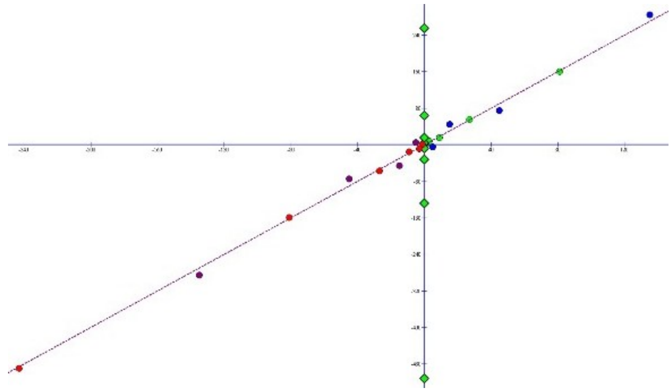
$$\begin{aligned}
y_1 &= \left[\begin{smallmatrix} 3 & 0 \\ 8 & -1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 3 \\ 8 \end{smallmatrix}\right] \\
y_2 &= \left[\begin{smallmatrix} 3 & 0 \\ 8 & -1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 3 \\ 8 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 9 \\ 16 \end{smallmatrix}\right] \\
y_3 &= \left[\begin{smallmatrix} 3 & 0 \\ 8 & -1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 9 \\ 16 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 27 \\ 56 \end{smallmatrix}\right] \\
y_4 &= \left[\begin{smallmatrix} 3 & 0 \\ 8 & -1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 27 \\ 48 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 81 \\ 160 \end{smallmatrix}\right]
\end{aligned}$$

Now, what happens for arbitrary k? Can you find a formula for y_k? This point is not as simple as x_o. The vector x_o is apparently a special vector for this matrix. Next study, $z_o = \left[\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right]$,

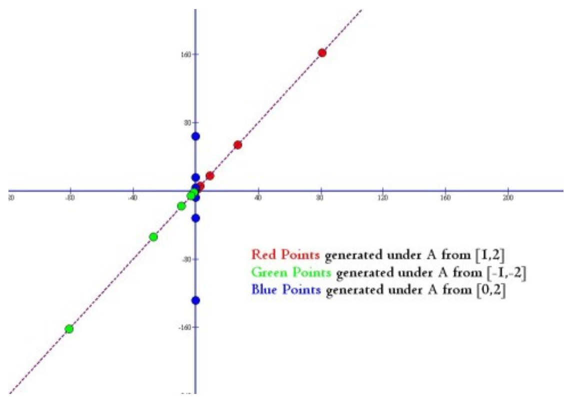
$$\begin{aligned}
z_1 &= \left[\begin{smallmatrix} 3 & 0 \\ 8 & -1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 \\ 2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ -2 \end{smallmatrix}\right] \\
z_2 &= \left[\begin{smallmatrix} 3 & 0 \\ 8 & -1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 \\ -2 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 4 \end{smallmatrix}\right] \\
z_3 &= \left[\begin{smallmatrix} 3 & 0 \\ 8 & -1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 \\ 4 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ -8 \end{smallmatrix}\right] \\
z_4 &= \left[\begin{smallmatrix} 3 & 0 \\ 8 & -1 \end{smallmatrix}\right]\left[\begin{smallmatrix} 0 \\ -8 \end{smallmatrix}\right] = \left[\begin{smallmatrix} 0 \\ 16 \end{smallmatrix}\right]
\end{aligned}$$

Let me illustrate what is happening with a picture. I have used color to track the motion of a particular point. You can see that all points tend to get drawn into the line with direction vector x_o with the sole exception of the points along the y-axis which I have denoted via diamonds in the picture below:

[6] ask Dr. Mavinga and he will show you how a recursively defined linear difference equation can be converted into a matrix equation of the form $x_{k+1} = Ax_k$, this is much the same idea as saying that an $n-th$ order ODE can be converted into a system of n- first order ODEs.



The directions $[1,2]$ *and* $[0,1]$ *are special, the following picture illustrates the motion of those points under* A:



The line with direction vector $[1,2]$ *seems to attract almost all states to itself. On the other hand, if you could imagine yourself a solution walking along the y-axis then if you took the slightest mis-step to the right or left then before another dozen or so steps you'd find yourself stuck along the line in the* $[1,2]$*-direction. There is a connection of the system* $x_{k+1} = Ax_k$ *and the system of differential equations* $dx/dt = Bx$ *if we have* $B = I + A$. *Perhaps we'll have time to explore the questions posed in this example from the viewpoint of the corresponding system of differential equations. In this case the motion is very discontinuous. I think you can connect the dots here to get a rough picture of what the corresponding system's solutions look like. In the differential equations Chapter we develop these ideas a bit further. For now we are simply trying to get a feeling for how one might discover that there are certain special vector(s) associated with a given matrix. We call these vectors the "eigenvectors" of A.*

The next matrix will generate rather different motions on points in the plane.

Example 7.1.6. *Let* $A = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$. *Consider the trajectory of* $x_o = (1, 0)$,

$$x_1 = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

$$x_2 = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$$

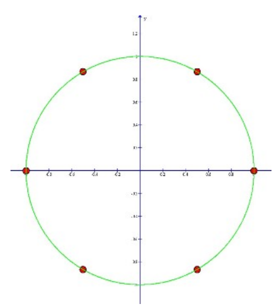
$$x_3 = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

$$x_4 = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix}$$

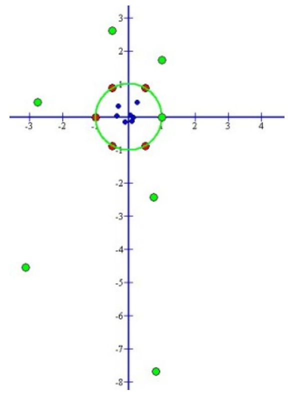
$$x_5 = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} -\frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix}$$

$$x_6 = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} \frac{1}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Past this point we just cycle back to the same points, clearly $x_k = x_{k+6}$ *for all* $k \geq 0$. *If we started with a different initial point we would find this pattern again. The reason for this is that* A *is the matrix which rotates vectors by* $\pi/3$ *radians. The trajectories generated by this matrix are quite different then the preceding example, there is no special direction in this case.*



Although, generally this type of matrix generates elliptical orbits and then there are two special directions. Namely the major and minor axis of the ellipitical orbits. Finally, this sort of matrix could have a scaling factor built in so that the trajectories spiral in or out of the origin. I provide a picture illustrating the various possibilities. The red dots in the picture below are generated from A as was given in the preceding example, the blue dots are generated from the matrix $[\frac{1}{2}col_1(A)|col_2(A)]$ whereas the green dots are obtained from the matrix $[2col_1(A)|col_2(A)]$. In each case I started with the point $(1, 0)$ and studied the motion of the point under repeated multiplications of matrix:



Let's summarize our findings so far: if we study the motion of a given point under successive multiplications of a matrix it may be pushed towards one of several directions or it may go in a circular/spiral-type motion.

7.2 eigenvector foundations

The preceding section was motivational. We now begin the real[7] material. Given our experience on coordinate change in the preceding chapter it should begin to be clear to you that the fundamental objects of linear algebra are linear transformations. A matrix usually give us just one picture of a more fundamental concept which is stated in terms of linear transformations and vector spaces[8]. In view of this wisdom we cast the definition of the eigenvalue and vector in terms of an abstract linear transformation on a vector space.

Definition 7.2.1.

Let $T : V \to V$ be a linear transformation on a vector space V. If there exists $v \in V$ such that $v \neq 0$ such that $T(v) = \lambda v$ for some constant λ then we say v is an **eigenvector** of T with **eigenvalue** λ.

Usually we work with real vector spaces so the scalar λ is taken from $\mathbb{R}$, however it is both interesting and useful to consider the extension to $\mathbb{C}$. We do so at the conclusion of this chapter. For now let me just introduce a little langauge. If $\lambda \in \mathbb{R}$ then I say λ is a **real eigenvalue with real eigenvector** v. On the other hand, my typical notation is that if $\lambda = \alpha + i\beta \in \mathbb{C}$ with $\beta \neq 0$ then I say λ is a **complex eigenvalue with complex eigenvector** $v = a + ib$.

Example 7.2.2. *Let $T(f) = Df$ where D is the derivative operator. This defines a linear transformation on function space $\mathcal{F}$. An eigenvector for T would be a function which is proportional to its own derivative fucntion... in other words solve $\frac{dy}{dt} = \lambda y$. Separation of variables yields $y = ce^{\lambda t}$. The eigenfunctions for T are simply exponential functions.*

[7] I should mention that your text insists that e-vectors have real e-values. I make no such restriction. If we want to insist the e-values are real I will say that explicitly.

[8] forgive me for this bit of abstract linear algebra here, I think it helps even if we don't have homework on it

Example 7.2.3. *Let $T(A) = A^T$ for $A \in \mathbb{R}^{n\times n}$. To find an eigenvector for T we want a matrix $V \in \mathbb{R}^{n\times n}$ and a constant λ such that $T(V) = V^T = \lambda V$. An obvious choice is $\lambda = 1$ and choose a symmetric matrix V so $V^T = V$. Another slightly less obvious guess exists. Can you think of it?*

Notice that there are infinitely many eigenvectors for a given eigenvalue in both of the examples above. The number of eigenvalues for the function space example is infinite since any $\lambda \in \mathbb{R}$ will do. On the other hand, the matrix example only had two eigenvalues. The distinction between these examples is that function space is infinite dimensional whereas the matrix example is finite-dimensional. For the most part we focus on less abstract eigenvector examples. The following definition dovetails with our definition above if you think about $L_A : \mathbb{R}^n \to \mathbb{R}^n$. An eigenvector of L_A is an eigenvector of A if we accept the definition that follows:

Definition 7.2.4.

Let $A \in \mathbb{R}^{n\times n}$. If $v \in \mathbb{R}^n$ is **nonzero** and $Av = \lambda v$ for some $\lambda \in \mathbb{C}$ then we say v is an **eigenvector** with **eigenvalue** λ of the matrix A.

We identify that the eigenvectors of A pointed in the direction where trajectories were asymptotically attracted in the examples of the preceding section. Although, the case of the circular trajectories broke from that general pattern. We'll discover those circular orbits correspond to the complex case.

Our goal at this point is to find a clear and concise method to calculate eigenvalues and their corresponding eigenvector(s). Fortunately, we soon find that guessing and solving differential equations are not the usual method to calculate eigenvectors (at least not in Math 321)

Proposition 7.2.5.

Let $A \in \mathbb{R}^{n\times n}$ then λ is an eigenvalue of A iff $det(A-\lambda I) = 0$. We say $P(\lambda) = det(A-\lambda I)$ the **characteristic polynomial** and $det(A - \lambda I) = 0$ is the **characteristic equation**.

Proof: Suppose λ is an eigenvalue of A then there exists a nonzero vector v such that $Av = \lambda v$ which is equivalent to $Av - \lambda v = 0$ which is precisely $(A - \lambda I)v = 0$. Notice that $(A - \lambda I)0 = 0$ thus the matrix $(A - \lambda I)$ is singular as the equation $(A - \lambda I)x = 0$ has more than one solution. Consequently $det(A - \lambda I) = 0$.

Conversely, suppose $det(A - \lambda I) = 0$. It follows that $(A - \lambda I)$ is singular. Clearly the system $(A - \lambda I)x = 0$ is consistent as $x = 0$ is a solution hence we know there are infinitely many solutions. In particular there exists at least one vector $v \neq 0$ such that $(A-\lambda I)v = 0$ which means the vector v satisfies $Av = \lambda v$. Thus v is an eigenvector with eigenvalue λ for A [9]. $\square$

Let's collect the observations of the above proof for future reference.

[9] It is worth mentioning that the theorems on uniqueness of solution and singular matrices and determinant hold for linear systems with complex coefficients and variables. We don't need a separate argument for the complex case

Proposition 7.2.6.

The following are equivalent for $A \in \mathbb{R}^{n\times n}$ and $\lambda \in \mathbb{C}$,

1. λ is an eigenvalue of A
2. there exists $v \neq 0$ such that $Av = \lambda v$
3. there exists $v \neq 0$ such that $(A - \lambda I)v = 0$
4. λ is a solution to $det(A - \lambda I) = 0$
5. $(A - \lambda I)v = 0$ has infinitely many solutions.

7.2.1 characteristic equations

Example 7.2.7. *Let* $A = \left[\begin{smallmatrix} 3 & 0 \\ 8 & -1 \end{smallmatrix}\right]$*. Find the eigenvalues of* A *from the characteristic equation:*

$$det(A - \lambda I) = det\begin{bmatrix} 3-\lambda & 0 \\ 8 & -1-\lambda \end{bmatrix} = (3-\lambda)(-1-\lambda) = (\lambda+1)(\lambda-3) = 0$$

Hence the eigenvalues are $\lambda_1 = -1$ *and* $\lambda_2 = 3$*. Notice this is precisely the factor of 3 we saw scaling the vector in the first example of the preceding section.*

Example 7.2.8. *Let* $A = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$*. Find the eigenvalues of* A *from the characteristic equation:*

$$det(A - \lambda I) = det\begin{bmatrix} \frac{1}{2}-\lambda & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2}-\lambda \end{bmatrix} = (\tfrac{1}{2}-\lambda)^2 + \tfrac{3}{4} = (\lambda - \tfrac{1}{2})^2 + \tfrac{3}{4} = 0$$

Well how convenient is that? The determinant completed the square for us. We find: $\lambda = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$*. It would seem that elliptical orbits somehow arise from complex eigenvalues*

Proposition 5.3.3 proved that taking the determinant of a triagular matrix was easy. We just multiply the diagonal entries together. This has interesting application in our discussion of eigenvalues.

Example 7.2.9. *Given* A *below, find the eigenvalues. Use Proposition 5.3.3 to calculate the determinant,*

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 0 & 5 & 6 \\ 0 & 0 & 7 \end{bmatrix} \quad \Rightarrow \quad det(A - \lambda I) = \begin{bmatrix} 2-\lambda & 3 & 4 \\ 0 & 5-\lambda & 6 \\ 0 & 0 & 7-\lambda \end{bmatrix} = (2-\lambda)(5-\lambda)(7-\lambda)$$

Therefore, $\lambda_1 = 2, \lambda_2 = 5$ *and* $\lambda_3 = 7$*.*

Remark 7.2.10. *eigenwarning*

Calculation of eigenvalues does not need to be difficult. However, I urge you to be careful in solving the characteristic equation. More often than not I see students make a mistake in calculating the eigenvalues. If you do that wrong then the eigenvector calculations will often turn into inconsistent equations. This should be a clue that the eigenvalues were wrong, but often I see what I like to call the "principle of minimal calculation" take over and students just adhoc set things to zero, hoping against all logic that I will somehow not notice this. Don't be this student. If the eigenvalues are correct, the eigenvector equations are consistent and you will be able to find nonzero eigenvectors.

7.2.2 eigenvector examples

Example 7.2.11. *Let* $A = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$ *find the e-values and e-vectors of* A*.*

$$det(A - \lambda I) = det \begin{bmatrix} 3-\lambda & 1 \\ 3 & 1-\lambda \end{bmatrix} = (3-\lambda)(1-\lambda) - 3 = \lambda^2 - 4\lambda = \lambda(\lambda - 4) = 0$$

We find $\lambda_1 = 0$ *and* $\lambda_2 = 4$*. Now find the e-vector with e-value* $\lambda_1 = 0$*, let* $u_1 = (u, v)$ *denote the e-vector we wish to find. Calculate,*

$$(A - 0I)u_1 = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 3u+v \\ 3u+v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Obviously the equations above are redundant and we have infinitely many solutions of the form $3u + v = 0$ *which means* $v = -3u$ *so we can write,* $u_1 = \begin{bmatrix} u \\ -3u \end{bmatrix} = u \begin{bmatrix} 1 \\ -3 \end{bmatrix}$*. In applications we often make a choice to select a particular e-vector. Most modern graphing calculators can calculate e-vectors. It is customary for the e-vectors to be chosen to have length one. That is a useful choice for certain applications as we will later discuss. If you use a calculator it would likely give* $u_1 = \frac{1}{\sqrt{10}} \begin{bmatrix} 1 \\ -3 \end{bmatrix}$ *although the* $\sqrt{10}$ *would likely be approximated unless your calculator is smart.*

Continuing we wish to find eigenvectors $u_2 = (u, v)$ *such that* $(A - 4I)u_2 = 0$*. Notice that* u, v *are disposable variables in this context, I do not mean to connect the formulas from the* $\lambda = 0$ *case with the case considered now.*

$$(A - 4I)u_1 = \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -u+v \\ 3u-3v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Again the equations are redundant and we have infinitely many solutions of the form $v = u$*. Hence,* $u_2 = \begin{bmatrix} u \\ u \end{bmatrix} = u \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ *is an eigenvector for any* $u \in \mathbb{R}$ *such that* $u \neq 0$*.*

Remark 7.2.12.

It was obvious the equations were redundant in the example above. However, we need not rely on pure intuition. The problem of calculating all the e-vectors is precisely the same as finding all the vectors in the null space of a matrix. We already have a method to do that without ambiguity. We find the rref of the matrix and the general solution falls naturally from that matrix. I don't bother with the full-blown theory for simple examples because there is no need. However, with 3×3 examples it may be advantageous to keep our earlier null space theorems in mind.

Example 7.2.13. *Let* $A = \begin{bmatrix} 0 & 0 & -4 \\ 2 & 4 & 2 \\ 2 & 0 & 6 \end{bmatrix}$ *find the e-values and e-vectors of A.*

$$\begin{aligned} 0 = det(A - \lambda I) &= det \begin{bmatrix} -\lambda & 0 & -4 \\ 2 & 4-\lambda & 2 \\ 2 & 0 & 6-\lambda \end{bmatrix} \\ &= (4-\lambda)\big[-\lambda(6-\lambda)+8\big] \\ &= (4-\lambda)\big[\lambda^2 - 6\lambda + 8\big] \\ &= -(\lambda-4)(\lambda-4)(\lambda-2) \end{aligned}$$

Thus we have a repeated e-value of $\lambda_1 = \lambda_2 = 4$ and $\lambda_3 = 2$. Let's find the eigenvector $u_3 = (u, v, w)$ such that $(A - 2I)u_3 = 0$, we find the general solution by row reduction

$$rref \left[\begin{array}{ccc|c} -2 & 0 & -4 & 0 \\ 2 & 2 & 2 & 0 \\ 2 & 0 & 4 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow \begin{array}{c} u + 2w = 0 \\ v - w = 0 \end{array} \Rightarrow \boxed{u_3 = w \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}}$$

Next find the e-vectors with e-value 4. Let $u_1 = (u, v, w)$ satisfy $(A - 4I)u_1 = 0$. Calculate,

$$rref \left[\begin{array}{ccc|c} -4 & 0 & -4 & 0 \\ 2 & 0 & 2 & 0 \\ 2 & 0 & 2 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \Rightarrow u + w = 0$$

Notice this case has two free variables, we can use v, w as parameters in the solution,

$$u_1 = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} -w \\ v \\ w \end{bmatrix} = v \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \boxed{u_1 = v \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ and } u_2 = w \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}$$

I have boxed two linearly independent eigenvectors u_1, u_2. These vectors will be linearly independent for any pair of nonzero constants v, w.

You might wonder if it is always the case that repeated e-values get multiple e-vectors. In the preceding example the e-value 4 had *multiplicity* two and there were likewise two linearly independent e-vectors. The next example shows that is not the case.

Example 7.2.14. *Let* $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ *find the e-values and e-vectors of* A.

$$det(A - \lambda I) = det \begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix} = (1-\lambda)(1-\lambda) = 0$$

Hence we have a repeated e-value of $\lambda_1 = 1$. *Find all e-vectors for* $\lambda_1 = 1$, *let* $u_1 = (u, v)$,

$$(A - I)u_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad v = 0 \quad \Rightarrow \quad \boxed{u_1 = u \begin{bmatrix} 1 \\ 0 \end{bmatrix}}$$

We have only one e-vector for this system.

Incidentally, you might worry that we could have an e-value (in the sense of having a zero of the characteristic equation) and yet have no e-vector. Don't worry about that, we always get at least one e-vector for each distinct e-value. See Proposition 7.2.6

Example 7.2.15. *Let* $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ *find the e-values and e-vectors of* A.

$$\begin{aligned} 0 = det(A - \lambda I) &= det \begin{bmatrix} 1-\lambda & 2 & 3 \\ 4 & 5-\lambda & 6 \\ 7 & 8 & 9-\lambda \end{bmatrix} \\ &= (1-\lambda)\big[(5-\lambda)(9-\lambda) - 48\big] - 2\big[4(9-\lambda) - 42\big] + 3\big[32 - 7(5-\lambda)\big] \\ &= -\lambda^3 + 15\lambda^2 + 18\lambda \\ &= -\lambda(\lambda^2 - 15\lambda - 18) \end{aligned}$$

Therefore, using the quadratic equation to factor the ugly part,

$$\lambda_1 = 0, \quad \lambda_2 = \frac{15 + 3\sqrt{33}}{2}, \quad \lambda_3 = \frac{15 - 3\sqrt{33}}{2}$$

The e-vector for e-value zero is not too hard to calculate. Find $u_1 = (u, v)$ *such that* $(A - 0I)u_1 = 0$. *This amounts to row reducing* A *itself:*

$$rref \left[\begin{array}{ccc|c} 1 & 2 & 3 & 0 \\ 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad \begin{array}{c} u - w = 0 \\ v + 2w = 0 \end{array} \quad \Rightarrow \quad \boxed{u_1 = w \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}}$$

The e-vectors corresponding e-values λ_2 *and* λ_3 *are hard to calculate without numerical help. Let's discuss Texas Instrument calculator output. To my knowledge, TI-85 and higher will calculate both e-vectors and e-values. For example, my ancient TI-89, displays the following if I define our matrix* $A = mat2$,

$$eigVl(mat2) = \{16.11684397, \ -1.11684397, \ 1.385788954e - 13\}$$

Calculators often need a little interpretation, the third entry is really zero in disguise. The e-vectors will be displayed in the same order, they are given from the "eigVc" command in my TI-89,

$$eigVc(mat2) = \begin{bmatrix} .2319706872 & .7858302387 & .4082482905 \\ .5253220933 & .0867513393 & -.8164965809 \\ .8186734994 & -.6123275602 & .4082482905 \end{bmatrix}$$

From this we deduce that eigenvectors for λ_1, λ_2 and λ_3 are

$$u_1 = \begin{bmatrix} .2319706872 \\ .5253220933 \\ .8186734994 \end{bmatrix} \qquad u_2 = \begin{bmatrix} .7858302387 \\ .0867513393 \\ -.6123275602 \end{bmatrix} \qquad u_3 = \begin{bmatrix} .4082482905 \\ -.8164965809 \\ .4082482905 \end{bmatrix}$$

Notice that $1/\sqrt{6} \approxeq 0.408248905$ so you can see that u_3 above is simply the $u = 1/\sqrt{6}$ case for the family of e-vectors we calculated by hand already. The calculator chooses e-vectors so that the vectors have length one.

While we're on the topic of calculators, perhaps it is worth revisiting the example where there was only one e-vector. How will the calculator respond in that case? Can we trust the calculator?

Example 7.2.16. *Recall Example 7.2.14. We let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and found a repeated e-value of $\lambda_1 = 1$ and single e-vector $u_1 = u \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Hey now, it's time for technology, let $A = a$,*

$$eigVl(a) = \{1, 1\} \quad and \quad eigVc(a) = \begin{bmatrix} 1. & -1. \\ 0. & 1.e-15 \end{bmatrix}$$

Behold, the calculator has given us two alleged e-vectors. The first column is the genuine e-vector we found previously. The second column is the result of machine error. The calculator was tricked by round-off error into claiming that $[-1, 0.000000000000001]$ is a distinct e-vector for A. It is not. Moral of story? When using calculator you must first master the theory or else you'll stay mired in ignorance as presribed by your robot masters.

Finally, I should mention that TI-calculators may or may not deal with complex e-vectors adequately. There are doubtless many web resources for calculating e-vectors/values. I would wager if you Googled[10] it you'd find an online calculator that beats any calculator. Many of you have a laptop with wireless so there is almost certainly a way to check your answers if you just take a minute or two. I don't mind you checking your answers. If I assign it in homework then I do want you to work it out without technology. Otherwise, you could get a false confidence before the test. Technology is to supplement not replace calculation.

Remark 7.2.17.

I would also remind you that there are oodles of examples beyond these lecture notes in the homework solutions from previous year(s). If these notes do not have enough examples on some topic then you should seek additional examples elsewhere, ask me, etc... Do not suffer in silence, ask for help. Thanks.

[10] or perhaps chatgpt'd or grok'd it etc.

7.3 theory for eigenvalues and eigenvectors

In this subsection we collect a number of general results on eigenvalues and eigenvectors. To begin, we prepare to argue a seemingly obvious proposition, namely that an $n \times n$ matrix will have n eigenvalues. From the three examples in the earlier section that's pretty obvious, however we should avoid proof by example in as much is possible.

Theorem 7.3.1.

> Fundamental Theorem of Algebra: if $P(x)$ is an n-th order polynomial complex coefficients then the equation $P(x) = 0$ has n-solutions where some of the solutions may be repeated. Moreover, if $P(x)$ is an n-th order polynomial with real coefficients then complex solutions to $P(x) = 0$ come in conjugate pairs. It follows that any polynomial with real coefficients can be factored into a unique product of repeated real and irreducible quadratic factors.

A proof of this theorem would take us far of topic here[11]. I state it to remind you what the possibilities are for the characteristic equation. Recall that the determinant is simply a product and sum of the entries in the matrix. Notice that $A - \lambda I$ has n-copies of λ and the determinant formula never repeats the same entry twice in the same summand. It follows that solving the characterictic equation for $A \in \mathbb{R}^{\,n\times n}$ boils down to factoring an n-th order polynomial in λ.

Proposition 7.3.2.

> If $A \in \mathbb{R}^{\,n\times n}$ then A has n eigenvalues, however, some may be repeated and/or complex.

Proof: follows from definition of determinant and the Fundamental Theorem of Algebra [12] $\square$

Notice that if $P(\lambda) = det(A - \lambda I)$ then λ_j is an e-value of the square matrix A iff $(\lambda - \lambda_j)$ divides[13] the characteristic polynomial $P(\lambda)$.

Proposition 7.3.3.

> The constant term in the characteristic polynomial $P(\lambda) = det(A - \lambda I)$ is the determinant of A.

Proof: Suppose the characteristic polynomial P of A has coefficients c_i:

$$P(\lambda) = det(A - \lambda I) = c_n\lambda^n + c_{n-1}\lambda^{n-1} + \cdots + c_1\lambda + c_0.$$

Notice that if $\lambda = 0$ then $A - \lambda I = A$ hence

$$P(0) = det(A) = c_n 0^n + \cdots + c_1 0 + c_0.$$

Thus $det(A) = c_0$. $\square$

[11] there is a nice proof which can be given in our complex variables course

[12] properties of eigenvalues and the characteristic equation can be understood from studying the *minimal* and *characteristic* polynomials. We take a less sophsiticated approach in this course

[13] the term "divides" is a technical term. If $f(x)$ divides $g(x)$ then there exists $h(x)$ such that $g(x) = h(x)f(x)$. In other words, $f(x)$ is a factor of $g(x)$.

Proposition 7.3.4.

Zero is an eigenvalue of A iff A is a singular matrix.

Proof: Let $P(\lambda)$ be the characteristic polynomial of A. If zero is an eigenvalue then λ must factor the characteristic polynomial. Moreover, the factor theorem tells us that $P(0) = 0$ since $(\lambda - 0)$ factors $P(\lambda)$. Thus $c_0 = 0$ and we deduce using the previous proposition that $det(A) = c_0 = 0$. Which shows that A is singular.

Conversely, assume A is singular then $det(A) = 0$. Again, using the previous proposition, $det(A) = c_0$ hence $c_0 = 0$. But, this means we can factor out a λ in $P(\lambda)$ hence $P(0) = 0$ and we find zero is an e-value of A. $\square$.

Proposition 7.3.5.

If $A \in \mathbb{R}^{n\times n}$ then A has n eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ then $det(A) = \lambda_1\lambda_2\cdots\lambda_n$.

Proof: If $A \in \mathbb{R}^{n\times n}$ then A has n eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ then the characteristic polynomial factors over $\mathbb{C}$:

$$det(A - \lambda I) = k(\lambda - \lambda_1)(\lambda - \lambda_2)\cdots(\lambda - \lambda_n)$$

Moreover, if you think about $A - \lambda I$ it is clear that the leading term obtains a coefficient of $(-1)^n$ hence $k = (-1)^n$. If c_0 is the constant term in the characteristic polynomial then algbera reveals that $c_0 = (-1)^n(-\lambda_1)(-\lambda_2)\cdots(-\lambda_n) = \lambda_1\lambda_2\dots\lambda_n$. Therefore, using Proposition 7.3.3, $det(A) = \lambda_1\lambda_2\dots\lambda_n$. $\square$.

Proposition 7.3.6.

If $A \in \mathbb{R}^{n\times n}$ has e-vector v with eigenvalue λ then v is a e-vector of A^k with e-value λ^k.

Proof: let $A \in \mathbb{R}^{n\times n}$ have e-vector v with eigenvalue λ. Consider,

$$A^k v = A^{k-1}Av = A^{k-1}\lambda v = \lambda A^{k-2}Av = \lambda^2 A^{k-2}v = \cdots = \lambda^k v.$$

The $\cdots$ is properly replaced by a formal induction argument. $\square$.

Proposition 7.3.7.

Let A be a upper or lower triangular matrix then the eigenvalues of A are the diagonal entries of the matrix.

Proof: follows immediately from Proposition 5.3.3 since the diagonal entries of $A - \lambda I$ are of the form $A_{ii} - \lambda$ hence the characteristic equation has the form $det(A - \lambda I) = (A_{11} - \lambda)(A_{22} - \lambda)\cdots(A_{nn} - \lambda)$ which has solutions $\lambda = A_{ii}$ for $i = 1, 2, \dots, n$. $\square$

We saw how this is useful in Example 5.4.5. The LU-factorization together with the proposition above gives a calculationally superior method for calculation the determinant. In addition, once you have the LU-factorization of A there are many other questions about A which are easier to answer. See your text for more on this if you are interested.

7.4 linear independendence of real eigenvectors

You might have noticed that e-vectors with distinct e-values are linearly independent. This is no accident.

Proposition 7.4.1.

If $A \in \mathbb{R}^{\,n\times n}$ has e-vector v_1 with e-value λ_1 and e-vector v_2 with e-value λ_2 such that $\lambda_1 \neq \lambda_2$ then $\{v_1, v_2\}$ is linearly independent.

Proof: Let v_1, v_2 have e-values λ_1, λ_2 respective and assume towards a contradction that $v_2 = kv_2$ for some nonzero constant k. Multiply by the matrix A,

$$Av_1 = A(kv_2) \quad \Rightarrow \quad \lambda_1 v_1 = k\lambda_2 v_2$$

But we can replace v_1 on the l.h.s. with kv_2 hence,

$$\lambda_1 k v_2 = k\lambda_2 v_2 \quad \Rightarrow \quad k(\lambda_1 - \lambda_2)v_2 = 0$$

Note, $k \neq 0$ and $v_2 \neq 0$ by assumption thus the equation above indicates $\lambda_1 - \lambda_2 = 0$ therefore $\lambda_1 = \lambda_2$ which is a contradiction. Therefore there does not exist such a k and the vectors are linearly independent. $\square$

Proposition 7.4.2.

If $A \in \mathbb{R}^{\,n\times n}$ has e-vectors $v_1, v_2, \dots, v_k$ with e-values $\lambda_1, \lambda_2, \dots, \lambda_k$ such that $\lambda_i \neq \lambda_j$ for all $i \neq j$ then $\{v_1, v_2, \dots, v_k\}$ is linearly independent.

Proof: Let e-vectors $v_1, v_2, \dots, v_k$ have e-values $\lambda_1, \lambda_2, \dots, \lambda_k$ with respect to A and assume towards a contradction that there is some vector v_j which is a nontrivial linear combination of the other vectors:

$$v_j = c_1 v_1 + c_2 v_2 + \cdots + \widehat{c_j v_j} + \cdots + c_k v_k$$

Multiply by A,

$$Av_j = c_1 Av_1 + c_2 Av_2 + \cdots + \widehat{c_j Av_j} + \cdots + c_k Av_k$$

Which yields,

$$\lambda_j v_j = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \cdots + \widehat{c_j \lambda_j v_j} + \cdots + c_k \lambda_k v_k$$

But, we can replace v_j on the l.h.s with the linear combination of the other vectors. Hence

$$\lambda_j \big[c_1 v_1 + c_2 v_2 + \cdots + \widehat{c_j v_j} + \cdots + c_k v_k\big] = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \cdots + \widehat{c_j \lambda_j v_j} + \cdots + c_k \lambda_k v_k$$

Consequently,

$$c_1(\lambda_j - \lambda_1)v_1 + c_2(\lambda_j - \lambda_2)v_2 + \cdots + c_j(\widehat{\lambda_j - \lambda_j})v_j + \cdots + c_k(\lambda_j - \lambda_k)v_k = 0$$

Since $v_i \neq 0$ and c_i are not all zero it follows at least one factor $\lambda_j - \lambda_i = 0$ for $i \neq j$ but this is a contradiction since we assumed the e-values were distinct. $\square$
Notice the proof of the preceding two propositions was essentially identical. I provided the $k = 2$ proof to help make the second proof more accessible.

Definition 7.4.3.

Let $A \in \mathbb{R}^{n \times n}$ then a basis $\{v_1, v_2, \dots, v_n\}$ for $\mathbb{R}^n$ is called an **eigenbasis** if each vector in the basis is an e-vector for A. Notice we assume these are real vectors since they form a basis for $\mathbb{R}^n$.

Example 7.4.4. *We calculated in Example 7.2.13 the e-values and e-vectors of* $A = \begin{bmatrix} 0 & 0 & -4 \\ 2 & 4 & 2 \\ 2 & 0 & 6 \end{bmatrix}$ *were* $\lambda_1 = \lambda_2 = 4$ *and* $\lambda_3 = 2$ *with e-vectors*

$$u_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

Linear indpendence of u_3 *from* u_1, u_2 *is given from the fact the e-values of* u_3 *and* u_1, u_2 *are distinct. Then is is clear that* u_1 *is not a multiple of* u_2 *thus they are linearly independent. It follows that* $\{u_1, u_2, u_3\}$ *form a linearly independent set of vectors in* $\mathbb{R}^3$*, therefore* $\{u_1, u_2, u_3\}$ *is an eigenbasis.*

Definition 7.4.5.

Let $A \in \mathbb{R}^{n \times n}$ then we call the set of all real e-vectors with real e-value λ unioned with the zero-vector the **λ-eigenspace** and we denote this set by W_λ.

Example 7.4.6. *Again using Example 7.2.13 we have two eigenspaces,*

$$W_4 = span\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \} \qquad W_2 = span\{ \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \}$$

Proposition 7.4.7.

Eigenspaces are subspaces of $\mathbb{R}^n$. Moreoever, $dim(W_\lambda) \leq m$ where m is multiplicity of the λ solution in the characteristic equation.

Proof: By definition zero is in the eigenspace W_λ. Suppose $x, y \in W_\lambda$ note that $A(x + cy) = Ax + cAy = \lambda x + c\lambda y = \lambda(x + cy)$ hence $x + cy \in W_\lambda$ for all $x, y \in W_\lambda$ and $c \in \mathbb{R}$ therefore $W_\lambda \leq \mathbb{R}^n$. To prove $dim(W_\lambda) \leq m$ we simply need to show that $dim(W_\lambda) > m$ yields a contradiction. This can be seen from showing that if there were more than m e-vectors with e-value λ then the chacteristic equation would likewise more than m solutions of λ. The question then is why does each linearly independent e-vector give a factor in the characteristic equation? Answer this question for bonus points. $\square$

Definition 7.4.8.

Let A be a real square matrix with real e-value λ. The dimension of W_λ is called the **geometric multiplicity** of λ. The number of times the λ solution is repeated in the characteristic equation's solution is called the **algebraic multiplicity** of λ.

We already know from the examples we've considered thus far that there will not always be an eigenbasis for a given matrix A. In general, here are the problems we'll face:

1. we could have complex e-vectors (see Example 7.6.2)

2. we could have less e-vectors than needed for a basis (see Example 7.2.14)

We can say case 2 is caused from the geometric multiplicity being less than the algebraic multiplicity. What can we do about this? If we want to adjoin vectors to make-up for the lack of e-vectors then how should we find them in case 2?

Definition 7.4.9.

A **generalized eigenvector** of order k with eigenvalue λ with respect to a matrix $A \in \mathbb{R}^{n\times n}$ is a nonzero vector v such that

$$(A-\lambda I)^k v = 0$$

and $(A-\lambda I)^{k-1}v \neq 0$.

Notice $k=1$ simply reduces to a plain old eigenvector. It's useful to construct generalized e-vectors from a *chain-condition* if possible.

Proposition 7.4.10.

Suppose $A \in \mathbb{R}^{n\times n}$ has e-value λ and e-vector v_1 then if $(A-\lambda I)v_2 = v_1$ has a solution then v_2 is a generalized e-vector of order 2 which is linearly independent from v_1.

Proof: Suppose $(A-\lambda I)v_2 = v_1$ is consistent then multiply by $(A-\lambda I)$ to find $(A-\lambda I)^2 v_2 = (A-\lambda I)v_1$. However, we assumed v_1 was an e-vector hence $(A-\lambda I)v_1 = 0$ and we find v_2 is a generalized e-vector of order 2. Suppose $v_2 = kv_1$ for some nonzero k then $Av_2 = Akv_1 = k\lambda v_1 = \lambda v_2$ hence $(A-\lambda I)v_2 = 0$ but this contradicts the construction of v_2 as the solution to $(A-\lambda I)v_2 = v_1$. Consequently, v_2 is linearly independent from v_1 by virtue of its construction. $\square$.

Example 7.4.11. *Let's return to Example 7.2.14 and look for a generalized e-vector of order 2. Recall* $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ *and we found a repeated e-value of* $\lambda_1 = 1$ *and single e-vector* $u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ *(fix* $u = 1$ *for convenience). Let's complete the chain: find* $v_2 = [u, v]^T$ *such that* $(A - I)u_2 = u_1$,

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \Rightarrow \quad v = 1, \ u \text{ is free}$$

Any choice of u will do, in this case we can even set $u = 0$ to find

$$\boxed{u_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$$

Clearly, $\{u_1, u_2\}$ forms a basis of $\mathbb{R}^2$. It is not an eigenbasis with respect to A, however it is what is known as a **Jordan basis** *for A.*

Theorem 7.4.12.

> Any matrix with real eigenvalues has a Jordan basis. We can always find enough generalized e-vectors to form a basis for $\mathbb{R}^n$ with respect to A in the case that the e-values are all real.

I might prove this in Math 321, but here we simply wish to appreciate the insight of this structure.

Proposition 7.4.13.

> Let $A \in \mathbb{R}^{\,n\times n}$ and suppose λ is an e-value of A with e-vector v_1 then if $(A - \lambda I)v_2 = v_1$, $(A - \lambda I)v_3 = v_2$, $\dots$, $(A - \lambda I)v_k = v_{k-1}$ then $\{v_1, v_2, \dots, v_k\}$ is a linearly independent set of vectors and v_j is a generalized vector of order j for each $j = 1, 2, \dots, k$.

This proposition says a k-chain of generalized eigenvectors is linearly independent. The set $\{v_1, \dots, v_k\}$ as described above is known as a k-chain with eigenvalue λ for A.

Usually we can find a chain of generalized e-vectors for each e-value and that will product a Jordan basis. However, there is a trap that you will not likely get caught in for a while. It is not always possible to use a single chain for each e-value. Sometimes it takes a couple chains for a single e-value. That said, the chain condition is very nice in that it automatically insures linear independence down the chain. This is important since the solution to $(A-\lambda I)^k v = 0$ and the solution to $(A-\lambda I)v = 0$ do not automatically provide a LI set. I do not attempt to describe the general algorithm to find the Jordan basis for a given matrix, I merely wish to introduce you to the idea of the Jordan form and perhaps convince you it's interesting.

Example 7.4.14. *Suppose* $A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ *it is not hard to show that* $\det(A - \lambda I) = (\lambda - 1)^4 = 0$. *We have a quadruple e-value $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = 1$.*

$$0 = (A - I)\vec{u} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \Rightarrow \quad \vec{u} = \begin{bmatrix} s_1 \\ 0 \\ s_3 \\ 0 \end{bmatrix}$$

Any nonzero choice of s_1 or s_3 gives us an e-vector. Let's define two e-vectors which are clearly linearly independent, $\vec{u}_1 = (1, 0, 0, 0)$ and $\vec{u}_2 = (0, 0, 1, 0)$. We'll find a generalized e-vector to go *with each of these. There are two length two chains to find here. In particular,*

$$(A-I)\vec{u}_3=\vec{u}_1 \quad \Rightarrow \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} s_1 \\ s_2 \\ s_3 \\ s_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad s_2=1, s_4=0, \ s_1, s_3 \ free$$

I choose $s_1 = 0$ *and* $s_3 = 1$ *since I want a new vector, define* $\vec{u}_3 = (0, 0, 1, 0)$. *Finally solving* $(A - I)\vec{u}_4 = \vec{u}_2$ *for* $\vec{u}_4 = (s_1, s_2, s_3, s_4)$ *yields conditions* $s_4 = 1, s_2 = 0$ *and* s_1, s_3 *free. I choose* $\vec{u}_4 = (0, 0, 0, 1)$. *To summarize we have four linearly independent vectors which form two chains:*

$$(A-I)\vec{(u)}_3=\vec{u}_1, \ (A-I)\vec{u}_1=0 \qquad (A-I)\vec{u}_4=\vec{u}_2, \ (A-I)\vec{u}_2=0$$

The matrix above was in an example of a matrix in Jordan form. When the matrix is in Jordan form then the each elemement of then standard basis is an e-vector or generalized e-vector.

Example 7.4.15.

$$A = \begin{bmatrix} 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 3 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

Here we have the chain $\{e_1, e_2, e_3\}$ *with e-value* $\lambda_1 = 2$, *the chain* $\{e_4.e_5, e_6, e_7\}$ *with e-value* $\lambda_2 = 3$ *and finally a lone e-vector* e_8 *with e-value* $\lambda_3 = 4$

7.5 diagonalization

If a matrix has n-linearly independent e-vectors then we'll find that we can perform a similarity transformation to transform the matrix into a diagonal form. Let me briefly summarize what is required for us to have n-LI e-vectors. This is the natural extension of Proposition 7.4.2 to the case of repeated e-values.

Proposition 7.5.1. *criteria for real diagonalizability*

Suppose that $A \in \mathbb{R}^{\,n\times n}$ has distinct eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_k \in \mathbb{R}$ such that the characteristic polynomial factors as follows:

$$P_A(\lambda) = \pm(\lambda - \lambda_1)^{m_1}(\lambda - \lambda_2)^{m_2} \cdots (\lambda - \lambda_k)^{m_k}.$$

We identify $m_1, m_2, \dots, m_k$ are the **algebraic mulitplicities** of $\lambda_1, \lambda_2, \dots, \lambda_k$ respective and $m_1 + m_2 + \cdots m_k = n$. Furthermore, suppose we say that the j-th eigenspace $W_{\lambda_j} = \{x \in \mathbb{R} \mid Ax = \lambda_j x\}$ has $dim(W_{\lambda_j}) = n_j$ for $j = 1, 2, \dots k$. The values $n_1, n_2, \dots, n_k$ are called the **geometric mulitplicities** of $\lambda_1, \lambda_2, \dots, \lambda_k$ respective. With all of the language above in mind we can state that if $m_j = n_j$ for all $j = 1, 2, \dots k$ then A is diagonalizable.

All the proposition above really says is that if there exists an eigenbasis for A then it is diagonalizable. Simply take the union of the basis for each eigenspace and note the LI of this union follows immediately from Proposition 7.4.2 and the fact they are bases[14]. Once we have an eigenbasis we still need to prove diagonalizability follows. Since that is what is most interesting I'll restate it once more. Note in the proposition below the e-values may be repeated.

Proposition 7.5.2.

Suppose that $A \in \mathbb{R}^{\,n\times n}$ has e-values $\lambda_1, \lambda_2, \dots, \lambda_n$ with linearly independent e-vectors $v_1, v_2, \dots, v_n$. If we define $V = [v_1|v_2|\cdots|v_n]$ then $D = V^{-1}AV$ where D is a diagonal matrix with the eigenvalues down the diagonal: $D = [\lambda_1 e_1|\lambda_2 e_2|\cdots|\lambda_n e_n]$.

Proof: Notice that V is invertible since we assume the e-vectors are linearly independent. Moreover, $V^{-1}V = I$ in terms of columns translates to $V^{-1}[v_1|v_2|\cdots|v_n] = [e_1|e_2|\cdots|e_n]$. From which we deduce that $V^{-1}v_j = e_j$ for all j. Also, since v_j has e-value λ_j we have $Av_j = \lambda_j v_j$. Observe,

$$\begin{aligned} V^{-1}AV &= V^{-1}A[v_1|v_2|\cdots|v_n] \\ &= V^{-1}[Av_1|Av_2|\cdots|Av_n] \\ &= V^{-1}[\lambda_1 v_1|\lambda_2 v_2|\cdots|\lambda_n v_n] \\ &= V^{-1}[\lambda_1 v_1|\lambda_2 v_2|\cdots|\lambda_n v_n] \\ &= [\lambda_1 V^{-1}v_1|\lambda_2 V^{-1}v_2|\cdots|\lambda_n V^{-1}v_n] \\ &= [\lambda_1 e_1|\lambda_2 e_2|\cdots|\lambda_n e_n] \qquad \square. \end{aligned}$$

[14] actually there is something to show here but I leave it to the reader for now

Remark 7.5.3.

In general, it is always possible to take a matrix with real e-values and perform a similarity transformation to a matrix in Jordan form. The similarity transformation is constructed in basically the same way as before; we calculate a Jordan basis then transform by its matrix. This is precisely what we just did in the diagonalizable case. Incidentally, a diagonal matrix is also in Jordan form but obviously the converse is not true in general. Finally, if there are complex e-values you can still perform a similarity transformation to a matrix with a complex Jordan form.

Example 7.5.4. *Revisit Example 7.2.11 where we learned* $A = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$ *had e-values* $\lambda_1 = 0$ *and* $\lambda_2 = 4$ *with e-vectors:* $u_1 = (1, -3)$ *and* $u_2 = (1, 1)$*. Let's follow the advice of the proposition above and diagonalize the matrix. We need to construct* $U = [u_1|u_2]$ *and calculate* U^{-1}*, which is easy since this is a* 2×2 *case:*

$$U = \begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} \qquad \Rightarrow \qquad U^{-1} = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}$$

Now multiply,

$$U^{-1}AU = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ -3 & 1 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 1 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 0 & 4 \\ 0 & 4 \end{bmatrix} = \frac{1}{4}\begin{bmatrix} 0 & 0 \\ 0 & 16 \end{bmatrix}$$

Therefore, we find confirmation of the proposition, $U^{-1}AU = \begin{bmatrix} 0 & 0 \\ 0 & 4 \end{bmatrix}$.

Notice there is one very unsettling aspect of diagonalization; we need to find the inverse of a matrix. Generally this is not pleasant. Orthogonality will offer an insight to help us here. We'll develop additional tools to help with this topic in the next chapter.

Calculational inconvieniences aside, we have all the tools we need to diagonalize a matrix. What then is the point? Why would we care if a matrix is diagonalized? One reason is that we can calculate arbitrary powers of the matrix with a simple calculation. Note that: if $A \sim D$ then $A^k \sim D^k$. In particular: if $D = P^{-1}AP$ then $A = PDP^{-1}$ thus:

$$A^k = \underbrace{AA \cdots A}_{k-factors} = (PDP^{-1})(PDP^{-1}) \cdots (PDP^{-1}) = PD^kP^{-1}.$$

Note, D^k is easy to calculate. Try this formula out on the last example. Try calculating A^{25} directly and then indirectly via this similarity transformation idea.

7.5.1 linear differential equations and e-vectors: diagonalizable case

Any system of linear differential equations with constant coefficients[15] can be reformulated into a single system of linear differential equations in **normal form** $\frac{d\vec{x}}{dt} = A\vec{x} + \vec{f}$ where $A \in \mathbb{R}^{\,n\times n}$ and $\vec{f} : \mathbb{R} \to \mathbb{R}^n$ is a vector-valued function of a real variable which is usually called the **inhomogeneous term**. To begin suppose $\vec{f} = 0$ so the problem becomes the homogeneous system $\frac{d\vec{x}}{dt} = A\vec{x}$. We wish to find a vector-valued function $\vec{x}(t) = (x_1(t), x_2(t), \dots, x_n(t)$ such that when we differentiate it we obtain the same result as if we multiplied it by A. This is what it means to solve the differential equation $\frac{d\vec{x}}{dt} = A\vec{x}$. Essentially, solving this DEqn is like performing n-integrations at once. For each integration we get a constant, these constants are fixed by initial conditions if we have n of them. In any event, the general solution has the form:

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \cdots + c_n\vec{x}_n(t)$$

where $\{\vec{x}_1(t), \vec{x}_2(t), \dots, \vec{x}_n(t)\}$ is a LI set of solutions to $\frac{d\vec{x}}{dt} = A\vec{x}$ meaning $\frac{d\vec{x}_j}{dt} = A\vec{x}_j$ for each $j = 1, 2, \dots, n$. Therefore, if we can find these n-LI solutions then we've solved the problem. It turns out that the solutions are particularly simple if the matrix is diagonalizable: suppose $\{\vec{u}_1, \vec{u}_2, \dots, \vec{u}_n\}$ is an eigenbasis with e-values $\lambda_1, \lambda_2, \dots, \lambda_n$. Let $\vec{x}_j = e^{\lambda_j t}\vec{u}_j$ and observe that

$$\frac{d\vec{x}_j}{dt} = \frac{d}{dt}\big[e^{\lambda_j t}\vec{u}_j\big] = \frac{d}{dt}\big[e^{\lambda_j t}\big]\vec{u}_j = e^{\lambda_j t}\lambda_j\vec{u}_j = e^{\lambda_j t}A\vec{u}_j = Ae^{\lambda_j t}\vec{u}_j = A\vec{x}_j.$$

We find that each eigenvector $\vec{u}_j$ yields a solution $\vec{x}_j = e^{\lambda_j t}\vec{u}_j$. If there are n-LI e-vectors then we obtain n-LI solutions.

Example 7.5.5. *Consider for example, the system*

$$x' = x + y, \qquad y' = 3x - y$$

We can write this as the matrix problem

$$\underbrace{\begin{bmatrix} x' \\ y' \end{bmatrix}}_{d\vec{x}/dt} = \underbrace{\begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}}_{A}\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{\vec{x}}$$

It is easily calculated that A has eigenvalue $\lambda_1 = -2$ with e-vector $\vec{u}_1 = (-1, 3)$ and $\lambda_2 = 2$ with e-vectors $\vec{u}_2 = (1, 1)$. The general solution of $d\vec{x}/dt = A\vec{x}$ is thus

$$\vec{x}(t) = c_1e^{-2t}\begin{bmatrix} -1 \\ 3 \end{bmatrix} + c_2e^{t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -c_1e^{-2t} + c_2e^{2t} \\ 3c_1e^{-2t} + c_2e^{2t} \end{bmatrix}$$

So, the **scalar solutions** *are simply* $\boxed{x(t) = -c_1e^{-2t} + c_2e^{2t}}$ *and* $\boxed{y(t) = 3c_1e^{-2t} + c_2e^{2t}}$.

[15] there are many other linear differential equations which are far more subtle than the ones we consider here, however, this case is of central importance to a myriad of applications

Thus far I have simply told you how to solve the system $d\vec{x}/dt = A\vec{x}$ with e-vectors, it is interesting to see what this means geometrically. For the sake of simplicity we'll continue to think about the preceding example. In it's given form the DEqn is **coupled** which means the equations for the derivatives of the dependent variables x, y cannot be solved one at a time. We have to solve both at once. In the next example I solve the same problem we just solved but this time using a change of variables approach.

Example 7.5.6. *Suppose we change variables using the diagonalization idea: introduce new variables $\bar{x}, \bar{y}$ by $P(\bar{x}, \bar{y}) = (x, y)$ where $P = [\vec{u}_1|\vec{u}_2]$. Note $(\bar{x}, \bar{y}) = P^{-1}(x, y)$. We can diagonalize A by the similarity transformation by P; $D = P^{-1}AP$ where $Diag(D) = (-2, 2)$. Note that $A = PDP^{-1}$ hence $d\vec{x}/dt = A\vec{x} = PDP^{-1}\vec{x}$. Multiply both sides by P^{-1}:*

$$P^{-1}\frac{d\vec{x}}{dt} = P^{-1}PDP^{-1}\vec{x} \quad \Rightarrow \quad \frac{d(P^{-1}\vec{x})}{dt} = D(P^{-1}\vec{x}).$$

You might not recognize it but the equation above is decoupled. In particular, using the notation $(\bar{x}, \bar{y}) = P^{-1}(x, y)$ we read from the matrix equation above that

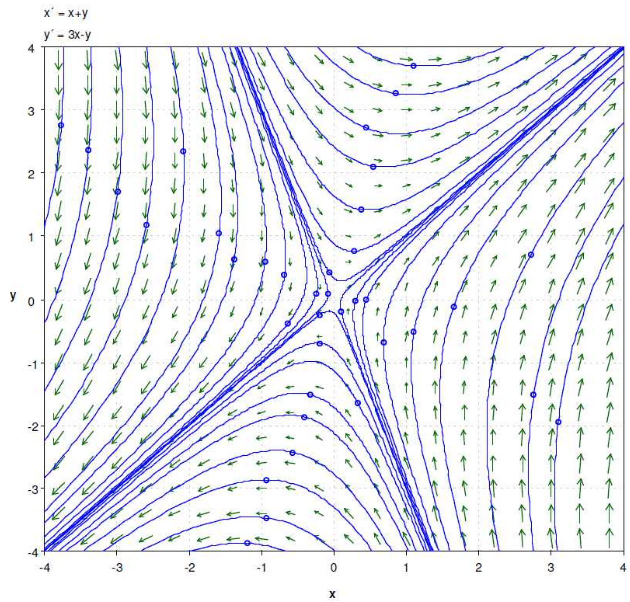
$$\frac{d\bar{x}}{dt} = -2\bar{x}, \qquad \frac{d\bar{y}}{dt} = 2\bar{y}.$$

Separation of variables and a little algebra yields that $\bar{x}(t) = c_1e^{-2t}$ and $\bar{y}(t) = c_2e^{2t}$. Finally, to find the solution back in the original coordinate system we multiply $P^{-1}\vec{x} = (c_1e^{-2t}, c_2e^{2t})$ by P to isolate $\vec{x}$,

$$\vec{x}(t) = \begin{bmatrix} -1 & 1 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} c_1e^{-2t} \\ c_2e^{2t} \end{bmatrix} = \begin{bmatrix} -c_1e^{-2t} + c_2e^{2t} \\ 3c_1e^{-2t} + c_2e^{2t} \end{bmatrix}.$$

This is the same solution we found in the last example. Usually linear algebra texts present this solution because it shows more interesting linear algebra, however, from a pragmatic viewpoint the first method is clearly faster.

Finally, we can better appreciate the solutions we found if we plot the direction field $(x', y') = (x+y, 3x-y)$ via the "pplane" tool in Matlab. I have clicked on the plot to show a few representative trajectories (solutions):



7.5.2 linear differential equations and e-vectors: non-diagonalizable case

Generally, there does not exist an eigenbasis for the matrix in $d\vec{x}/dt = A\vec{x}$. If the e-values are all real then the remaining solutions are obtained from the matrix exponential. It turns out that $X = exp(tA)$ is a solution matrix for $d\vec{x}/dt = A\vec{x}$ thus each column in the matrix exponential is a solution. However, direct computation of the matrix exponential is not usually tractable. Instead, an indirect approach is used. One calculates generalized e-vectors which when multiplied on $exp(tA)$ yield very simple solutions. For example, if $(A-3I)\vec{u}_1 = 0$ and $A-3I)\vec{u}_2 = \vec{u}_1$ and $(A-3I)\vec{u}_3 = \vec{u}_2$ is a chain of generalized e-vectors with e-value $\lambda = 3$ we obtain solutions to $d\vec{x}/dt = A\vec{x}$ of the form:

$$\vec{x}_1(t) = e^{3t}\vec{u}_1, \qquad \vec{x}_2(t) = e^{3t}(\vec{u}_2 + t\vec{u}_1), \qquad \vec{x}_3(t) = e^{3t}(\vec{u}_3 + t\vec{u}_2 + \frac{1}{2}t^2\vec{u}_1).$$

All these formulas stem from a simplification of $\vec{x}_j = e^{tA}\vec{u}_j$ which I call the *the magic formula*. That said, if you'd like to understand what in the world this subsection really means then you probably should read the DEqns chapter. There is one case left, if we have complex e-valued then A is not real-diagonalizable and the solutions actually have the form $\vec{x}(t) = Re(e^{tA}\vec{u})$ or $\vec{x}(t) = Im(e^{tA}\vec{u})$ where $\vec{u}$ is either a complex e-vector or a generalized complex e-vector. Again, I leave the details for the later chapter. My point here is mostly to alert you to the fact that there are deep and interesting connections between diagonalization and the Jordan form and the solutions to corresponding differential equations.

7.6 complex eigenvector examples

Before I begin the material concerning complex eigenvectors I suppose I owe the reader a little background on matrices with complex number entries.

7.6.1 concerning matrices and vectors with complex entries

To begin, we denote the complex numbers by $\mathbb{C}$. As a two-dimensional real vector space we can decompose the complex numbers into the direct sum of the real and pure-imaginary numbers; $\mathbb{C} = \mathbb{R} \oplus i\mathbb{R}$. In other words, any complex number $z \in \mathbb{R}$ can be written as $z = a + ib$ where $a, b \in \mathbb{R}$. It is convenient to define

$$\boxed{\text{If } \lambda = \alpha + i\beta \in \mathbb{C} \quad \text{for } \alpha, \beta \in \mathbb{R} \quad \text{then} \quad Re(\lambda) = \alpha, \quad Im(\lambda) = \beta}$$

The projections onto the real or imaginary part of a complex number are actually linear transformations from $\mathbb{C}$ to $\mathbb{R}$; $Re : \mathbb{C} \to \mathbb{R}$ and $Im : \mathbb{C} \to \mathbb{R}$. Next, complex vectors are simply n-tuples of complex numbers:

$$\boxed{\mathbb{C}^{\ n} = \{(z_1, z_2, \dots, z_n) \mid z_j \in \mathbb{C}\ \}}.$$

Definitions of scalar multiplication and vector addition follow the obvious rules: if $z, w \in \mathbb{C}^{\ n}$ and $c \in \mathbb{C}$ then

$$(z + w)_j = z_j + w_j \qquad (cz)_j = cz_j$$

for each $j = 1, 2, \dots, n$. The complex n-space is again naturally decomposed into the direct sum of two n-dimensional real spaces; $\mathbb{C}^{\ n} = \mathbb{R}^n \oplus i\mathbb{R}^n$. In particular, any complex n-vector can be written uniquely as the sum of real vectors are known as the real and imaginary vector components:

$$\boxed{\text{If } v = a + ib \in \mathbb{C}^{\ n} \text{ for } a, b \in \mathbb{R}^n \text{ then} \quad Re(v) = a, \quad Im(v) = b.}$$

Recall $z = x + iy \in \mathbb{C}$ has complex conjugate $z^* = x - iy$. Let $v \in \mathbb{C}^{\ n}$ we define the complex conjugate of the vector v to be v^* which is the vector of complex conjugates;

$$(v^*)_j = (v_j)^*$$

for each $j = 1, 2, \dots, n$. For example, $[1 + i, 2, 3 - i]^* = [1 - i, 2, 3 + i]$. It is easy to verify the following properties for complex conjugation of numbers and vectors:

$$(v + w)^* = v^* + w^*, \qquad (cv)^* = c^*v^*, \qquad v^{*}* = v.$$

Complex matrices $\mathbb{C}^{\,m\times n}$ can be added, subtracted, multiplied and scalar multiplied in precisely the same ways as real matrices in $\mathbb{R}^{\,m\times n}$. However, we can also identify them as $\mathbb{C}^{\,m\times n} = \mathbb{R}^{\,m\times n} \oplus i\mathbb{R}^{\,m\times n}$ via the real and imaginary part maps $(Re(Z))_{ij} = Re(Z_{ij})$ and $(Im(Z))_{ij} = Im(Z_{ij})$ for all i, j. There is an obvious isomorphism $\mathbb{C}^{\ m\times n} \approxeq \mathbb{R}^{\ 2m\times 2n}$ which follows from stringing out all the real and imaginary parts. Again, complex conjugation is also defined component-wise: $((X + iY)^*)_{ij} = X_{ij} - iY_{ij}$. It's easy to verify that

$$(Z + W)^* = Z^* + W^*, \qquad (cZ)^* = c^*Z^*, \qquad (ZW)^* = Z^*W^*$$

for appropriately sized complex matrices Z, W and $c \in \mathbb{C}$. Conjugation gives us a natural operation to characterize the *reality* of a variable. Let $c \in \mathbb{C}$ then c is **real** iff $c^* = c$. Likewise, if $v \in \mathbb{C}^{\,n}$ then we say that v is **real** iff $v^* = v$. If $Z \in \mathbb{C}^{m\times n}$ then we say that Z is **real** iff $Z^* = Z$. In short, an object is real if all its imaginary components are zero. Finally, while there is of course much more to say we will stop here for now.

7.6.2 complex eigenvectors

Proposition 7.6.1.

If $A \in \mathbb{R}^{\,n\times n}$ has e-value λ and e-vector v then λ^* is likewise an e-value with e-vector v^* for A.

Proof: We assume $Av = \lambda v$ for some $\lambda \in \mathbb{C}$ and $v \in \mathbb{C}^{\,n\times 1}$ with $v \neq 0$. We can write $v = a + ib$ and $\lambda = \alpha + i\beta$ for some $a, b \in \mathbb{R}^n$ and $\alpha, \beta \in \mathbb{R}$. Take the complex conjugate of $Av = \lambda v$ to find $A^*v^* = \lambda^* v^*$. But, $A \in \mathbb{R}^{\,n\times n}$ thus $A^* = A$ and we find $Av^* = \lambda^* v^*$. Moreover, if $v = a + ib$ and $v \neq 0$ then we cannot have $a = 0$ and $b = 0$. Thus $v = a - ib \neq 0$. Therefore, v^* is an e-vector with e-value λ^*. $\square$

This is a useful proposition. It means that once we calculate one complex e-vectors we almost automatically get a second e-vector merely by taking the complex conjugate.

Example 7.6.2. *Let* $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ *and find the e-values and e-vectors of the matrix. Observe that* $det(A - \lambda I) = \lambda^2 + 1$ *hence the eigevalues are* $\lambda = \pm i$*. Find* $u_1 = (u, v)$ *such that* $(A - iI)u_1 = 0$

$$0 = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -iu + v \\ -u - iv \end{bmatrix} \Rightarrow \begin{array}{c} -iu + v = 0 \\ -u - iv = 0 \end{array} \Rightarrow v = iu \Rightarrow \boxed{u_1 = u\begin{bmatrix} 1 \\ i \end{bmatrix}}$$

We find infinitely many complex eigenvectors, one for each nonzero complex constant u*. In applications, in may be convenient to set* $u = 1$ *so we can write,* $u_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Let's generalize the last example.

Example 7.6.3. *Let* $\theta \in \mathbb{R}$ *and define* $A = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$ *and find the e-values and e-vectors of the matrix. Observe*

$$\begin{aligned} 0 = det(A - \lambda I) &= det\begin{bmatrix} \cos\theta - \lambda & \sin\theta \\ -\sin\theta & \cos\theta - \lambda \end{bmatrix} \\ &= (\cos\theta - \lambda)^2 + \sin^2\theta \\ &= \cos^2\theta - 2\lambda\cos\theta + \lambda^2 + \sin^2\theta \\ &= \lambda^2 - 2\lambda\cos\theta + 1 \\ &= (\lambda - \cos\theta)^2 - \cos^2\theta + 1 \\ &= (\lambda - \cos\theta)^2 + \sin^2\theta \end{aligned}$$

Thus $\lambda = \cos\theta \pm i\sin\theta = e^{\pm i\theta}$. *Find* $u_1 = (u, v)$ *such that* $(A - e^{i\theta} I)u_1 = 0$

$$0 = \left[\begin{array}{cc} -i\sin\theta & \sin\theta \\ -\sin\theta & -i\sin\theta \end{array}\right]\left[\begin{array}{c} u \\ v \end{array}\right] = \left[\begin{array}{c} 0 \\ 0 \end{array}\right] \quad \Rightarrow \quad -iu\sin\theta + v\sin\theta = 0$$

If $\sin\theta \neq 0$ *then we divide by* $\sin\theta$ *to obtain* $v = iu$ *hence* $u_1 = (u, iu) = u(1, i)$ *which is precisely what we found in the preceding example. However, if* $\sin\theta = 0$ *we obtain no condition what-so-ever on the matrix. That special case is not complex. Moreover, if* $\sin\theta = 0$ *it follows* $\cos\theta = 1$ *and in fact* $A = I$ *in this case. The identity matrix has the repeated eigenvalue of* $\lambda = 1$ *and every vector in* $\mathbb{R}^2$ *is an e-vector.*

Example 7.6.4. *Let* $A = \left[\begin{array}{ccc} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{array}\right]$ *find the e-values and e-vectors of* A.

$$0 = det(A - \lambda I) = \left[\begin{array}{ccc} 1-\lambda & 1 & 0 \\ -1 & 1-\lambda & 0 \\ 0 & 0 & 3-\lambda \end{array}\right]$$
$$= (3-\lambda)\big[(1-\lambda)^2 + 1\big]$$

Hence $\lambda_1 = 3$ *and* $\lambda_2 = 1 \pm i$. *We have a pair of complex e-values and one real e-value. Notice that for any* $n \times n$ *matrix we must have at least one real e-value since all odd polynomials possess at least one zero. Let's begin with the real e-value. Find* $u_1 = (u, v, w)$ *such that* $(A - 3I)u_1 = 0$:

$$rref\left[\begin{array}{ccc|c} -2 & 1 & 0 & 0 \\ -1 & -2 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad \boxed{u_1 = w\left[\begin{array}{c} 0 \\ 0 \\ 1 \end{array}\right]}$$

Next find e-vector with $\lambda_2 = 1 + i$. *We wish to find* $u_2 = (u, v, w)$ *such that* $(A - (1+i)I)u_2 = 0$:

$$\left[\begin{array}{ccc|c} -i & 1 & 0 & 0 \\ -1 & -i & 0 & 0 \\ 0 & 0 & -1-i & 0 \end{array}\right] \xrightarrow{r_2 + ir_1 \to r_2} \xrightarrow{\frac{1}{-1-i}r_3 \to r_3} \left[\begin{array}{ccc|c} -i & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

One more row-swap and a rescaling of row 1 and it's clear that

$$rref\left[\begin{array}{ccc|c} -i & 1 & 0 & 0 \\ -1 & -i & 0 & 0 \\ 0 & 0 & -1-i & 0 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \Rightarrow \quad \begin{array}{c} u + iv = 0 \\ w = 0 \end{array} \quad \Rightarrow \quad \boxed{u_2 = v\left[\begin{array}{c} i \\ 1 \\ 0 \end{array}\right]}$$

I chose the free parameter to be v. *Any choice of a nonzero complex constant* v *will yield an e-vector with e-value* $\lambda_2 = 1 + i$. *For future reference, it's worth noting that if we choose* $v = 1$ *then we find*

$$u_2 = \left[\begin{array}{c} 0 \\ 1 \\ 0 \end{array}\right] + i\left[\begin{array}{c} 1 \\ 0 \\ 0 \end{array}\right]$$

We identify that $Re(u_2) = e_2$ *and* $Im(u_2) = e_1$

Example 7.6.5. *Let* $B = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ *and let* $C = \begin{bmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{bmatrix}$. *Define* A *to be the* **block matrix**

$$A = \left[\begin{array}{c|c} B & 0 \\ \hline 0 & C \end{array}\right] = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ \hline 0 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\ 0 & 0 & -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{array}\right]$$

find the e-values and e-vectors of the matrix. Block matrices have nice properties: the blocks behave like numbers. Of course there is something to prove here, and I have yet to discuss block multiplication in these notes.

$$det(A - \lambda I) = det \begin{bmatrix} B - \lambda I & 0 \\ 0 & C - \lambda I \end{bmatrix} = det(B - \lambda I)det(C - \lambda I)$$

Notice that both B *and* C *are rotation matrices.* B *is the rotation matrix with* $\theta = \pi/2$ *whereas* C *is the rotation by* $\theta = \pi/3$. *We already know the e-values and e-vectors for each of the blocks if we ignore the other block. It would be nice if a block matrix allowed for analysis of each block one at a time. This turns out to be true, I can tell you without further calculation that we have e-values* $\lambda_1 = \pm i$ *and* $\lambda_2 = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ *which have complex e-vectors*

$$u_1 = \begin{bmatrix} 1 \\ i \\ 0 \\ 0 \end{bmatrix} = e_1 + ie_2 \qquad u_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ i \end{bmatrix} = e_3 + ie_4.$$

Finally, you might wonder are there matrices which have a repeated complex e-value. And if so are there always as many complex e-vectors as there are complex e-values? The answer: sometimes. Take for instance $A = \left[\begin{array}{c|c} B & 0 \\ \hline 0 & B \end{array}\right]$ (where B is the same B as in the preceding example) this matrix will have a repeated e-value of $\lambda = \pm i$ and you'll be able to calculate $u_1 = e_1 \pm ie_2$ and $u_2 = e_3 \pm ie_4$ are linearly independent e-vectors for A. However, there are other matrices for which only one complex e-vector is available despite a repeat of the e-value.

7.7 linear independendence of complex eigenvectors

The complex case faces essentially the same difficulties. Complex e-vectors give us pair of linearly independent vectors with which we are welcome to form a basis. However, the complex case can also fail to provide a sufficient number of complex e-vectors to fill out a basis. In such a case we can still look for generalized complex e-vectors. Each generalized complex e-vector will give us a pair of linearly independent real vectors which are linearly independent from the pairs already constructed from the complex e-vectors. Although many of the arguments transfer directly from pervious sections there are a few features which are uniquely complex.

Proposition 7.7.1.

If $A \in \mathbb{R}^{m\times n}$ has complex e-value $\lambda = \alpha + i\beta$ such that $\beta \neq 0$ and e-vector $v = a + ib \in \mathbb{C}^{n\times 1}$ such that $a, b \in \mathbb{R}^n$ then $\lambda^* = \alpha - i\beta$ is a complex e-value with e-vector $v^* = a - ib$ and $\{v, v^*\}$ is a linearly independent set of vectors over $\mathbb{C}$.

Proof: Proposition 7.6.1 showed that v^* is an e-vector with e-value $\lambda^* = \alpha - i\beta$. Notice that $\lambda \neq \lambda^*$ since $b \neq 0$. Therefore, v and v^* are e-vectors with distinct e-values. Note that Proposition 7.4.2 is equally valid for complex e-values and e-vectors. Hence, $\{v, v^*\}$ is linearly independent since these are e-vectors with distinct e-values. □

Proposition 7.7.2.

If $A \in \mathbb{R}^{m\times n}$ has complex e-value $\lambda = \alpha + i\beta$ such that $\beta \neq 0$ and e-vector $v = a + ib \in \mathbb{C}^{n\times 1}$ such that $a, b \in \mathbb{R}^n$ then $a \neq 0$ and $b \neq 0$.

Proof: Expand $Av = \lambda v$ into the real components,

$$\lambda v = (\alpha + i\beta)(a + ib) = \alpha a - \beta b + i(\beta a + \alpha b)$$

and

$$Av = A(a + ib) = Aa + iAb$$

Equating real and imaginary components yeilds two real matrix equations,

$$Aa = \alpha a - \beta b \quad \text{and} \quad Ab = \beta a + \alpha b$$

Suppose $a = 0$ towards a contradiction, note that $0 = -\beta b$ but then $b = 0$ since $\beta \neq 0$ thus $v = 0 + i0 = 0$ but this contradicts v being an e-vector. Likewise if $b = 0$ we find $\beta a = 0$ which implies $a = 0$ and again $v = 0$ which contradicts v being an e-vector. Therefore, $a, b \neq 0$. □

Proposition 7.7.3.

If $A \in \mathbb{R}^{n\times n}$ and $\lambda = \alpha + i\beta \in \mathbb{C}$ with $\alpha, \beta \in \mathbb{R}$ and $\beta \neq 0$ is an e-value with e-vector $v = a + ib \in \mathbb{C}^{n\times 1}$ and $a, b \in \mathbb{R}^n$ then $\{a, b\}$ is a linearly independent set of real vectors.

Proof: Add and subtract the equations $v = a + ib$ and $v^* = a - ib$ to deduce

$$a = \tfrac{1}{2}(v + v*) \qquad \text{and} \qquad b = \tfrac{1}{2i}(v - v*)$$

Let $c_1, c_2 \in \mathbb{R}$ then consider,

$$\begin{aligned} c_1 a + c_2 b = 0 \quad &\Rightarrow \quad c_1[\tfrac{1}{2}(v + v*)] + c_2[\tfrac{1}{2i}(v - v*)] = 0 \\ &\Rightarrow \quad [c_1 - ic_2]v + [c_1 + ic_2]v^* = 0 \end{aligned}$$

But, $\{v, v^*\}$ is linearly independent hence $c_1 - ic_2 = 0$ and $c_1 + ic_2 = 0$. Adding these equations gives $2c_1 = 0$. Subtracting yields $2ic_2 = 0$. Thus $c_1 = c_2 = 0$ and we conclude $\{a, b\}$ is linearly independent. □

Proposition 7.7.4.

If $A \in \mathbb{R}^{m\times n}$ has complex e-value $\lambda = \alpha + i\beta$ such that $\beta \neq 0$ and chain of generalized e-vectors $v_k = a_k + ib_k \in \mathbb{C}^{\,n\times 1}$ of orders $k = 1, 2, \dots, m$ such that $a_k, b_k \in \mathbb{R}$ then $\{a_1, b_1, a_2, b_2, \dots a_m, b_m\}$ is linearly independent.

Proof: will earn bonus points. Give it to me it soon please. □

7.8 diagonalization in complex case

Given a matrix $A \in \mathbb{R}^{\,n\times n}$, we restrict our attention to the case that there are enough e-vectors both real and complex to complete a basis for $\mathbb{R}^n$. We have seen that each complex e-vector yields two LI real vectors so if we have k-complex e-vectors we assume that there are another $n-2k$-real e-vectors to complex a basis for $\mathbb{R}^n$. This is not an e-basis, but it's close. We seek to analyze how this basis will transform a given matrix. These notes loosely follow Lay's Text pages 339-341.

To begin let's try an experiment using the e-vector and complex e-vectors for found in Example 7.6.4. We'll perform a similarity transformation based on this complex basis: $\beta = \{(i, 1, 0), (-i, 1, 0), (0, 0, 1)\}$. Notice that

$$[\beta] = \begin{bmatrix} i & -i & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \Rightarrow \quad [\beta]^{-1} = \frac{1}{2}\begin{bmatrix} -i & 1 & 0 \\ i & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

Then, we can calculate that

$$[\beta]^{-1}A[\beta] = \frac{1}{2}\begin{bmatrix} -i & 1 & 0 \\ i & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} i & -i & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1+i & 0 & 0 \\ 0 & 1-i & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

I would say that A is complex-diagonalizable in this case. However, usually we are interested in obtaining factorizations in terms of real matrices so we should continue thinking.

Example 7.8.1. *Suppose* $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. *We can calculate that* $det(A-\lambda I) = (a-\lambda)^2 + b^2 = 0$ *hence we have two(one), typically complex, e-value* $\lambda = a \pm ib$. *Denoting* $r = \sqrt{a^2+b^2}$ *(the modulus of* $a+ib$*). We can work out that*

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} = r\begin{bmatrix} a/r & -b/r \\ b/r & a/r \end{bmatrix} = \begin{bmatrix} r & 0 \\ 0 & r \end{bmatrix}\begin{bmatrix} \cos(\beta) & -\sin(\beta) \\ \sin(\beta) & \cos(\beta) \end{bmatrix}$$

Therefore, a 2×2 *matrix with complex-evalue will factor into a dilation by the modulus of the e-value* $|\lambda|$ *times a rotation by the arugment of the e-value. If we write* $\lambda = rexp(i\beta)$ *then we can identify that* $r > 0$ *is the modulus and* β *is an arugment (there is degeneracy here because angle are multiply defined).*

Continuing to think about the 2×2 case, note that our complex e-vector yields two real LI-vectors and hence a basis for $\mathbb{R}^2$. Performing a similarity transformation by $P = [Re(\vec{u})|Im(\vec{u})]$ will uncover the rotation hiding inside the matrix. We may work this out in lecture if there is interest.

Chapter 8

systems of differential equations

This chapter is an application linear algebra to the problem of differential equations. We'll see how to use primarily algebraic arguments to solve systems of linear differential equations. The big idea of this chapter is centered around the matrix exponential. It turns out that once we can extract all our solutions from the matrix exponential by simple multiplication of e-vectors and generalized e-vectors. Both complex and real e-vectors are generally of interest. This is one of the reasons I do not limit our discussion earlier in this course to merely real e-vectors. The algebra allows for real and complex e-values and both have physical significance in basic applications of linear algebra.

For the interested reader: Chapter 4 of my Spring 2012 Math 334 notes have a much more readable treatment of systems of differential equations. I include more examples, less theory, and some discussion of the geometric meaning of e-values as they relate to critical points of ODEs. In fact, I was tempted to transfer many examples from those notes to these for the sake of illustrating how linear algebra is used in DEqns. I resisted this temptation since many of you have not had DEqns at this point. I urge you to think about these connections sometime before you graduate. Linear algebra and differential equations complement one another in many ways. One great textbook to read about differential equations with linear algebra is Finney and Ostberg's *Elementary Differential Equations With Linear Algebra*.

8.1 calculus of matrices

A more apt title would be "calculus of matrix-valued functions of a real variable".

Definition 8.1.1.

A matrix-valued function of a real variable is a function from $I \subseteq \mathbb{R}$ to $\mathbb{R}^{m \times n}$. Suppose $A : I \subseteq \mathbb{R} \to \mathbb{R}^{m \times n}$ is such that $A_{ij} : I \subseteq \mathbb{R} \to \mathbb{R}$ is differentiable for each i, j then we define

$$\frac{dA}{dt} = \left[\frac{dA_{ij}}{dt}\right]$$

which can also be denoted $(A')_{ij} = A'_{ij}$. We likewise define $\int A dt = [\int A_{ij} dt]$ for A with integrable components. Definite integrals and higher derivatives are also defined componentwise.

Example 8.1.2. *Suppose* $A(t) = \begin{bmatrix} 2t & 3t^2 \\ 4t^3 & 5t^4 \end{bmatrix}$. *I'll calculate a few items just to illustrate the definition above. calculate; to differentiate a matrix we differentiate each component one at a time:*

$$A'(t) = \begin{bmatrix} 2 & 6t \\ 12t^2 & 20t^3 \end{bmatrix} \qquad A''(t) = \begin{bmatrix} 0 & 6 \\ 24t & 60t^2 \end{bmatrix} \qquad A'(0) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}$$

Integrate by integrating each component:

$$\int A(t)dt = \begin{bmatrix} t^2 + c_1 & t^3 + c_2 \\ t^4 + c_3 & t^5 + c_4 \end{bmatrix} \qquad \int_0^2 A(t)dt = \begin{bmatrix} t^2\big|_0^2 & t^3\big|_0^2 \\ t^4\big|_0^2 & t^5\big|_0^2 \end{bmatrix} = \begin{bmatrix} 4 & 8 \\ 16 & 32 \end{bmatrix}$$

Proposition 8.1.3.

Suppose A, B are matrix-valued functions of a real variable, f is a function of a real variable, c is a constant, and C is a constant matrix then

1. $(AB)' = A'B + AB'$ (product rule for matrices)
2. $(AC)' = A'C$
3. $(CA)' = CA'$
4. $(fA)' = f'A + fA'$
5. $(cA)' = cA'$
6. $(A + B)' = A' + B'$

where each of the functions is evaluated at the same time t and I assume that the functions and matrices are differentiable at that value of t and of course the matrices A, B, C are such that the multiplications are well-defined.

Proof: Suppose $A(t) \in \mathbb{R}^{m\times n}$ and $B(t) \in \mathbb{R}^{n\times p}$ consider,

$$\begin{aligned}
{(AB)'}_{ij} &= \tfrac{d}{dt}((AB)_{ij}) && \text{defn. derivative of matrix}\\
&= \tfrac{d}{dt}(\textstyle\sum_k A_{ik}B_{kj}) && \text{defn. of matrix multiplication}\\
&= \textstyle\sum_k \tfrac{d}{dt}(A_{ik}B_{kj}) && \text{linearity of derivative}\\
&= \textstyle\sum_k \big[\tfrac{dA_{ik}}{dt}B_{kj} + A_{ik}\tfrac{dB_{kj}}{dt}\big] && \text{ordinary product rules}\\
&= \textstyle\sum_k \tfrac{dA_{ik}}{dt}B_{kj} + \sum_k A_{ik}\tfrac{dB_{kj}}{dt} && \text{algebra}\\
&= (A'B)_{ij} + (AB')_{ij} && \text{defn. of matrix multiplication}\\
&= (A'B + AB')_{ij} && \text{defn. matrix addition}
\end{aligned}$$

this proves (1.) as i, j were arbitrary in the calculation above. The proof of (2.) and (3.) follow quickly from (1.) since C constant means $C' = 0$. Proof of (4.) is similar to (1.):

$$\begin{aligned}
{(fA)'}_{ij} &= \tfrac{d}{dt}((fA)_{ij}) && \text{defn. derivative of matrix}\\
&= \tfrac{d}{dt}(fA_{ij}) && \text{defn. of scalar multiplication}\\
&= \tfrac{df}{dt}A_{ij} + f\tfrac{dA_{ij}}{dt} && \text{ordinary product rule}\\
&= (\tfrac{df}{dt}A + f\tfrac{dA}{dt})_{ij} && \text{defn. matrix addition}\\
&= (\tfrac{df}{dt}A + f\tfrac{dA}{dt})_{ij} && \text{defn. scalar multiplication.}
\end{aligned}$$

The proof of (5.) follows from taking $f(t) = c$ which has $f' = 0$. I leave the proof of (6.) as an exercise for the reader. $\square$.

To summarize: the calculus of matrices is the same as the calculus of functions with the small qualifier that we must respect the rules of matrix algebra. The noncommutativity of matrix multiplication is the main distinguishing feature.

Since we're discussing this type of differentiation perhaps it would be worthwhile for me to insert a comment about complex functions here. Differentiation of functions from $\mathbb{R}$ to $\mathbb{C}$ is defined by splitting a given function into its real and imaginary parts then we just differentiate with respect to the real variable one component at a time. For example:

$$\begin{aligned}
\frac{d}{dt}(e^{2t}\cos(t) + ie^{2t}\sin(t)) &= \frac{d}{dt}(e^{2t}\cos(t)) + i\frac{d}{dt}(e^{2t}\sin(t))\\
&= (2e^{2t}\cos(t) - e^{2t}\sin(t)) + i(2e^{2t}\sin(t) + e^{2t}\cos(t)) \qquad (8.1)\\
&= e^{2t}(2+i)(\cos(t) + i\sin(t))\\
&= (2+i)e^{(2+i)t}
\end{aligned}$$

where we made use of the identity[1] $e^{x+iy} = e^x(\cos(y) + i\sin(y))$. We just saw that $\frac{d}{dt}e^{\lambda t} = \lambda e^{\lambda t}$ which seems obvious enough until you appreciate that we just proved it for $\lambda = 2 + i$. We make use of this calculation in the next section in the case we have complex e-values.

[1]or definition, depending on how you choose to set-up the complex exponential, I take this as the definition in calculus II

8.2 introduction to systems of linear differential equations

A differential equation (DEqn) is simply an equation that is stated in terms of derivatives. The highest order derivative that appears in the DEqn is called the *order* of the DEqn. In calculus we learned to integrate. Recall that $\int f(x)dx = y$ iff $\frac{dy}{dx} = f(x)$. Everytime you do an integral you are solving a first order DEqn. In fact, it's an *ordinary* DEnq (ODE) since there is only one indpendent variable (it was x). A system of ODEs is a set of differential equations with a common independent variable. It turns out that any linear differential equation can be written as a system of ODEs in *normal form*. I'll define *normal form* then illustrate with a few examples.

Definition 8.2.1.

Let t be a real variable and suppose $x_1, x_2, \dots, x_n$ are functions of t. If A_{ij}, f_i are functions of t for all $1 \leq i \leq m$ and $1 \leq j \leq n$ then the following set of differential equations is defined to be a system of linear differential equations in **normal form**:

$$\begin{aligned} \tfrac{dx_1}{dt} &= A_{11}x_1 + A_{12}x_2 + \cdots A_{1n}x_n + f_1 \\ \tfrac{dx_2}{dt} &= A_{21}x_1 + A_{22}x_2 + \cdots A_{2n}x_n + f_2 \\ \vdots &= \vdots \qquad \vdots \quad \cdots \quad \vdots \\ \tfrac{dx_m}{dt} &= A_{m1}x_1 + A_{m2}x_2 + \cdots A_{mn}x_n + f_m \end{aligned}$$

In matrix notation, $\frac{dx}{dt} = Ax + f$. The system is called homogeneous if $f = 0$ whereas the system is called nonhomogeneous if $f \neq 0$. The system is called **constant coefficient** if $\frac{d}{dt}(A_{ij}) = 0$ for all i, j. If $m = n$ and a set of intial conditions $x_1(t_0) = y_1, x_2(t_0) = y_2, \dots, x_n(t_0) = y_n$ are given then this is called an **initial value problem** (IVP).

Example 8.2.2. *If x is the number of tigers and y is the number of rabbits then*

$$\frac{dx}{dt} = x + y \qquad\qquad \frac{dy}{dt} = -100x + 20y$$

is a model for the population growth of tigers and bunnies in some closed environment. My logic for my made-up example is as follows: the coefficient 1 is the growth rate for tigers which don't breed to quickly. Whereas the growth rate for bunnies is 20 since bunnies reproduce like, well bunnies. Then the y in the $\frac{dx}{dt}$ equation goes to account for the fact that more bunnies means more tiger food and hence the tiger reproduction should speed up (this is probably a bogus term, but this is my made up example so deal). Then the $-100x$ term accounts for the fact that more tigers means more tigers eating bunnies so naturally this should be negative. In matrix form

$$\begin{bmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -100 & 20 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

How do we solve such a system? This is the question we seek to answer.

The preceding example is a *predator-prey* model. There are many other terms that can be added to make the model more realistic. Ultimately all population growth models are only useful if they can account for all significant effects. History has shown population growth models are of only limited use for humans.

Example 8.2.3. Reduction of Order *in calculus II you may have studied how to solve* $y''+by'+cy=0$ *for any choice of constants* b,c. *This is a second order ODE. We can reduce it to a system of first order ODEs by introducing new variables:* $x_1=y$ *and* $x_2=y'$ *then we have*

$$x_1' = y' = x_2$$

and,

$$x_2' = y'' = -by' - cy = -bx_2 - cx_1$$

As a matrix DEqn,

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}' = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Similarly if $y'''' + 2y''' + 3y'' + 4y' + 5y = 0$ *we can introduce variables to reduce the order:* $x_1 = y, x_2 = y', x_3 = y'', x_4 = y'''$ *then you can show:*

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -5 & -4 & -3 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

is equivalent to $y'''' + 2y''' + 3y'' + 4y' + 5y = 0$. *We call the matrix above the* **companion matrix** *of the* n*-th order constant coefficient ODE. There is a beautiful interplay between solutions to* n*-th order ODEs and the linear algebra of the compansion matrix.*

Example 8.2.4. *Suppose* $y'' + 4y' + 5y = 0$ *and* $x'' + x = 0$. *The is a system of linear second order ODEs. It can be recast as a system of 4 first order ODEs by introducing new variables:* $x_1 = y, x_2 = y'$ *and* $x_3 = x, x_4 = x'$. *In matrix form the given system in normal form is:*

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -5 & -4 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

The companion matrix above will be found to have eigenvalues $\lambda = -2 \pm i$ *and* $\lambda = \pm i$. *I know this without further calculation purely on the basis of what I know from DEqns and the interplay I alluded to in the last example.*

Example 8.2.5. *If* $y'''' + 2y'' + y = 0$ *we can introduce variables to reduce the order:* $x_1 = y, x_2 = y', x_3 = y'', x_4 = y'''$ *then you can show:*

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & -2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$$

is equivalent to $y'''' + 2y'' + y = 0$. If we solve the matrix system then we solve the equation in y and vice-versa. I happen to know the solution to the y equation is $y = c_1 \cos t + c_2 \sin t + c_3 t \cos t + c_4 t \sin t$. From this I can deduce that the companion matrix has a repeated e-value of $\lambda = \pm i$ and just one complex e-vector and its conjugate. This matrix would answer the bonus point question I posed a few sections back. I invite the reader to verify my claims.

Remark 8.2.6.

For those of you who will or have taken math 334 my guesswork above is predicated on two observations:

1. the "auxillarly" or "characteristic" equation in the study of the constant coefficient ODEs is identical to the characteristic equation of the companion matrix.

2. ultimately eigenvectors will give us exponentials and sines and cosines in the solution to the matrix ODE whereas solutions which have multiplications by t stem from generalized e-vectors. Conversely, if the DEqn has a t or t^2 multiplying cosine, sine or exponential functions then the companion matrix must in turn have generalized e-vectors to account for the t or t^2 etc...

I will not explain (1.) in this course, however we will hopefully make sense of (2.) by the end of this section.

8.3 the matrix exponential

Perhaps the most important first order ODE is $\frac{dy}{dt} = ay$. This DEqn says that the rate of change in y is simply proportional to the amount of y at time t. Geometrically, this DEqn states the solutions value is proportional to its slope at every point in its domain. The solution[2] is the exponential function $y(t) = e^{at}$.

We face a new differential equation; $\frac{dx}{dt} = Ax$ where x is a vector-valued function of t and $A \in \mathbb{R}^{n \times n}$. Given our success with the exponential function for the scalar case is it not natural to suppose that $x = e^{tA}$ is the solution to the matrix DEqn? The answer is yes. However, we need to define a few items before we can understand the true structure of the claim.

Definition 8.3.1.

Let $A \mathbb{R}^{n \times n}$ define $e^A \in \mathbb{R}^{n \times n}$ by the following formula

$$e^A = \sum_{n=0}^{\infty} \tfrac{1}{n!} A^n = I + A + \tfrac{1}{2} A^2 + \tfrac{1}{3!} A^3 + \cdots .$$

We also denote $e^A = exp(A)$ when convenient.

[2] ok, technically separation of variables yields the general solution $y = ce^{at}$ but I'm trying to focus on the exponential function for the moment.

This definition is the natural extension of the Taylor series formula for the exponential function we derived in calculus II. Of course, you should be skeptical of this definition. How do I even know the series converges for an arbitrary matrix A? And, what do I even mean by "converge" for a series of matrices? (skip the next subsection if you don't care)

8.3.1 analysis for matrices

Remark 8.3.2.

The purpose of this section is to alert the reader to the gap in the development here. We will use the matrix exponential despite our inability to fully grasp the underlying analysis. Much in the same way we calculate series in calculus without proving every last theorem. I will attempt to at least sketch the analytical underpinnings of the matrix exponential. The reader will be happy to learn this is not part of the required material.

We use the Frobenius norm for $A \in \mathbb{R}^{n\times n}$, $||A|| = \sqrt{\sum_{i,j}(A_{ij})^2}$. We already proved this was a norm in a previous chapter. A sequence of square matrices is a function from $\mathbb{N}$ to $\mathbb{R}^{n\times n}$. We say the sequence $\{A_n\}_{n=1}^{\infty}$ converges to $L \in \mathbb{R}^{n\times n}$ iff for each $\epsilon > 0$ there exists $M \in \mathbb{N}$ such that $||A_n - L|| < \epsilon$ for all $n > M$. This is the same definition we used in calculus, just now the norm is the Frobenius norm and the functions are replaced by matrices. The definition of a series is also analogus to the definition you learned in calculus II.

Definition 8.3.3.

Let $A_k \in \mathbb{R}^{m\times m}$ for all k, the sequence of partial sums of $\sum_{k=0}^{\infty} A_k$ is given by $S_n = \sum_{k=1}^{n} A_k$. We say the series $\sum_{k=0}^{\infty} A_k$ converges to $L \in \mathbb{R}^{m\times m}$ iff the sequence of partial sums converges to L. In other words,

$$\sum_{k=1}^{\infty} A_k = \lim_{n\to\infty} \sum_{k=1}^{n} A_k.$$

Many of the same theorems hold for matrices:

Proposition 8.3.4.

Let $t \to S_A(t) = \sum A_k(t)$ and $t \to S_B(t) = \sum_k B_k(t)$ be matrix valued functions of a real variable t where the series are uniformly convergent and $c \in \mathbb{R}$ then

1. $\sum_k cA_k = c\sum_k A_k$
2. $\sum_k (A_k + B_k) = \sum_k A_k + \sum_k B_k$
3. $\frac{d}{dt}\left[\sum_k A_k\right] = \sum_k \frac{d}{dt}\left[A_k\right]$
4. $\int\left[\sum_k A_k\right]dt = C + \sum_k \int A_k dt$ where C is a constant matrix.

The summations can go to infinity and the starting index can be any integer.

Uniform convergence means the series converge without regard to the value of t. Let me just refer you to the analysis course, we should discuss uniform convergence in that course, the concept equally well applies here. It is the crucial fact which one needs to interchange the limits which are implicit within $\sum_k$ and $\frac{d}{dt}$. There are counterexamples in the case the series is not uniformly convergent. Fortunately,

Proposition 8.3.5.

Let A be a square matrix then $exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$ is a uniformly convergent series of matrices.

Basically, the argument is as follows: The set of square matrices with the Frobenius norm is isometric to $\mathbb{R}^{n^2}$ which is a complete space. A complete space is one in which every Cauchy sequence converges. We can show that the sequence of partial sums for $exp(A)$ is a Cauchy sequence in $\mathbb{R}^{\,n\times n}$ hence it converges. Obviously I'm leaving some details out here. You can look at the excellent *Calculus* text by Apostle to see more gory details. Also, if you don't like my approach to the matrix exponential then he has several other ways to look it.
(Past this point I expect you to start following along again.)

8.3.2 formulas for the matrix exponential

Now for the fun part.

Proposition 8.3.6.

Let A be a square matrix then $\frac{d}{dt}\big[exp(tA)\big] = Aexp(tA)$

Proof: I'll give the proof in two notations. First,

$$\begin{aligned}
\tfrac{d}{dt}\big[exp(tA)\big] &= \tfrac{d}{dt}\Big[\sum_{k=0}^{\infty} \tfrac{1}{k!} t^k A^k\Big] && \text{defn. of matrix exponential} \\
&= \sum_{k=0}^{\infty} \tfrac{d}{dt}\big[\tfrac{1}{k!} t^k A^k\big] && \text{since matrix exp. uniformly conv.} \\
&= \sum_{k=0}^{\infty} \tfrac{k}{k!} t^{k-1} A^k && A^k \text{ constant and } \tfrac{d}{dt}(t^k) = kt^{k-1} \\
&= A\sum_{k=1}^{\infty} \tfrac{1}{(k-1)!} t^{k-1} A^{k-1} && \text{since } k! = k(k-1)! \text{ and } A^k = AA^{k-1}. \\
&= Aexp(tA) && \text{defn. of matrix exponential.}
\end{aligned}$$

I suspect the following argument is easier to follow:

$$\begin{aligned}
\tfrac{d}{dt}(exp(tA)) &= \tfrac{d}{dt}(I + tA + \tfrac{1}{2}t^2A^2 + \tfrac{1}{3!}t^3A^3 + \cdots) \\
&= \tfrac{d}{dt}(I) + \tfrac{d}{dt}(tA) + \tfrac{1}{2}\tfrac{d}{dt}(t^2A^2) + \tfrac{1}{3\cdot 2}\tfrac{d}{dt}(t^3A^3) + \cdots \\
&= A + tA^2 + \tfrac{1}{2}t^2A^3 + \cdots \\
&= A(I + tA + \tfrac{1}{2}t^2A^2 + \cdots) \\
&= Aexp(tA). \qquad \square
\end{aligned}$$

Notice that we have all we need to see that $exp(tA)$ is a matrix of solutions to the differential equation $x' = Ax$. The following prop. follows from the preceding prop. and Prop. 2.3.12.

Proposition 8.3.7.

If $X = exp(tA)$ then $X' = Aexp(tA) = AX$. This means that each column in X is a solution to $x' = Ax$.

Let us illustrate this proposition with a particularly simple example.

Example 8.3.8. *Suppose $x' = x, y' = 2y, z' = 3z$ then in matrix form we have:*

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

The coefficient matrix is diagonal *which makes the k-th power particularly easy to calculate,*

$$A^k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}^k = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{bmatrix}$$

$$\Rightarrow \quad exp(tA) = \sum_{k=0}^{\infty} \frac{t^k}{k!} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^k & 0 \\ 0 & 0 & 3^k \end{bmatrix} = \begin{bmatrix} \sum_{k=0}^{\infty} \frac{t^k}{k!}1^k & 0 & 0 \\ 0 & \sum_{k=0}^{\infty} \frac{t^k}{k!}2^k & 0 \\ 0 & 0 & \sum_{k=0}^{\infty} \frac{t^k}{k!}3^k \end{bmatrix}$$

$$\Rightarrow \quad exp(tA) = \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{3t} \end{bmatrix}$$

Thus we find three solutions to $x' = Ax$,

$$x_1(t) = \begin{bmatrix} e^t \\ 0 \\ 0 \end{bmatrix} \qquad x_2(t) = \begin{bmatrix} 0 \\ e^{2t} \\ 0 \end{bmatrix} \qquad x_3(t) = \begin{bmatrix} 0 \\ 0 \\ e^{3t} \end{bmatrix}$$

In turn these vector solutions amount to the solutions $x = e^t, y = 0, z = 0$ or $x = 0, y = e^{2t}, z = 0$ or $x = 0, y = 0, z = e^{3t}$. It is easy to check these solutions.

Usually we cannot calculate the matrix exponential explicitly by such a straightforward calculation. We need e-vectors and sometimes generalized e-vectors to reliably calculate the solutions of interest.

Proposition 8.3.9.

If A, B are square matrices such that $AB = BA$ then $e^{A+B} = e^A e^B$

Proof: I'll show how this works for terms up to quadratic order,

$$e^A e^B = (1 + A + \tfrac{1}{2}A^2 + \cdots)(1 + B + \tfrac{1}{2}B^2 + \cdots) = 1 + (A + B) + \tfrac{1}{2}A^2 + AB + \tfrac{1}{2}B^2 + \cdots .$$

However, since $AB = BA$ and

$$(A + B)^2 = (A + B)(A + B) = A^2 + AB + BA + B^2 = A^2 + 2AB + B^2.$$

Thus,

$$e^A e^B = 1 + (A + B) + \tfrac{1}{2}(A + B)^2 + \cdots = e^{A+B} \qquad \square$$

You might wonder what happens if $AB \neq BA$. In this case we can account for the departure from commutativity by the **commutator** of A and B.

Definition 8.3.10.

Let $A, B \in \mathbb{R}^{\,n\times n}$ then the **commutator** of A and B is $[A, B] = AB - BA$.

Proposition 8.3.11.

Let $A, B, C \in \mathbb{R}^{\,n\times n}$ then

1. $[A, B] = -[B, A]$
2. $[A + B, C] = [A, C] + [B, C]$
3. $[AB, C] = A[B, C] + [A, C]B$
4. $[A, BC] = B[A, C] + [A, B]C$
5. $[[A, B], C] + [[B, C], A] + [[C, A], B] = 0$

The proofs of the properties above are not difficult. In contrast, the following formula known as the Baker-Campbell-Hausdorff (BCH) relation takes considerably more calculation:

$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\frac{1}{12}[[A,B],B]++\frac{1}{12}[[B,A],A]+\cdots} \qquad \text{BCH-formula}$$

The higher order terms can also be written in terms of nested commutators. What this means is that if we know the values of the commutators of two matrices then we can calculate the product of their exponentials with a little patience. This connection between multiplication of exponentials and commutators of matrices is at the heart of Lie theory. Actually, mathematicians have greatly abstracted the idea of Lie algebras and Lie groups way past matrices but the concrete example of matrix Lie groups and algebras is perhaps the most satisfying. If you'd like to know more just ask. It would make an excellent topic for an independent study that extended this course.

Remark 8.3.12.

In fact the BCH holds in the abstract as well. For example, it holds for the Lie algebra of derivations on smooth functions. A *derivation* is a linear differential operator which satisfies the product rule. The derivative operator is a derivation since $D[fg] = D[f]g + fD[g]$. The commutator of derivations is defined by $[X, Y][f] = X(Y(f)) - Y(X(f))$. It can be shown that $[D, D] = 0$ thus the BCH formula yields

$$e^{aD}e^{bD} = e^{(a+b)D}.$$

If the coefficient of D is thought of as position then multiplication by e^{bD} generates a translation in the position. By the way, we can state Taylor's Theorem rather compactly in this operator notation: $f(x+h) = exp(hD)f(x) = f(x) + hf'(x) + \frac{h^2}{2}f''(x) + \frac{h^3}{3!}f'''(x) + \cdots$.

Proposition 8.3.13.

Let $A, P \in \mathbb{R}^{\ n\times n}$ and assume P is invertible then

$$exp(P^{-1}AP) = P^{-1}exp(A)P$$

Proof: this identity follows from the following observation:

$$(P^{-1}AP)^k = P^{-1}APP^{-1}APP^{-1}AP \cdots P^{-1}AP = P^{-1}A^kP.$$

Thus $exp(P^{-1}AP) = \sum_{k=0}^{\infty}\frac{1}{k!}(P^{-1}AP)^k = P^{-1}(\sum_{k=0}^{\infty}\frac{1}{k!}A^k)P = P^{-1}exp(A)P$. $\square$

Proposition 8.3.14.

Let A be a square matrix, $det(exp(A)) = exp(trace(A))$.

Proof: If the matrix A is diagonalizable then the proof is simple. Diagonalizability means there exists invertible$P = [v_1|v_2|\cdots|v_n]$ such that $P^{-1}AP = D = [\lambda_1 v_1|\lambda_2 v_2|\cdots|\lambda_n v_n]$ where v_i is an e-vector with e-value λ_i for all i. Use the preceding proposition to calculate

$$\det(exp(D)) = \det(exp(P^{-1}AP) = \det(P^{-1}exp(A)P) = \det(P^{-1}P)\det(exp(A)) = det(exp(A))$$

On the other hand, the trace is cyclic $trace(ABC) = trace(BCA)$

$$trace(D) = trace(P^{-1}AP) = trace(PP^{-1}A) = trace(A)$$

But, we also know D is diagonal with eigenvalues on the diagonal hence $exp(D)$ is diagonal with e^{λ_i} on the corresponding diagonals

$$\det(exp(D)) = e^{\lambda_1}e^{\lambda_2}\cdots e^{\lambda_n} \quad \text{and} \quad trace(D) = \lambda_1 + \lambda_2 + \cdots + \lambda_n$$

Finally, use the laws of exponents to complete the proof,

$$e^{trace(A)} = e^{trace(D)} = e^{\lambda_1+\lambda_2+\cdots+\lambda_n} = e^{\lambda_1}e^{\lambda_2}\cdots e^{\lambda_n} = det(exp(D)) = det(exp(A)).$$

I've seen this proof in texts presented as if it were the general proof. But, not all matrices are diagonalizable so this is a curious proof. I stated the proposition for an arbitrary matrix and I meant it. The proof, the real proof, is less obvious. Let me sketch it for you:

better proof: The preceding proof shows it may be hopeful to suppose that $det(exp(tA)) = exp(t\, trace(A))$ for $t \in \mathbb{R}$. Notice that $y = e^{kt}$ satisfies the differential equation $\frac{dy}{dt} = ky$. Conversely, if $\frac{dy}{dt} = ky$ for some constant k then the general solution is given by $y = c_o e^{kt}$ for some $c_o \in \mathbb{R}$. Let $f(t) = det(exp(tA))$. If we can show that $f'(t) = trace(A)f(t)$ then we can conclude $f(t) = c_0 e^{t\, trace(A)}$. Consider:

$$\begin{aligned}
f'(t) &= \frac{d}{dh}\left(f(t+h)\right|_{h=0} \\
&= \frac{d}{dh}\left(det(exp[(t+h)A])\right|_{h=0} \\
&= \frac{d}{dh}\left(det(exp[tA+hA])\right|_{h=0} \\
&= \frac{d}{dh}\left(det(exp[tA]exp[hA])\right|_{h=0} \\
&= det(exp[tA])\frac{d}{dh}\left(det(exp[hA])\right|_{h=0} \\
&= f(t)\frac{d}{dh}\left(det(I + hA + \tfrac{1}{2}h^2A^2 + \tfrac{1}{3!}h^3A^3 + \cdots)\right|_{h=0} \\
&= f(t)\frac{d}{dh}\left(det(I + hA))\right|_{h=0}
\end{aligned}$$

Let us discuss the $\frac{d}{dh}(det(I+hA))$ term seperately for a moment:[3]

$$\begin{aligned}
\tfrac{d}{dh}(det(I+hA)) &= \tfrac{d}{dh}[\sum_{i_1,\dots,i_n} \epsilon_{i_1 i_2 \dots i_n}(I+hA)_{i_1 1}(I+hA)_{i_2 2}\cdots(I+hA)_{i_n n}]_{h=0} \\
&= \sum_{i_1,\dots,i_n} \epsilon_{i_1 i_2 \dots i_n} \tfrac{d}{dh}[(I+hA)_{1 i_1}(I+hA)_{1 i_2}\cdots(I+hA)_{n i_n}]_{h=0} \\
&= \sum_{i_1,\dots,i_n} \epsilon_{i_1 i_2 \dots i_n}(A_{1i_1}I_{1i_2}\cdots I_{ni_n} + I_{1i_1}A_{2i_2}\cdots I_{ni_n} + \cdots + I_{1i_1}I_{2i_2}\cdots A_{ni_n}) \\
&= \sum_{i_1} \epsilon_{i_1 2 \dots n}A_{1i_1} + \sum_{i_2} \epsilon_{1 i_2 \dots n}A_{2i_2} + \cdots + \sum_{i_n} \epsilon_{12 \dots I_n}A_{ni_n} \\
&= A_{11} + A_{22} + \cdots + A_{nn} \\
&= trace(A)
\end{aligned}$$

Therefore, $f'(t) = trace(A)f(t)$ consequently, $f(t) = c_o e^{t\, trace(A)} = det(exp(tA))$. However, we can resolve c_o by calculating $f(0) = det(exp(0)) = det(I) = 1 = c_o$ hence

$$e^{t\, trace(A)} = det(exp(tA))$$

Take $t = 1$ to obtain the desired result. $\square$

Remark 8.3.15.

The formula $det(exp(A)) = exp(trace(A))$ is very important to the theory of matrix Lie groups and Lie algebras. Generically, if G is the Lie group and $\mathfrak{g}$ is the Lie algebra then they are connected via the matrix exponential: $exp : \mathfrak{g} \to G_o$ where I mean G_o to denoted the connected component of the identity. For example, the set of all nonsingular matrices $GL(n)$ forms a Lie group which is disconnected. Half of $GL(n)$ has positive determinant whereas the other half has negative determinant. The set of all $n \times n$ matrices is denoted $gl(n)$ and it can be shown that $exp(gl(n))$ maps into the part of $GL(n)$ which has positive determinant. One can even define a matrix logarithm map which serves as a local inverse for the matrix exponential near the identity. Generally the matrix exponential is not injective thus some technical considerations must be discussed before we could put the matrix log on a solid footing. This would take us outside the scope of this course. However, this would be a nice topic to do a follow-up independent study. The theory of matrix Lie groups and their representations is ubiqitious in modern quantum mechanical physics.

[3] I use the definition of the identity matrix $I_{ij} = \delta_{ij}$ in eliminating all but the last summation in the fourth line. Then the levi-civita symbols serve the same purpose in going to the fifth line as $\epsilon_{i_1 2 \dots n} = \delta_{1i_1}$,$\epsilon_{1i_2 \dots n} = \delta_{2i_2}$ etc...

Finally, we come to the formula that is most important to our study of systems of DEqns. Let's call this the magic formula.

Proposition 8.3.16.

Let $\lambda \in \mathbb{C}$ and suppose $A \in \mathbb{R}^{\ n\times n}$ then

$$exp(tA) = e^{\lambda t}(I + t(A - \lambda I) + \frac{t^2}{2}(A - \lambda I)^2 + \frac{t^3}{3!}(A - \lambda I)^3 + \cdots).$$

Proof: Notice that $tA = t(A - \lambda I) + t\lambda I$ and $t\lambda I$ commutes with all matrices thus,

$$\begin{aligned} exp(tA) &= exp(t(A - \lambda I) + t\lambda I) \\ &= exp(t(A - \lambda I))exp(t\lambda I) \\ &= e^{\lambda t} exp(t(A - \lambda I)) \\ &= e^{\lambda t}\big(I + t(A - \lambda I) + \tfrac{t^2}{2}(A - \lambda I)^2 + \tfrac{t^3}{3!}(A - \lambda I)^3 + \cdots\big) \end{aligned}$$

In the third line I used the identity proved below,

$$exp(t\lambda I) = I + t\lambda I + \frac{1}{2}(t\lambda)^2 I^2 + \cdots = I(1 + t\lambda + \frac{(t\lambda)^2}{2} + \cdots) = Ie^{t\lambda}. \qquad \square$$

While the proofs leading up to the magic formula only dealt with real matrices it is not hard to see the proofs are easily modified to allow for complex matrices.

8.4 solutions for systems of DEqns with real eigenvalues

Let us return to the problem of solving $\vec{x}\,' = A\vec{x}$ for a constant square matrix A where $\vec{x} = [x_1, x_2, \dots, x_n]$ is a vector of functions of t. I'm adding the vector notation to help distinguish the scalar function x_1 from the vector function $\vec{x}_1$ in this section. Let me state one theorem from the theory of differential equations. The existence of solutions theorem which is the heart of of this theorem is fairly involved to prove, you'll find it in one of the later chapters of the differential equations text by Nagel Saff and Snider.

Theorem 8.4.1.

If $\vec{x}\,' = A\vec{x}$ and A is a constant matrix then any solution to the system has the form

$$\vec{x}(t) = c_1\vec{x}_1(t) + c_2\vec{x}_2(t) + \cdots + c_n\vec{x}_n(t)$$

where $\{\vec{x}_1, \vec{x}_2, \dots, \vec{x}_n\}$ is a linearly independent set of solutions defined on $\mathbb{R}$ (this is called the **fundamental solution set**). Moreover, these fundamental solutions can be concatenated into a single invertible solution matrix called the **fundamental matrix** $X = [\vec{x}_1|\vec{x}_2|\cdots|\vec{x}_n]$ and the general solution can be expressed as $\vec{x}(t) = X(t)\vec{c}$ where $\vec{c}$ is an arbitrary vector of real constants. If an initial condtion $\vec{x}(t_o) = \vec{x}_o$ is given then the solution to the IVP is $\vec{x}(t) = X^{-1}(t_o)X(t)\vec{x}_o$.

We proved in the previous section that the matrix exponential $exp(tA)$ is a solution matrix and the inverse is easy enought to guess: $exp(tA)^{-1} = exp(-tA)$. This proves the columns of $exp(tA)$ are solutions to $\vec{x}\,' = A\vec{x}$ which are linearly independent and as such form a fundamental solution set.

<u>Problem:</u> we cannot directly calculate $exp(tA)$ for most matrices A. We have a solution we can't calculate. What good is that?

When can we explicitly calculate $exp(tA)$ without much thought? Two cases come to mind: (1.) if A is diagonal then it's easy, saw this in Example 8.3.8, (2.) if A is a **nilpotent** matrix then there is some finite power of the matrix which is zero; $A^k = 0$. In the nilpotent case the infinite series defining the matrix exponential truncates at order k:

$$exp(tA) = I + tA + \tfrac{t^2}{2}A^2 + \cdots + \tfrac{t^{k-1}}{(k-1)!}A^{k-1}$$

Example 8.4.2. *Let* $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ *we calculate* $A^2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ *thus*

$$exp(tA) = I + tA = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + t\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

Incidentally, the solution to $\vec{x}\,' = A\vec{x}$ *is generally* $\vec{x}(t) = c_1\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} t \\ 1 \end{bmatrix}$*. In other words,* $x_1(t) = c_2 + c_2 t$ *whereas* $x_2(t) = c_2$*. These solutions are easily seen to solve the system* $x_1' = x_2$ *and* $x_2' = 0$*.*

Unfortunately, the calculation we just did in the last example almost never works. For example, try to calculate an arbitrary power of $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, let me know how it works out. We would like for all examples to truncate. The **magic formula** gives us a way around this dilemma:

Proposition 8.4.3.

Let $A \in \mathbb{R}^{\,n\times n}$. Suppose v is an e-vector with e-value λ then $exp(tA)v = e^{\lambda t}v$.

Proof: we are given that $(A - \lambda I)v = 0$ and it follows that $(A - \lambda I)^k v = 0$ for all $k \geq 1$. Use the magic formula,

$$exp(tA)v = e^{\lambda t}(I + t(A - \lambda I) + \cdots)v = e^{\lambda t}(Iv + t(A - \lambda I)v + \cdots = e^{\lambda t}v$$

noting all the higher order terms vanish since $(A - \lambda I)^k v = 0$. $\square$

We can't hope for the matrix exponential itself to truncate, but when we multiply $exp(tA)$ on an e-vector something special happens. Since $e^{\lambda t} \neq 0$ the set of vector functions $\{e^{\lambda_1 t}v_1, e^{\lambda_2 t}v_2, \ldots, e^{\lambda_k t}v_k\}$ will be linearly independent if the e-vectors v_i are linearly independent. If the matrix A is diagonalizable then we'll be able to find enough e-vectors to construct a fundamental solution set using e-vectors alone. However, if A is not diagonalizable, and has only real e-values, then we can still find a Jordan basis $\{v_1, v_2, \dots, v_n\}$ which consists of generalized e-vectors and it follows that $\{e^{tA}v_1, e^{tA}v_2, \dots, e^{tA}v_n\}$ forms a fundamental solution set. Moreover, this is not just of theoretical use. We can actually calculate this solution set.

Proposition 8.4.4.

Let $A \in \mathbb{R}^{\,n\times n}$. Suppose A has a chain $\{v_1, v_2, \dots, v_k\}$ is of generalized e-vectors with e-value λ, meaning $(A-\lambda)v_1 = 0$ and $(A-\lambda)v_{k-1} = v_k$ for $k \geq 2$, then

1. $e^{tA}v_1 = e^{\lambda t}v_1$,
2. $e^{tA}v_2 = e^{\lambda t}(v_2 + tv_1)$,
3. $e^{tA}v_3 = e^{\lambda t}\big(v_3 + tv_2 + \frac{t^2}{2}v_1\big)$,
4. $e^{tA}v_k = e^{\lambda t}\big(v_k + tv_{k-1} + \cdots + \frac{t^{k-1}}{(k-1)!}v_1\big)$.

Proof: Study the chain condition,

$$(A-\lambda I)v_2 = v_1 \quad \Rightarrow \quad (A-\lambda)^2 v_2 = (A-\lambda I)v_1 = 0$$

$$(A-\lambda I)v_3 = v_2 \quad \Rightarrow \quad (A-\lambda I)^2 v_3 = (A-\lambda I)v_2 = v_1$$

Continuing with such calculations[4] we find $(A-\lambda I)^j v_i = v_{i-j}$ for all $i > j$ and $(A-\lambda I)^i v_i = 0$. The magic formula completes the proof:

$$e^{tA}v_2 = e^{\lambda t}\big(v_2 + t(A-\lambda I)v_2 + \tfrac{t^2}{2}(A-\lambda I)^2 v_2 \cdots\big) = e^{\lambda t}\big(v_2 + tv_1\big)$$

likewise,

$$\begin{aligned} e^{tA}v_3 &= e^{\lambda t}\big(v_3 + t(A-\lambda I)v_3 + \tfrac{t^2}{2}(A-\lambda I)^2 v_3 + \tfrac{t^3}{3!}(A-\lambda I)^3 v_3 + \cdots\big) \\ &= e^{\lambda t}\big(v_3 + tv_2 + \tfrac{t^2}{2}(A-\lambda I)v_2\big) \\ &= e^{\lambda t}\big(v_3 + tv_2 + \tfrac{t^2}{2}v_1\big). \end{aligned}$$

We already proved the e-vector case in the preceding proposition and the general case follows from essentially the same calculation. $\square$

We have all the theory we need to solve systems of homogeneous constant coefficient ODEs.

[4]keep in mind these conditions hold because of our current labling scheme, if we used a different indexing system then you'd have to think about how the chain conditions work out, to test your skill perhaps try to find the general solution for the system with the matrix from Example 7.4.14

Example 8.4.5. *Recall Example 7.2.11 we found* $A = \begin{bmatrix} 3 & 1 \\ 3 & 1 \end{bmatrix}$ *had e-values* $\lambda_1 = 0$ *and* $\lambda_2 = 4$ *and corresponding e-vectors*

$$\vec{u}_1 = \begin{bmatrix} 1 \\ -3 \end{bmatrix} \qquad and \qquad \vec{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

thus we find the general solution to $\vec{x}\,' = A\vec{x}$ *is simply,*

$$\boxed{\vec{x}(t) = c_1 \begin{bmatrix} 1 \\ -3 \end{bmatrix} + c_2 e^{4t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

just to illustrate the terms: we have fundmamental solution set and matrix:

$$\left\{ \begin{bmatrix} 1 \\ -3 \end{bmatrix}, \begin{bmatrix} e^{4t} \\ e^{4t} \end{bmatrix} \right\} \qquad X = \begin{bmatrix} 1 & e^{4t} \\ -3 & e^{4t} \end{bmatrix}$$

Notice that a different choice of e-vector scaling would just end up adjusting the values of c_1, c_2 in the event an initial condition was given. This is why different choices of e-vectors still gives us the same general solution. It is the flexibility to change c_1, c_2 that allows us to fit any initial condition.

Example 8.4.6. *We can modify Example 8.2.2 and propose a different model for a tiger/bunny system. Suppose* x *is the number of tigers and* y *is the number of rabbits then*

$$\tfrac{dx}{dt} = x - 4y \qquad \qquad \tfrac{dy}{dt} = -10x + 19y$$

is a model for the population growth of tigers and bunnies in some closed environment. Suppose that there is initially 2 *tigers and* 100 *bunnies.* **Find the populations of tigers and bunnies at time** $t > 0$**:**

Solution: notice that we must solve $\vec{x}\,' = A\vec{x}$ *where* $A = \begin{bmatrix} 1 & -4 \\ -10 & 19 \end{bmatrix}$ *and* $\vec{x}(0) = (2, 100)$*. We can calculate the eigenvalues and corresponding eigenvectors:*

$$det(A - \lambda I) = 0 \;\Rightarrow\; \lambda_1 = -1, \; \lambda_2 = 21 \;\Rightarrow\; u_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}, \; u_2 = \begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

Therefore, using Proposition 8.4.4, the general solution has the form:

$$\vec{x}(t) = c_1 e^{-t} \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 e^{21t} \begin{bmatrix} -1 \\ 5 \end{bmatrix}.$$

However, we also know that $\vec{x}(0) = (2, 100)$ *hence*

$$\begin{bmatrix} 2 \\ 100 \end{bmatrix} = c_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 5 \end{bmatrix} \;\Rightarrow\; \begin{bmatrix} 2 \\ 100 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 5 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 5 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 100 \end{bmatrix} = \frac{1}{11} \begin{bmatrix} 110 \\ 198 \end{bmatrix}$$

Finally, we find the vector-form of the solution to the given initial value problem:

$$\vec{x}(t) = 10e^{-t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \tfrac{198}{11}e^{21t}\begin{bmatrix} -1 \\ 5 \end{bmatrix}$$

Which means that $x(t) = 20e^{-t} - \frac{198}{11}e^{21t}$ *and* $y(t) = 1020e^{-t} + 90e^{21t}$ *are the number of tigers and bunnies respective at time* t*.*

Notice that a different choice of e-vectors would have just made for a different choice of c_1, c_2 in the preceding example. Also, notice that when an initial condition is given there ought not be any undetermined coefficients in the final answer[5].

Example 8.4.7. *We found that in Example 7.2.13 the matrix* $A = \begin{bmatrix} 0 & 0 & -4 \\ 2 & 4 & 2 \\ 2 & 0 & 6 \end{bmatrix}$ *has e-values* $\lambda_1 = \lambda_2 = 4$ *and* $\lambda_3 = 2$ *with corresponding e-vectors*

$$\vec{u}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \vec{u}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad \vec{u}_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

Hence, using Proposition 8.4.4 and Theorem 8.4.1 the general solution of $\frac{d\vec{x}}{dt} = A\vec{x}$ *is simply:*

$$\vec{x}(t) = c_1e^{4t}\vec{u}_1 + c_2e^{4t}\vec{u}_2 + c_3e^{2t}\vec{u}_3 = c_1e^{4t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + c_2e^{4t}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3e^{2t}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

Example 8.4.8. *Find the general solution of* $\frac{d\vec{x}}{dt} = A\vec{x}$ *given that:*

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We analyzed this matrix in Example 7.4.14. We found a pair of chains of generalized e-vectors all with eigenvalue $\lambda = 1$ *which satisfied the following conditions:*

$$(A - I)\vec{u}_3 = \vec{u}_1, \ (A - I)\vec{u}_1 = 0 \qquad (A - I)\vec{u}_4 = \vec{u}_2, \ (A - I)\vec{u}_2 = 0$$

In particular, $\vec{u}_j = e_j$ *for* $j = 1, 2, 3, 4$*. We can use the magic formula to extract 4 solutions from the matrix exponential, by Proposition 8.4.4 we find:*

$$\begin{aligned} \vec{x}_1 &= e^{At}\vec{u}_1 = e^t\vec{u}_1 = e^te_1 \qquad (8.2) \\ \vec{x}_2 &= e^{At}\vec{u}_2 = e^t(e_2 + te_1) \\ \vec{x}_3 &= e^{At}\vec{u}_3 = e^te_3 \\ \vec{x}_4 &= e^{At}\vec{u}_4 = e^t(e_4 + te_3) \end{aligned}$$

[5] Assuming of course that there are enough initial conditions given to pick a unique solution from the family of solutions which we call the "general solution".

Let's write the general solution in vector and scalar form, by Theorem 8.4.1,

$$\vec{x}(t) = c_1\vec{x}_1 + c_2\vec{x}_2 + c_3\vec{x}_3 + c_4\vec{x}_4 = c_1e^te_1 + c_2e^t(e_2 + te_1) + c_3e^te_3 + c_4e^t(e_4 + te_3) = \begin{bmatrix} c_1e^t + tc_2e^t \\ c_2e^t \\ c_3e^t + tc_4e^t \\ c_4e^t \end{bmatrix}$$

In other words, $x_1(t) = c_1e^t + tc_2e^t$, $x_2(t) = c_2e^t$, $x_3(t) = c_3e^t + tc_4e^t$ *and* $x_4(t) = c_4e^t$ *form the general solution to the given system of differential equations.*

Example 8.4.9. *Find the general solution of* $\frac{d\vec{x}}{dt} = A\vec{x}$ *given (generalized)eigenvectors* $\vec{u}_i$, $i = 1, 2, 3, 4, 5, 6, 7, 8, 9$ *such that:*

$$(A - I)\vec{u}_1 = 0, \quad A\vec{u}_2 = \vec{u}_2, \quad A\vec{u}_3 = 7\vec{u}_3, \quad (A - I)\vec{u}_4 = \vec{u}_1$$

$$(A + 5I)\vec{u}_5 = 0, \ (A - 3I)\vec{u}_6 = \vec{u}_7 \quad A\vec{u}_7 = 3\vec{u}_7, \ A\vec{u}_8 = 0, \quad (A - 3I)\vec{u}_9 = \vec{u}_6$$

We can use the magic formula to extract 9 solutions from the matrix exponential, by Proposition 8.4.4 we find:

$$\begin{aligned}
\vec{x}_1 &= e^{At}\vec{u}_1 = e^t\vec{u}_1 = e^t\vec{u}_1 \\
\vec{x}_2 &= e^{At}\vec{u}_2 = e^t\vec{u}_2 \\
\vec{x}_3 &= e^{At}\vec{u}_3 = e^{7t}\vec{u}_3 \\
\vec{x}_4 &= e^{At}\vec{u}_4 = e^t(\vec{u}_4 + t\vec{u}_1) \qquad \textit{can you see why?} \\
\vec{x}_5 &= e^{At}\vec{u}_5 = e^{-5t}\vec{u}_5 \\
\vec{x}_6 &= e^{At}\vec{u}_6 = e^{3t}(\vec{u}_6 + t\vec{u}_7) \qquad \textit{can you see why?} \\
\vec{x}_7 &= e^{At}\vec{u}_7 = e^{3t}\vec{u}_7 \\
\vec{x}_8 &= e^{At}\vec{u}_8 = \vec{u}_8 \\
\vec{x}_9 &= e^{At}\vec{u}_9 = e^{3t}(\vec{u}_9 + t\vec{u}_6 + \tfrac{1}{2}t^2\vec{u}_7) \qquad \textit{can you see why?}
\end{aligned} \tag{8.3}$$

Let's write the general solution in vector and scalar form, by Theorem 8.4.1,

$$\vec{x}(t) = \sum_{i=1}^{9} c_i\vec{x}_i$$

where the formulas for each solution $\vec{x}_i$ *was given above. If I was to give an explicit matrix A with the eigenvectors given above it would be a* 9×9 *matrix.* **Challenge:** *find the matrix exponential* e^{At} *in terms of the given (generalized)eigenvectors.*

8.5 solutions for systems of DEqns with complex eigenvalues

The calculations in the preceding section still make sense for a complex e-value and complex e-vector. However, we usually need to find real solutions. How to fix this? The same way as always. We extract real solutions from the complex solutions. Fortunately, our previous work on linear independence of complex e-vectors insures that the resulting solution set will be linearly independent.

Proposition 8.5.1.

Let $A \in \mathbb{R}^{\,n\times n}$. Suppose A has a chain $\{v_1, v_2, \dots, v_k\}$ is of generalized complex e-vectors with e-value $\lambda = \alpha + i\beta$, meaning $(A-\lambda)v_1 = 0$ and $(A-\lambda)v_{k-1} = v_k$ for $k \geq 2$ and $v_j = a_j + ib_j$ for $a_j, b_j \in \mathbb{R}^n$ for each j, then

1. $e^{tA}v_1 = e^{\lambda t}v_1$,
2. $e^{tA}v_2 = e^{\lambda t}(v_2 + tv_1)$,
3. $e^{tA}v_3 = e^{\lambda t}\big(v_3 + tv_2 + \frac{t^2}{2}v_1\big)$,
4. $e^{tA}v_k = e^{\lambda t}\big(v_k + tv_{k-1} + \cdots + \frac{t^{k-1}}{(k-1)!}v_1\big)$.

Furthermore, the following are the $2k$ linearly independent real solutions that are implicit within the complex solutions above,

1. $x_1 = Re(e^{tA}v_1) = e^{\alpha t}\big[(\cos\beta t)a_1 - (\sin\beta t)b_1\big]$,
2. $x_2 = Im(e^{tA}v_1) = e^{\alpha t}\big[(\sin\beta t)a_1 + (\cos\beta t)b_1\big]$),
3. $x_3 = Re(e^{tA}v_2) = e^{\alpha t}\big[(\cos\beta t)(a_2 + ta_1) - (\sin\beta t)(b_2 + tb_1)\big]$,
4. $x_4 = Im(e^{tA}v_2) = e^{\alpha t}\big[(\sin\beta t)(a_2 + ta_1) + (\cos\beta t)(b_2 + tb_1)\big]$,
5. $x_5 = Re(e^{tA}v_3) = e^{\alpha t}\big[(\cos\beta t)(a_3 + ta_2 + \frac{t^2}{2}a_1) - (\sin\beta t)(b_3 + tb_2 + \frac{t^2}{2}b_1)\big]$,
6. $x_6 = Im(e^{tA}v_3) = e^{\alpha t}\big[(\cos\beta t)(a_3 + ta_2 + \frac{t^2}{2}a_1) - (\sin\beta t)(b_3 + tb_2 + \frac{t^2}{2}b_1)\big]$.

Proof: the magic formula calculations of the last section just as well apply to the complex case. Furthermore, we proved that

$$Re\big[e^{\alpha t + i\beta t}(v + iw)\big] = e^{\alpha t}\big[(\cos\beta t)v - (\sin\beta t)w\big]$$

and

$$Im\big[e^{\alpha t + i\beta t}(v + iw)\big] = e^{\alpha t}\big[(\sin\beta tv + (\cos\beta t)w\big],$$

the proposition follows. □

Example 8.5.2. *This example uses the results derived in Example 7.6.2. Let* $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ *and find the e-values and e-vectors of the matrix. Observe that* $det(A-\lambda I) = \lambda^2+1$ *hence the eigevalues are* $\lambda = \pm i$. *We find* $u_1 = (1, i)$. *Notice that*

$$u_1 = \begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

This means that $\vec{x}\,' = A\vec{x}$ *has general solution:*

$$\vec{x}(t) = c_1\Bigg(\cos(t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin(t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\Bigg) + c_2\Bigg(\sin(t)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \cos(t)\begin{bmatrix} 0 \\ 1 \end{bmatrix}\Bigg).$$

The solution above is the "vector-form of the solution". We can add the terms together to find the scalar solutions: denoting $\vec{x}(t) = (x(t), y(t))$,

$$x(t) = c_1\cos(t) + c_2\sin(t) \qquad y(t) = -c_1\sin(t) + c_2\cos(t)$$

These are the parametric equations of a circle with radius $R = \sqrt{c_1^2 + c_2^2}$.

Example 8.5.3. *We solved the e-vector problem for* $A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ *in Example 7.6.4. We found one real e-value* $\lambda_1 = 3$ *and a pair of complex e-values* $\lambda_2 = 1 \pm i$. *The corresponding e-vectors were:*

$$\vec{u}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + i\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

We identify that $Re(\vec{u}_2) = e_2$ *and* $Im(\vec{u}_2) = e_1$. *The general solution of* $\vec{x}\,' = A\vec{x}$ *should have the form:*

$$\vec{x}(t) = c_1 e^{At}\vec{u}_1 + c_2 Re(e^{At}\vec{u}_2) + c_3 Im(e^{At}\vec{u}_2)$$

The vectors above are e-vectors so these solution simplify nicely:

$$\vec{x}(t) = c_1 e^{3t} e_3 + c_2 e^t(\cos(t)e_2 - \sin(t)e_1) + c_3 e^t(\sin(t)e_2 + \cos(t)e_1)$$

For fun let's look at the scalar form of the solution. Denoting $\vec{x}(t) = (x(t), y(t), z(t))$,

$$x(t) = -c_2 e^t\sin(t) + c_3 e^t\cos(t), \qquad y(t) = c_2 e^t\cos(t) + c_3 e^t\sin(t), \qquad z(t) = c_1 e^{3t}$$

Believe it or not this is a spiral helix which has an exponentially growing height and radius.

Example 8.5.4. *Let's suppose we have a chain of* 2 *complex eigenvectors* $\vec{u}_1, \vec{u}_2$ *with eigenvalue* $\lambda = 2 + i3$. *I'm assuming that*

$$(A - (2+i)I)\vec{u}_2 = \vec{u}_1, \qquad (A - (2+i)I)\vec{u}_1 = 0.$$

We get a pair of complex-vector solutions (using the magic formula which truncates since these are e-vectors):

$$\vec{z}_1(t) = e^{At}\vec{u}_1 = e^{(2+i)t}\vec{u}_1, \qquad \vec{z}_2(t) = e^{At}\vec{u}_2 = e^{(2+i)t}(\vec{u}_2 + t\vec{u}_1),$$

The real and imaginary parts of these solutions give us 4 *real solutions which form the general solution:*

$$\begin{aligned}\vec{x}(t) &= c_1 e^{2t}\big[\cos(3t)Re(\vec{u}_1) - \sin(3t)Im(\vec{u}_1)\big] \\ &+ c_2 e^{2t}\big[\sin(3t)Re(\vec{u}_1) + \cos(3t)Im(\vec{u}_1)\big] \\ &+ c_3 e^{2t}\big[\cos(3t)[Re(\vec{u}_2) + tRe(\vec{u}_1)] - \sin(3t)[Im(\vec{u}_2) + tIm(\vec{u}_1)]\big] \\ &+ c_4 e^{2t}\big[\sin(3t)[Re(\vec{u}_2) + tRe(\vec{u}_1)] + \cos(3t)[Im(\vec{u}_2) + tIm(\vec{u}_1)]\big].\end{aligned}$$

8.6 geometry and difference equations revisited

In Example 7.1.5 we studied $A = \left[\begin{smallmatrix} 3 & 0 \\ 8 & -1 \end{smallmatrix}\right]$ and how it pushed the point $x_o = \left[\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right]$ around the plane. We found x_i for $i = 1, 2, 3, 4$ by multiplication by A directly. That method is fine for small i but what is we wished to know the formula for the 1000-th state? We should hope there is some way to find that state without direct multiplication repeated 1000 times. One method is to make use of the diagonalization of the matrix. We know that e-vectors (if they exist) can be glued together to make the diagonalizing similarity transforming matrix; there exists $P \in \mathbb{R}^{\,n\times n}$ such that $P^{-1}AP = D$ where D is a diagonal matrix. Notice that D^k is easy to calculate. We can solve for $A = PDP^{-1}$ and find that $A^2 = PDP^{-1}PDP^{-1} = PD^2P^{-1}$. The you can prove inductively that $A^k = PD^kP^{-1}$. It is much easier to calculate PD^kP^{-1} when $k >> 1$.

8.6.1 difference equations vs. differential equations

I mentioned that the equation $x_{k+1} = Ax_k$ is a *difference equation*. We can think of this as a differential equation where the time-step is always one-unit. To see this I should remind you how $\vec{x}\,' = B\vec{x}$ is defined in terms of a limiting process:

$$\vec{x}\,'(t) = \lim_{h\to 0} \frac{\vec{x}(t+h) - \vec{x}(t)}{h} = B\vec{x}(t)$$

A gross approximation to the continuous limiting process would be to just take $h = 1$ and drop the limit. That approximation yields:

$$B\vec{x}(t) = \vec{x}(t+1) - \vec{x}(t).$$

We then suppose $t \in \mathbb{N}$ and denote $\vec{x}(t) = \vec{x}_t$ to obtain:

$$\vec{x}_{t+1} = (B + I)\vec{x}_t.$$

We see that the differential equation $\vec{x}\,' = B\vec{x}$ is crudely approximated by the difference equation $\vec{x}_{t+1} = A\vec{x}_t$. where $A = B + I$. Since we now have tools to solve differential equations directly it should be interesting to contrast the motion generated by the difference equation to the exact parametric equations which follow from the e-vector solution of the corresponding differential equation.

(this section is a work in progress, if you think the thought is not complete, you're correct!)

Chapter 9

linear geometry

The concept of a geometry is very old. Philosophers in the nineteenth century failed miserably in their analysis of geometry and the physical world. They became mired in the popular misconception that mathematics must be physical. They argued that because 3 dimensional Eulcidean geometry was the only geometry familar to everyday experience it must surely follow that a geometry which differs from Euclidean geometry must be nonsensical. However, why should physical intuition factor into the argument? We understand now that geometry is a mathematical construct, not a physical one. There are many possible geometries. On the other hand, it would seem the geometry of space and time probably takes just one form. We are tempted by this misconception every time we ask "but what is this math really". That question is usually wrong-headed. A better question is "is this math logically consistent" and if so what physical systems is it known to model.

The modern view of geometry is stated in the langauge of manifolds, fiber bundles,algebraic geometry and perhaps even more fantastic structures. There is currently great debate as to how we should model the true intrinsic geometry of the universe. Branes, strings, quivers, noncommutative geometry, twistors, ... this list is endless. However, at the base of all these things we must begin by understanding what the geometry of a flat space entails.

Vector spaces are flat manifolds. They possess a global coordinate system once a basis is chosen. Up to this point we have only cared about algebraic conditions of linear independence and spanning. There is more structure we can assume. We can ask what is the length of a vector? Or, given two vectors we might want to know what is the angle bewtween those vectors? Or when are two vectors orthogonal?

If we desire we can also insist that the basis consist of vectors which are *orthogonal* which means "perpendicular" in a generalized sense. A geometry is a vector space plus an idea of orthogonality and length. The concepts of orthogonality and length are encoded by an inner-product. Inner-products are symmetric, positive definite, bilinear forms, they're like a dot-product. Once we have a particular geometry in mind then we often restrict the choice of bases to only those bases which preserve the length of vectors.

The mathematics of orthogonality is exhibited by the dot-products and vectors in calculus III. However, it turns out the concept of an *inner-product* allows us to extend the idea or perpendicular to abstract vectors such as functions. This means we can even ask interesting questions such as "how close is one function to another" or "what is the closest function to a set of functions". Least-squares curve fitting is based on this geometry.

This chapter begins by defining dot-products and the norm (a.k.a. length) of a vector in $\mathbb{R}^n$. Then we discuss orthogonality, the Gram Schmidt algorithm, orthogonal complements and finally the application to the problem of least square analysis. The chapter concludes with a consideration of the similar, but abstract, concept of an inner product space. We look at how least squares generalizes to that context and we see how Fourier analysis naturally flows from our finite dimensional discussions of orthogonality. [1]

Let me digress from linear algebra for a little while. In physics it is customary to only allow coordinates which fit the physics. In classical mechanics one often works with intertial frames which are related by a rigid motion. Certain quantities are the same in all intertial frames, notably force. This means Newtons laws have the same form in all intertial frames. The geometry of special relativity is 4 dimensional. In special relativity, one considers coordinates which preserve Einstein's three axioms. Allowed coordinates are related to other coordinates by Lorentz transformations. These Lorentz transformations include rotations and velocity boosts. These transformations are designed to make the speed of a light ray invariant in all frames. For a linear algebraist the vector space is the starting point and then coordinates are something we add on later. Physics, in contrast, tends to start with coordinates and if the author is kind he might warn you which transformations are allowed.

What coordinate transformations are allowed actually tells you what kind of physics you are dealing with. This is an interesting and nearly universal feature of modern physics. The allowed transformations form what is known to physicsists as a "group" (however, strictly speaking these groups do not always have the strict structure that mathematicians insist upon for a group). In special relativity the group of interest is the Poincaire group. In quantum mechanics you use unitary groups because unitary transformations preserve probabilities. In supersymmetric physics you use the super Poincaire group because it is the group of transformations on superspace which preserves supersymmetry. In general relativity you allow general coordinate transformations which are locally lorentzian because all coordinate systems are physical provided they respect special relativity in a certain approximation. In solid state physics there is something called the renormilzation group which plays a central role in physical predictions of field-theoretic models. My point? Transformations of coordinates are important if you care about physics. We study the basic case of vector spaces in this course. If you are interested in the more sophisticated topics just ask, I can show you where to start reading.

[1] we ignore analytical issues of convergence since we have only in mind a Fourier approximation, not the infinite series

9.1 Euclidean geometry of $\mathbb{R}^n$

The dot-product is a mapping from $\mathbb{R}^n \times \mathbb{R}^n$ to $\mathbb{R}$. We take in a pair of vectors and output a real number.

Definition 9.1.1.

Let $x, y \in \mathbb{R}^n$ we define $x \cdot y \in \mathbb{R}$ by

$$x \cdot y = x^T y = x_1 y_1 + x_2 y_2 + \cdots x_n y_n$$

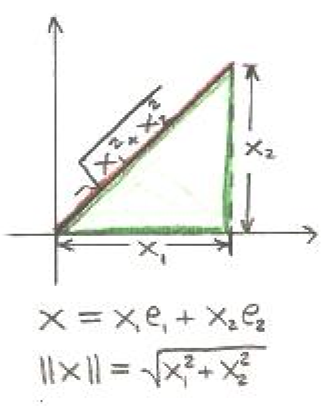
Example 9.1.2. *Let* $v = (1, 2, 3, 4, 5)$ *and* $w = (6, 7, 8, 9, 10)$

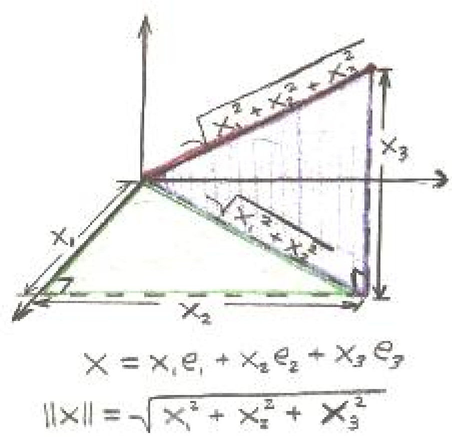
$$v \cdot w = 6 + 14 + 24 + 36 + 50 = 130$$

The dot-product can be used to define the length or norm of a vector and the angle between two vectors.

Definition 9.1.3.

The **length** or **norm** of $x \in \mathbb{R}^n$ is a real number which is defined by $||x|| = \sqrt{x \cdot x}$. Furthermore, let x, y be nonzero vectors in $\mathbb{R}^n$ we define the **angle** θ between x and y by $\cos^{-1}\left[\frac{x \cdot y}{||x|| \, ||y||}\right]$. $\mathbb{R}$ together with these defintions of length and angle forms a **Euclidean Geometry**.





Technically, before we make this definition we should make sure that the formulas given above even make sense. I have not shown that $x \cdot x$ is nonnegative and how do we know that the inverse cosine is well-defined? The first proposition below shows the norm of x is well-defined and establishes several foundational properties of the dot-product.

Proposition 9.1.4.

Suppose $x, y, z \in \mathbb{R}^n$ and $c \in \mathbb{R}$ then

1. $x \cdot y = y \cdot x$
2. $x \cdot (y + z) = x \cdot y + x \cdot z$
3. $c(x \cdot y) = (cx) \cdot y = x \cdot (cy)$
4. $x \cdot x \geq 0$ and $x \cdot x = 0$ iff $x = 0$

Proof: the proof of (1.) is easy, $x \cdot y = \sum_{i=1}^n x_i y_i = \sum_{i=1}^n y_i x_i = y \cdot x$. Likewise,

$$x \cdot (y + z) = \sum_{i=1}^n x_i (y + z)_i = \sum_{i=1}^n (x_i y_i + x_i z_i) = \sum_{i=1}^n x_i y_i + \sum_{i=1}^n x_i z_i = x \cdot y + x \cdot z$$

proves (2.) and since

$$c \sum_{i=1}^n x_i y_i = \sum_{i=1}^n c x_i y_i = \sum_{i=1}^n (cx)_i y_i = \sum_{i=1}^n x_i (cy)_i$$

we find $c(x \cdot y) = (cx) \cdot y = x \cdot (cy)$. Continuting to (4.) notice that $x \cdot x = {x_1}^2 + {x_2}^2 + \cdots + {x_n}^2$ thus $x \cdot x$ is the sum of squares and it must be nonnegative. Suppose $x = 0$ then $x \cdot x = x^T x = 0^T 0 = 0$. Conversely, suppose $x \cdot x = 0$. Suppose $x \neq 0$ then we find a contradiction since it would have a nonzero component which implies ${x_1}^2 + {x_2}^2 + \cdots + {x_n}^2 \neq 0$. This completes the proof of (4.). $\square$

The formula $\cos^{-1}\left[\frac{x \cdot y}{||x||\,||y||}\right]$ is harder to justify. The inequality that we need for it to be reasonable is $\left|\frac{x \cdot y}{||x||\,||y||}\right| \leq 1$, otherwise we would not have a number in the $dom(cos^{-1}) = range(cos) = [-1, 1]$. An equivalent inequality is $|x \cdot y| \leq ||x||\,||y||$ which is known as the **Cauchy-Schwarz** inequality.

Proposition 9.1.5.

If $x, y \in \mathbb{R}^n$ then $|x \cdot y| \leq ||x||\,||y||$

Proof: I've looked in a few linear algebra texts and I must say the proof given in Spence, Insel and Friedberg is probably the most efficient and clear. Other texts typically run up against a quadratic inequality in some part of their proof (for example the linear algebra texts by Apostle, Larson& Edwards, Anton & Rorres to name a few). That is somehow hidden in the proof that follows: let $x, y \in \mathbb{R}^n$. If either $x = 0$ or $y = 0$ then the inequality is clearly true. Suppose then that both x and y are nonzero vectors. It follows that $||x||, ||y|| \neq 0$ and we can define vectors of unit-length; $\hat{x} = \frac{x}{||x||}$ and $\hat{y} = \frac{y}{||y||}$. Notice that $\hat{x} \cdot \hat{x} = \frac{x}{||x||} \cdot \frac{x}{||x||} = \frac{1}{||x||^2} \hat{x} \cdot x = \frac{x \cdot x}{x \cdot x} = 1$ and likewise $\hat{y} \cdot \hat{y} = 1$.

Consider,

$$\begin{aligned} 0 \leq ||\hat{x} \pm \hat{y}||^2 &= (\hat{x} \pm \hat{y}) \cdot (\hat{x} \pm \hat{y}) \\ &= \hat{x} \cdot \hat{x} \pm 2(\hat{x} \cdot \hat{y}) + \hat{y} \cdot \hat{y} \\ &= 2 \pm 2(\hat{x} \cdot \hat{y}) \\ &\Rightarrow \quad -2 \leq \pm 2(\hat{x} \cdot \hat{y}) \\ &\Rightarrow \quad \pm \hat{x} \cdot \hat{y} \leq 1 \\ &\Rightarrow \quad |\hat{x} \cdot \hat{y}| \leq 1 \end{aligned}$$

Therefore, noting that $x = ||x||\hat{x}$ and $y = ||y||\hat{y}$,

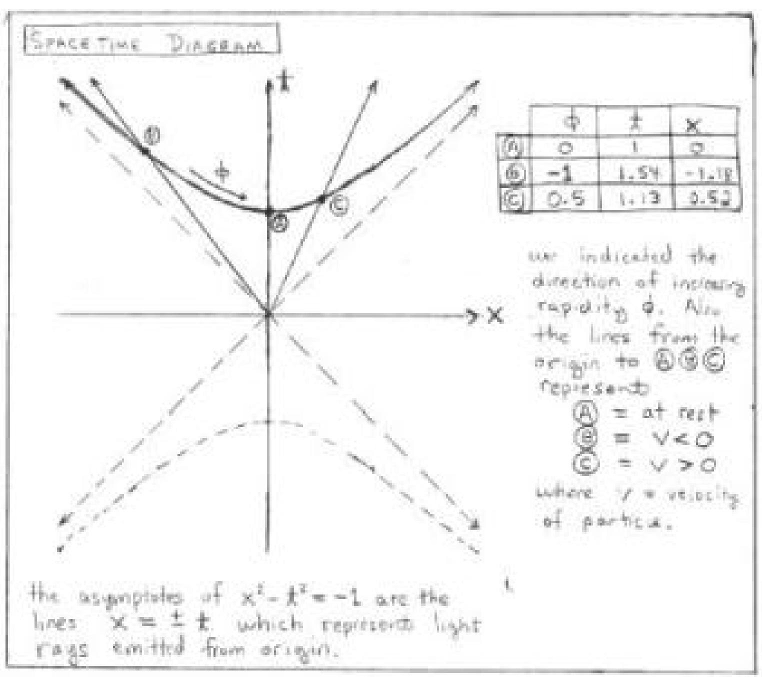
$$|x \cdot y| = |\, ||x||\hat{x} \cdot ||y||\hat{y}\, | = ||x||\, ||y||\, |\hat{x} \cdot \hat{y}| \leq ||x||\, ||y||.$$

The use of unit vectors is what distinguishes this proof from the others I've found. □

Remark 9.1.6.

The dot-product is but one of many geometries for $\mathbb{R}^n$. We will explore generalizations of the dot-product in a later section. However, in this section we will work exclusively with the standard dot-product on $\mathbb{R}^n$. Generally, unless explicitly indicated otherwise, we assume Euclidean geometry for $\mathbb{R}^n$.

Just for fun here's a picture of a circle in the hyperbolic geometry of special relativity, technically it's not a geometry since we have nonzero-vectors with zero length (so-called null-vectors). Perhaps we will offer a course in special relativity some time and we could draw these pictures with understanding in that course.



Example 9.1.7. *Let $v = (1,2,3,4,5)$ and $w = (6,7,8,9,10)$ find the angle between these vectors and calculate the unit vectors in the same directions as v and w. Recall that, $v \cdot w = 6+14+24+36+50 = 130$. Furthermore,*

$$|||v|| = \sqrt{1^2+2^2+3^2+4^2+5^2} = \sqrt{1+4+9+16+25} = \sqrt{55}$$

$$||w|| = \sqrt{6^2+7^2+8^2+9^2+10^2} = \sqrt{36+49+64+81+100} = \sqrt{330}$$

We find unit vectors via the standard trick, you just take the given vector and multiply it by the reciprocal of its length. This is called **normalizing** *the vector,*

$$\boxed{\hat{v} = \tfrac{1}{\sqrt{55}}(1,2,3,4,5) \qquad \hat{w} = \tfrac{1}{\sqrt{330}}(6,7,8,9,10)}$$

The angle is calculated from the definition of angle,

$$\boxed{\theta = \cos^{-1}\left(\frac{130}{\sqrt{55}\sqrt{330}}\right) = 15.21^o}$$

It's good we have this definition, 5-dimensional protractors are very expensive.

Proposition 9.1.8.

Let $x, y \in \mathbb{R}^n$ and suppose $c \in \mathbb{R}$ then

1. $||cx|| = |c|\,||x||$
2. $||x+y|| \leq ||x|| + ||y||$

Proof: let $x \in \mathbb{R}^n$ and $c \in \mathbb{R}$ then calculate,

$$||cx||^2 = (cx)\cdot(cx) = c^2 x\cdot x = c^2||x||^2$$

Since $||cx|| \geq 0$ the squareroot yields $||cx|| = \sqrt{c^2}||x||$ and $\sqrt{c^2} = |c|$ thus $||cx|| = |c|||x||$. Item (2.) is called the **triangle inequality** for reasons that will be clear when we later discuss the distance function. Let $x, y \in \mathbb{R}^n$,

$$\begin{aligned}
||x+y||^2 &= |(x+y)\cdot(x+y)| && \text{defn. of norm}\\
&= |x\cdot(x+y) + y\cdot(x+y)| && \text{prop. of dot-product}\\
&= |x\cdot x + x\cdot y + y\cdot x + y\cdot y| && \text{prop. of dot-product}\\
&= |\ ||x||^2 + 2x\cdot y + ||y||^2\ | && \text{prop. of dot-product}\\
&\leq ||x||^2 + 2|x\cdot y| + ||y||^2 && \text{triangle ineq. for } \mathbb{R}\\
&\leq ||x||^2 + 2||x||\,||y|| + ||y||^2 && \text{Cauchy-Schwarz ineq.}\\
&\leq (||x|| + ||y||)^2 && \text{algebra}
\end{aligned}$$

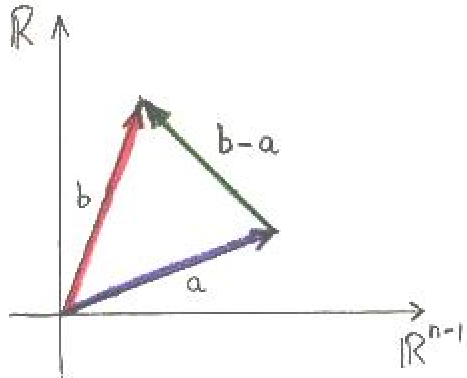
Notice that both $||x+y||$ and $||x||+||y||$ are nonnegative by (4.) of Proposition 9.1.4 hence the inequality above yields $||x+y|| \leq ||x|| + ||y||$. $\square$

Definition 9.1.9.

The **distance** between $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^n$ is defined to be $d(a,b) \equiv ||b-a||$.

If we draw a picture this definition is very natural. Here we are thinking of the points a, b as vectors from the origin then $b-a$ is the vector which points from a to b (this is algebraically clear since $a+(b-a)=b$). Then the distance between the points is the length of the vector that points from one point to the other. If you plug in two dimensional vectors you should recognize the distance formula from middle school math:

$$d((a_1,a_2),(b_1,b_2)) = \sqrt{(b_1-a_1)^2+(b_2-a_2)^2}$$



Proposition 9.1.10.

Let $d : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be the distance function then

1. $d(x,y) = d(y,x)$
2. $d(x,y) \geq 0$
3. $d(x,x) = 0$
4. $d(x,y) + d(y,z) \geq d(x,z)$

Proof: I leave the proof of (1.), (2.) and (3.) to the reader. Item (4.) is also known as the **triangle inequality**. Think of the points x, y, z as being the vertices of a triangle, this inequality says the sum of the lengths of two sides cannot be smaller than the length of the remaining side. Let $x, y, z \in \mathbb{R}^n$ and note by the triangle inequality for $||\cdot||$,

$$d(x,z) = ||z-x|| = ||z-y+y-x|| \leq ||z-y|| + ||y-x|| = d(y,z) + d(x,y). \qquad \square$$

We study the 2 and 3 dimensional case in some depth in calculus III. I would recommend you take that course, even if it's not "on your sheet".

9.2 orthogonality in $\mathbb{R}^n$

Two vectors are orthogonal if the vectors point in mutually exclusive directions. We saw in calculus III the dot-product allowed us to pick apart vectors into pieces. The same is true in n-dimensions: we can take a vector an disassemble it into component vectors which are orthogonal.

Definition 9.2.1.

Let $v, w \in \mathbb{R}^n$ then we say v and w are **orthogonal** iff $v \cdot w = 0$.

Example 9.2.2. *Let $v = (1, 2, 3)$ describe the set of all vectors which are orthogonal to v. Let $r = (x, y, z)$ be an arbitrary vector and consider the orthogonality condition:*

$$0 = v \bullet r = (1, 2, 3) \bullet (x, y, z) = x + 2y + 3z = 0.$$

If you've studied 3 dimensional Cartesian geometry you should recognize this as the equation of a plane through the origin with normal vector $\langle 1, 2, 3\rangle$.

Proposition 9.2.3. *Pythagorean Theorem in n-dimensions*

If $x, y \in \mathbb{R}^n$ are orthogonal vectors then $\|x\|^2 + \|y\|^2 = \|x + y\|^2$.

Proof: Calculuate $\|x + y\|^2$ from the dot-product,

$$\|x + y\|^2 = (x + y) \bullet (x + y) = x \bullet x + x \bullet y + y \bullet x + y \bullet y = \|x\|^2 + \|y\|^2. \qquad \Box$$

Proposition 9.2.4.

The zero vector is orthogonal to all other vectors in $\mathbb{R}^n$.

Proof: let $x \in \mathbb{R}^n$ note $2(0) = 0$ thus $0 \cdot x = 2(0) \cdot x = 2(0 \cdot x)$ which implies $0 \cdot x = 0$. $\Box$

Definition 9.2.5.

A set S of vectors in $\mathbb{R}^n$ is **orthogonal** iff every pair of vectors in the set is orthogonal. If S is orthogonal and all vectors in S have length one then we say S is **orthonormal**.

Example 9.2.6. *Let $u = [1, 1, 0]$, $v = [1, -1, 0]$ and $w = [0, 0, 1]$. We calculate*

$$u \cdot v = 0, \quad u \cdot w, \quad v \cdot w = 0$$

thus $S = \{u, v, w\}$ is an orthogonal set. However, it is not orthonormal since $||u|| = \sqrt{2}$. It is easy to create an orthonormal set, we just normalize the vectors; $T = \{\hat{u}, \hat{v}, \hat{w}\}$ meaning,

$$T = \left\{ \tfrac{1}{\sqrt{2}}[1, 1, 0],\ \tfrac{1}{\sqrt{2}}[1, -1, 0],\ [0, 0, 1] \right\}$$

Proposition 9.2.7. *Extended Pythagorean Theorem in n-dimensions*

If $x_1, x_2, \dots x_k$ are orthogonal then

$$||x_1||^2 + ||x_2||^2 + \cdots + ||x_k||^2 = ||x_1 + x_2 + \cdots + x_k||^2$$

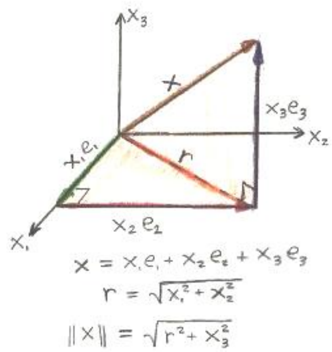
Proof: we can prove the second statement by applying the Pythagorean Theorem for two vectors repeatedly, starting with

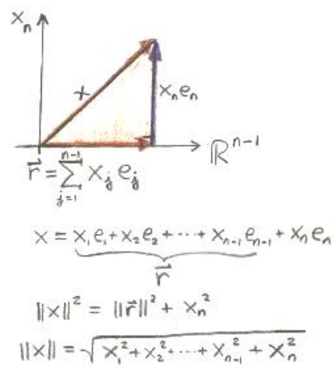
$$||x_1 + (x_2 + \cdots + x_k)||^2 = ||x_1||^2 + ||x_2 + \cdots + x_k||^2$$

but then we can apply the Pythagorean Theorem to the rightmost term

$$||x_2 + (x_3 + \cdots + x_k)||^2 = ||x_2||^2 + ||x_3 + \cdots + x_k||^2.$$

Continuing in this fashion until we obtain the Pythagorean Theorem for k-orthogonal vectors. $\square$





I have illustrated the proof above in the case of three dimensions and k-dimensions, however my k-dimensional diagram takes a little imagination. Another thing to think about: given $v = v_1e_1 + v_2e_2 + \cdots + v_ne_n$ if e_i are orthonormal then $||v||^2 = v_1^2 + v_2^2 + \cdots + v_n^2$. Therefore, if we use a basis which is orthonormal then we obtain the standard formula for length of a vector with respect to the coordinates. If we were to use a basis of vectors which were not orthogonal or normalizes then the formula for the length of a vector in terms of the coordinates could look quite different.

Example 9.2.8. *Let $v_1 = (1,1), v_2 = (2,0)$. Use the basis $\{v_1, v_2\}$ for $\mathbb{R}^2$. Notice that $\{v_1, v_2\}$ is not orthogonal or normal. Given $x, y \in \mathbb{R}$ we wish to find $a, b \in \mathbb{R}$ such that $r = (x,y) = av_1 + bv_2$, this amounts to the matrix calculation:*

$$rref[v_1|v_2|r] = rref\left[\begin{array}{cc|c} 1 & 2 & x \\ 1 & 0 & y \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & y \\ 0 & 1 & \frac{1}{2}(x-y) \end{array}\right]$$

Thus $a = y$ and $b = \frac{1}{2}(x-y)$. Let's check my answer,

$$av_1 + bv_2 = y(1,1) + \tfrac{1}{2}(x-y)(2,0) = (y + x - y, y + 0) = (x,y).$$

Furthermore, solving for x, y in terms of a, b yields $x = 2b + a$ and $y = a$. Therefore, $\|(x,y)\|^2 = x^2 + y^2$ is modified to

$$\|av_1 + bv_2\|^2 = (2b+a)^2 + a^2 \neq \|av_1\|^2 + \|bv_2\|^2.$$

If we use a basis which is not orthonormal then we should take care not to assume formulas given for the standard basis equally well apply. However, if we trade the standard basis for a new basis which is orthogonal then we have less to worry about. The Pythagorean Theorem only applies in the orthogonal case. For two normalized, but possibly non-orthogonal, vectors we can replace the Pythagorean Theorem with a generalization of the Law of Cosines in $\mathbb{R}^n$.

$$\|av_1 + bv_2\|^2 = a^2 + b^2 + 2ab\cos\theta$$

where $v_1 \bullet v_2 = \cos\theta$. (I leave the proof to the reader)

Proposition 9.2.9.

If $S = \{v_1, v_2, \dots, v_k\} \subset \mathbb{R}^n$ is an orthogonal set of nonzero vectors then S is linearly independent.

Proof: suppose $c_1, c_2, \dots, c_k \in \mathbb{R}$ such that

$$c_1v_1 + c_2v_2 + \cdots c_kv_k = 0$$

Take the dot-product of both sides with respect to $v_j \in S$,

$$c_1v_1 \cdot v_j + c_2v_2 \cdot v_j + \cdots + c_kv_k \cdot v_j = 0 \cdot v_j = 0$$

Notice all terms in the sum above vanish by orthogonality except for one term and we are left with $c_jv_j \cdot v_j = 0$. However, $v_j \neq 0$ thus $v_j \cdot v_j \neq 0$ and it follows we can divide by the nonzero scalar $v_j \cdot v_j$ leaving $c_j = 0$. But j was arbitrary hence $c_1 = c_2 = \cdots = c_k = 0$ and hence S is linearly independent. $\square$

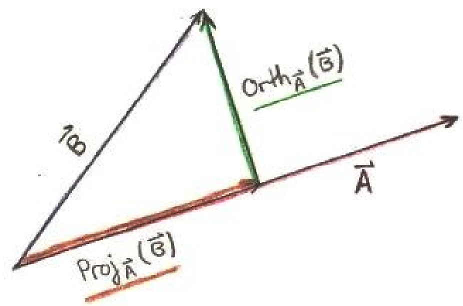
The converse of the proposition above is false. Given a linearly indepdent set of vectors it is not necessarily true that set is also orthogonal. However, we can modify any linearly independent set of vectors to obtain a linearly indepedent set. The procedure for this modification is known as the *Gram-Schmidt orthogonalization*. It is based on a generalization of the idea the vector projection from calculus III. Let me remind you: we found the projection operator to be a useful construction in calculus III. The projection operation allowed us to select the vector component of one vector that pointed in the direction of another given vector. We used this to find the distance from a point to a plane.

Definition 9.2.10.

Let $\vec{A} \neq 0, \vec{B}$ be vectors then we define

$$Proj_{\vec{A}}(\vec{B}) = (\vec{B} \cdot \hat{A})\hat{A}$$

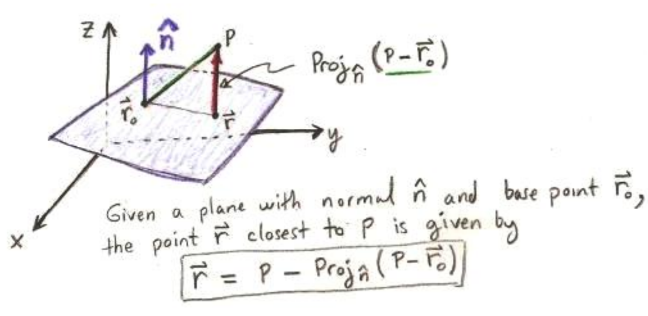
where $\hat{A} = \frac{1}{||A||}A$. Moreover, the length of $Proj_{\vec{A}}(\vec{B})$ is called the component of $\vec{B}$ in the $\vec{A}$-direction and is denoted $Comp_{\vec{A}}(\vec{B}) = ||Proj_{\vec{A}}(\vec{B})||$. Finally, the **orthogonal complement** is defined by $Orth_{\vec{A}}(\vec{B}) = \vec{B} - Proj_{\vec{A}}(\vec{B})$.



Example 9.2.11. *Suppose* $\vec{A} =< 2, 2, 1 >$ *and* $\vec{B} =< 2, 4, 6 >$ *notice that we can also express the projection opertation by* $Proj_{\vec{A}}(\vec{B}) = (\vec{B} \cdot \hat{A})\hat{A} = \frac{1}{||\vec{A}||^2}(\vec{B} \cdot \vec{A})\vec{A}$ *thus*

$$Proj_{\vec{A}}(\vec{B}) = \tfrac{1}{9}(< 2, 4, 6 > \cdot < 2, 2, 1 >) < 2, 2, 1 >= \tfrac{4+8+6}{9} < 2, 2, 1 >=< 4, 4, 2 >$$

The length of the projection vector gives $Comp_{\vec{A}}(\vec{B}) = \sqrt{16 + 16 + 4} = 6$. *One application of this algebra is to calculate the distance from the plane* $2x + 2y + z = 0$ *to the point* $(2, 4, 6)$. *The "distance" from a plane to a point is defined to be the shortest distance. It's geometrically clear that the shortest path from the plane is found along the normal to the plane. If you draw a picture its not hard to see that* $(2, 4, 6) - Proj_{\vec{A}}(\vec{B}) =< 2, 4, 6 > - < 4, 4, 2 >= (-2, 0, 4)$ *is the closest point to* $(2, 4, 6)$ *that lies on the plane* $2x + 2y + z = 0$. *Moreover the distance from the plane to the point is just* 6.



Example 9.2.12. *We studied* $\vec{A} =< 2,2,1 >$ *and* $\vec{B} =< 2,4,6 >$ *in the preceding example. We found that notice that* $Proj_{\vec{A}}(\vec{B}) =< 4,4,2 >$*. The projection of* $\vec{B}$ *onto* $\vec{A}$ *is the part of* $\vec{B}$ *which points in the direction of* $\vec{A}$*. It stands to reason that if we subtract away the projection then we will be left with the part of* $\vec{B}$ *which does not point in the direction of* $\vec{A}$*, it should be orthogonal.*

$$Orth_{\vec{A}}(\vec{B}) = \vec{B} - Proj_{\vec{A}}(\vec{B}) =< 2,4,6 > - < 4,4,2 >=< -2,0,4 >$$

Let's verify $Orth_{\vec{A}}(\vec{B})$ *is indeed orthogonal to* $\vec{A}$*,*

$$Orth_{\vec{A}}(\vec{B}) \cdot \vec{A} =< -2,0,4 > \cdot < 2,2,1 >= -4+4 = 0.$$

Notice that the projection operator has given us the following orthogonal decomposition of $\vec{B}$*:*

$$< 2,4,6 >= \vec{B} = Proj_{\vec{A}}(\vec{B}) + Orth_{\vec{A}}(\vec{B}) =< 4,4,2 > + < -2,0,4 > .$$

If $\vec{A}, \vec{B}$ are any two nonzero vectors it is probably clear that we can perform the decomposition outlined in the example above. It would not be hard to show that if $S = \{\vec{A}, \vec{B}\}$ is linearly indepedendent then $S' = \{\vec{A}, Orth_{\vec{A}}(\vec{B})\}$ is an orthogonal set, moreover they have the same span. This is a partial answer to the converse of Proposition 9.2.9. But, what if we had three vectors instead of two? How would we orthogonalize a set of three linearly independent vectors?

Remark 9.2.13.

I hope you can forgive me for reverting to calculus III notation in the last page or two. It should be clear enough to the reader that the orthogonalization and projection operations can be implemented on either rows or columns. I return to our usual custom of thinking primarily about column vectors at this point. We've already seen the definition from Calculus III, now we turn to the n-dimensional case in matrix notation.

Definition 9.2.14.

Suppose $a \neq 0 \in \mathbb{R}^n$, define the **projection of b onto a** to be the mapping $Proj_a : \mathbb{R}^n \to \mathbb{R}^n$ such that $Proj_a(b) = \frac{1}{a^T a}(a^T b)a$. Moreover, we define $Orth_a : \mathbb{R}^n \to \mathbb{R}^n$ by $Orth_a(b) = b - Proj_a(b) = b - \frac{1}{a^T a}(a^T b)a$ for all $b \in \mathbb{R}^n$.

Proposition 9.2.15.

If $a \neq 0 \in \mathbb{R}^n$ then $Proj_a$ and $Orth_a$ are linear transformations.

1. $Orth_a(b) \cdot a = 0$ for all $b \in \mathbb{R}^n$,
2. $Orth_a(b) \cdot Proj_a(y) = 0$ for all $b, y \in \mathbb{R}^n$,
3. the projection is idempotent; $Proj_a \circ Proj_a = Proj_a$.

I leave the proof of linearity as an exercise. Begin with (1.): let $a \neq 0 \in \mathbb{R}^n$ and let $b \in \mathbb{R}^n$,

$$\begin{aligned} a \cdot Orth_a(b) &= a^T(b - \tfrac{1}{a^Ta}(a^Tb)a) \\ &= a^Tb - a^T(\tfrac{1}{a^Ta}(a^Tb)a) \\ &= a^Tb - \tfrac{1}{a^Ta}(a^Tb)a^Ta \\ &= a^Tb - a^Tb = 0. \end{aligned}$$

notice I used the fact that a^Tb, a^Ta were scalars to commute the a^T to the end of the expression. Notice that (2.) follows since $Proj_a(y) = ka$ for some constant k. Next, let $b \in \mathbb{R}^n$ and consider:

$$\begin{aligned} (Proj_a \circ Proj_a)(b) &= Proj_a(Proj_a(b)) \\ &= Proj_a(\tfrac{1}{a^Ta}(a^Tb)a) \\ &= \tfrac{1}{a^Ta}(a^T[\tfrac{1}{a^Ta}(a^Tb)a])a \\ &= \tfrac{1}{a^Ta}(\tfrac{a^Tb}{a^Ta}a^Ta)a \\ &= \tfrac{1}{a^Ta}(a^Tb)a \\ &= Proj_a(b) \end{aligned}$$

since the above holds for all $b \in \mathbb{R}^n$ we find $Proj_a \circ Proj_a = Proj_a$. This can also be denoted $Proj_a^2 = Proj_a$. $\square$

Proposition 9.2.16.

If $S = \{a, b, c\}$ be a linearly independent set of vectors in $\mathbb{R}^n$ then $S' = \{a', b', c'\}$ is an orthogonal set of vectors in $\mathbb{R}^n$ if we define a', b', c' as follows:

$$a' = a, \qquad b' = Orth_{a'}(b), \qquad c' = Orth_{a'}(Orth_{b'}(c)).$$

Proof: to prove S' orthogonal we must show that $a' \cdot b' = 0, a' \cdot c' = 0$ and $b' \cdot c' = 0$. We already proved $a' \cdot b' = 0$ in the Proposition 9.2.15. Likewise, $a' \cdot c' = 0$ since $Orth_{a'}(x)$ is orthogonal to a' for any x. Consider:

$$\begin{aligned} b' \cdot c' &= b' \cdot Orth_{a'}(Orth_{b'}(c)) \\ &= b' \cdot \big[\ Orth_{b'}(c) - Proj_{a'}(Orth_{b'}(c))\ \big] \\ &= b' \cdot Orth_{b'}(c) - Orth_a(b) \cdot Proj_a(Orth_{b'}(c)) \\ &= 0 \end{aligned}$$

Where we again used (1.) and (2.) of Proposition 9.2.15 in the critical last step. The logic of the formulas is very natural. To construct b' we simply remove the part of b which points in the direction of a'. Then to construct c' we first remove the part of c in the b' direction and then the part in the a' direction. This means no part of c' will point in the a' or b' directions. In principle, one might worry we would subtract away so much that nothing is left, but the linear independence of the vectors insures that is not possible. If it were that would imply a linear dependence of the original set of vectors. □

For convenience let me work out the formulas we just discovered in terms of an explicit formula with dot-products. We can also perform the same process for a set of 4 or 5 or more vectors. I'll state the process for arbitrary order, you'll forgive me if I skip the proof this time. There is a careful proof on page 379 of Spence, Insel and Friedberg. The connection between my *Orth* operator approach and the formulas in the proposition that follows is *just algebra:*

$$\begin{aligned}
v_3' &= Orth_{v_1'}(Orth_{v_2'}(v_3)) \\
&= Orth_{v_2'}(v_3) - Proj_{v_1'}(Orth_{v_2'}(v_3)) \\
&= v_3 - Proj_{v_2'}(v_3) - Proj_{v_1'}(v_3 - Proj_{v_2'}(v_3)) \\
&= v_3 - Proj_{v_2'}(v_3) - Proj_{v_1'}(v_3) - Proj_{v_1'}(Proj_{v_2'}(v_3)) \\
&= v_3 - \frac{v_3 \cdot v_2'}{v_2' \cdot v_2'} v_2' - \frac{v_3 \cdot v_1'}{v_1' \cdot v_1'} v_1'
\end{aligned}$$

The last term vanished because $v_1' \cdot v_2' = 0$ and the projections are just scalar multiples of those vectors.

Proposition 9.2.17. The Gram-Schmidt Process

If $S = \{v_1, v_2, \dots, v_k\}$ is a linearly independent set of vectors in $\mathbb{R}^n$ then $S' = \{v_1', v_2', \dots, v_k'\}$ is an orthogonal set of vectors in $\mathbb{R}^n$ if we define v_i' as follows:

$$\begin{aligned}
v_1' &= v_1 \\
v_2' &= v_2 - \frac{v_2 \cdot v_1'}{v_1' \cdot v_1'} v_1' \\
v_3' &= v_3 - \frac{v_3 \cdot v_2'}{v_2' \cdot v_2'} v_2' - \frac{v_3 \cdot v_1'}{v_1' \cdot v_1'} v_1' \\
v_k' &= v_k - \frac{v_k \cdot v_{k-1}'}{v_{k-1}' \cdot v_{k-1}'} v_{k-1}' - \frac{v_k \cdot v_{k-2}'}{v_{k-2}' \cdot v_{k-2}'} v_{k-2}' - \cdots - \frac{v_k \cdot v_1'}{v_1' \cdot v_1'} v_1'.
\end{aligned}$$

Example 9.2.18. *Suppose* $v_1 = (1, 0, 0, 0), v_2 = (3, 1, 0, 0), v_3 = (3, 2, 0, 3)$. *Let's use the Gram-Schmidt Process to orthogonalize these vectors: let* $v_1' = v_1 = (1, 0, 0, 0)$ *and calculate:*

$$v_2' = v_2 - \frac{v_2 \bullet v_1}{v_1 \bullet v_1} v_1 = (3, 1, 0, 0) - 3(1, 0, 0, 0) = (0, 1, 0, 0).$$

Next,

$$v_3' = v_3 - \frac{v_3 \bullet v_2'}{v_2' \bullet v_2'} v_2' - \frac{v_3 \bullet v_1'}{v_1' \bullet v_1'} v_1' = (3, 2, 0, 3) - 2(0, 1, 0, 0) - 3(1, 0, 0, 0) = (0, 0, 0, 3).$$

We find the orthogonal set of vectors $\{e_1, e_2, e_4\}$. It just so happens this is also an orthonormal set of vectors.

Proposition 9.2.19. Normalization

If $S' = \{v_1', v_2', \dots, v_k'\}$ is an orthogonal subset of $\mathbb{R}^n$ then $S'' = \{v_1'', v_2'', \dots, v_k''\}$ is an orthonormal set if we define $v_i'' = \widehat{v_i'} = \frac{1}{||v_i'||}v_i'$ for each $i = 1, 2, \dots, k$.

Example 9.2.20. *Suppose $v_1 = (1,1,1), v_2 = (1,2,3), v_3 = (0,0,3)$ find an orthonormal set of vectors that spans span$\{v_1, v_2, v_3\}$. We can use Gram-Schmidt followed by a normalization, let $v_1' = (1,1,1)$ then calculate*

$$v_2' = (1,2,3) - \left(\frac{1+2+3}{3}\right)(1,1,1) = (1,2,3) - (2,2,2) = (-1,0,1).$$

as a quick check on my arthimetic note $v_1' \cdot v_2' = 0$ (good). Next,

$$v_3' = (0,0,3) - \left(\frac{0(-1)+0(0)+3(1)}{2}\right)(-1,0,1) - \left(\frac{0(1)+0(1)+3(1)}{3}\right)(1,1,1)$$

$$\Rightarrow v_3' = (0,0,3) + (\tfrac{3}{2}, 0, -\tfrac{3}{2}) - (1,1,1) = (\tfrac{1}{2}, -1, \tfrac{1}{2})$$

again it's good to check that $v_2' \bullet v_3' = 0$ and $v_1' \bullet v_3' = 0$ as we desire. Finally, note that $||v_1'|| = \sqrt{3}$, $||v_2'|| = \sqrt{2}$ and $||v_3'|| = \sqrt{3/2}$ hence

$$\boxed{v_1'' = \tfrac{1}{\sqrt{3}}(1,1,1), \quad v_2'' = \tfrac{1}{\sqrt{2}}(-1,0,1), \quad v_3'' = \sqrt{\tfrac{2}{3}}(\tfrac{1}{2}, -1, \tfrac{1}{2})}$$

are orthonormal vectors.

Definition 9.2.21.

A basis for a subspace W of $\mathbb{R}^n$ is an **orthogonal** basis for W iff it is an orthogonal set of vectors which is a basis for W. Likewise, an **orthonormal** basis for W is a basis which is **orthonormal**.

Proposition 9.2.22. Existence of Orthonormal Basis

If $W \leq \mathbb{R}^n$ then there exists an orthonormal basis of W

Proof: since W is a subspace it has a basis. Apply Gram-Schmidt to that basis then normalize the vectors to obtain an orthnormal basis. □

Example 9.2.23. *Let $W =$ span$\{(1,0,0,0), (3,1,0,0), (3,2,0,3)\}$. Find an orthonormal basis for $W \leq \mathbb{R}^{\ 4\times 1}$. Recall from Example 9.2.18 we applied Gram-Schmidt and found the orthonormal set of vectors $\{e_1, e_2, e_4\}$. That is an orthonormal basis for W.*

Example 9.2.24. *In Example 9.2.20 we found* $\{v_1'', \ v_2'', \ v_3''\}$ *is an orthonormal set of vectors. Since orthogonality implies linear independence it follows that this set is in fact a basis for* $\mathbb{R}^3$. *It is an* **orthonormal basis**. *Of course there are other bases which are orthogonal. For example, the standard basis is orthonormal.*

Example 9.2.25. *Let us define* $S = \{v_1, v_2, v_3, v_4\} \subset \mathbb{R}^4$ *as follows:*

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \ v_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \ v_3 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 3 \end{bmatrix}, \ v_4 = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 3 \end{bmatrix}$$

It is easy to verify that S *defined below is a linearly independent set vectors basis for* $span(S) \leq \mathbb{R}^4$. *Let's see how to find an orthonormal basis for* $span(S)$. *The procedure is simple: apply the Gram-Schmidt algorithm then normalize the vectors.*

$$v_1' = v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$v_2' = v_2 - \left(\frac{v_2 \bullet v_1'}{v_1' \bullet v_1'}\right) v_1' = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{3}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

$$v_3' = v_3 - \left(\frac{v_3 \bullet v_2'}{v_2' \bullet v_2'}\right) v_2' - \left(\frac{v_3 \bullet v_1'}{v_1' \bullet v_1'}\right) v_1' = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 3 \end{bmatrix} - \frac{0}{1}\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{5}{3}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} -5 \\ 0 \\ 1 \\ 4 \end{bmatrix}$$

$$v_4' = v_4 - \left(\tfrac{v_4 \bullet v_3'}{v_3' \bullet v_3'}\right) v_3' - \left(\tfrac{v_3 \bullet v_2'}{v_2' \bullet v_2'}\right) v_2' - \left(\tfrac{v_3 \bullet v_1'}{v_1' \bullet v_1'}\right) v_1' = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 3 \end{bmatrix} - \tfrac{1}{14}\begin{bmatrix} -5 \\ 0 \\ 1 \\ 4 \end{bmatrix} - \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 2 \\ 2 \end{bmatrix} = \tfrac{1}{14}\begin{bmatrix} 9 \\ 0 \\ -27 \\ 18 \end{bmatrix}$$

Then normalize to obtain the orthonormal basis for $Span(S)$ *below:*

$$\beta = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \ \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \ \frac{1}{\sqrt{42}}\begin{bmatrix} -5 \\ 0 \\ 1 \\ 4 \end{bmatrix}, \ \tfrac{1}{9\sqrt{14}}\begin{bmatrix} 9 \\ 0 \\ -27 \\ 18 \end{bmatrix}\Bigg\}.$$

Proposition 9.2.26. Coordinates with respect to an Orthonormal Basis

If $W \leq \mathbb{R}^n$ has an orthonormal basis $\{v_1, v_2, \dots, v_k\}$ and if $w = \sum_{i=1}^{k} w_i v_i$ then $w_i = w \cdot v_i$ for all $i = 1, 2, \dots, k$. In other words, each vector $w \in W$ may be expressed as

$$\boxed{w = (w \cdot v_1)v_1 + (w \cdot v_2)v_2 + \cdots + (w \cdots v_k)v_k}$$

Proof: Let $w = w_1v_1 + w_2v_2 + \cdots + w_kv_k$ and take the dot-product with v_j,

$$w \cdot v_j = (w_1v_1 + w_2v_2 + \cdots + w_kv_k) \cdot v_j = w_1(v_1 \cdot v_j) + w_2(v_2 \cdot v_j) + \cdots + w_k(v_k \cdot v_j)$$

Orthonormality of the basis is compactly expressed by the Kronecker Delta; $v_i \bullet v_j = \delta_{ij}$ this is zero if $i \neq j$ and it is 1 if they are equal. The whole sum collapses except for the j-th term which yields: $w \bullet v_j = w_j$. But, j was arbitrary hence the proposition follows. $\square$.

The proposition above reveals the real reason we like to work with orthonormal coordinates. It's easy to figure out the coordinates, we simply take dot-products. This technique was employed with great sucess in (you guessed it) Calculus III. The standard $\{\widehat{i}, \widehat{j}, \widehat{k}\}$ is an orthonormal basis and one of the first things we discuss is that if $\vec{v} = \langle A, B, C\rangle$ then $A = \vec{v} \bullet \widehat{i}$, $B = \vec{v} \bullet \widehat{j}$ and $C = \vec{v} \bullet \widehat{k}$.

Example 9.2.27. *For the record, the standard basis of $\mathbb{R}^n$ is an orthonormal basis and*

$$v = (v \cdot e_1)e_1 + (v \cdot e_2)e_2 + \cdots + (v \cdot e_n)e_n$$

for any vector v in $\mathbb{R}^n$.

Example 9.2.28. *Let $v = (1, 2, 3, 4)$. Find the coordinates of v with respect to the orthonormal basis β found in Example 9.2.25.*

$$\beta = \{f_1, f_2, f_3, f_4\} = \left\{ \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{42}} \begin{bmatrix} -5 \\ 0 \\ 1 \\ 4 \end{bmatrix}, \tfrac{1}{9\sqrt{14}} \begin{bmatrix} 9 \\ 0 \\ -27 \\ 18 \end{bmatrix} \right\}$$

Let us denote the coordinates vector $[v]_\beta = [w_1, w_2, w_3, w_4]$ we know we can calculate these by taking the dot-products with the vectors in the orthonormal basis β:

$$w_1 = v \bullet f_1 = \frac{1}{\sqrt{3}}(1, 2, 3, 4) \bullet (1, 0, 1, 1) = \frac{8}{\sqrt{3}}$$

$$w_2 = v \bullet f_2 = (1, 2, 3, 4) \bullet (0, 1, 0, 0) = 2$$

$$w_3 = v \bullet f_3 = \frac{1}{\sqrt{42}}(1, 2, 3, 4) \bullet (-5, 0, 1, 4) = \frac{14}{\sqrt{42}}$$

$$w_4 = v \bullet f_4 = \tfrac{1}{9\sqrt{14}}(1, 2, 3, 4) \bullet (9, 0, -27, 18) = \tfrac{0}{9\sqrt{14}} = 0$$

Therefore, $[v]_\beta = \left(\frac{8}{\sqrt{3}}, 2, \frac{14}{\sqrt{42}}, 0\right)$*. Now, let's check our answer. What should this mean if it is correct? We should be able verify* $v = w_1f_1 + w_2f_2 + w_3f_3 + w_4f_4$*:*

$$\begin{aligned}
w_1f_1 + w_2f_2 + w_3f_3 + w_4f_4 &= \frac{8}{\sqrt{3}}\frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{14}{\sqrt{42}}\frac{1}{\sqrt{42}}\begin{bmatrix} -5 \\ 0 \\ 1 \\ 4 \end{bmatrix} \\
&= \frac{8}{3}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} -5 \\ 0 \\ 1 \\ 4 \end{bmatrix} \\
&= \begin{bmatrix} 8/3 - 5/3 \\ 2 \\ 8/3 + 1/3 \\ 8/3 + 4/3 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}
\end{aligned}$$

Well, that's a relief.

9.3 orthogonal complements and projections

Upto now we have discussed projections with respect to one vector at a time, however we can just as well discuss the projection onto some subspace of $\mathbb{R}^n$. We need a few definitions to clarify and motivate the projection.

Definition 9.3.1.

Suppose $W_1, W_2 \subseteq \mathbb{R}^n$ then we say W_1 is **orthogonal** to W_2 iff $w_1 \cdot w_2 = 0$ for all $w_1 \in W_1$ and $w_2 \in W_2$. We denote orthogonality by writing $W_1 \perp W_2$.

Example 9.3.2. *Let* $W_1 = span\{e_1, e_2\}$ *and* $W_2 = span\{e_3\}$ *then* $W_1, W_2 \leq \mathbb{R}^n$*. Let* $w_1 = ae_1 + be_2 \in W_1$ *and* $w_2 = ce_3 \in W_2$ *calculate,*

$$w_1 \cdot w_2 = (ae_1 + be_2) \cdot (ce_3) = ace_1 \cdot e_3 + bce_2 \cdot e_3 = 0$$

Hence $W_1 \perp W_2$*. Geometrically, we have shown the xy-plane is orthogonal to the z-axis.*

We notice that orthogonality relative to the basis will naturally extend to the span of the basis since the dot-product has nice linearity properties.

Proposition 9.3.3.

Suppose $W_1, W_2 \leq \mathbb{R}^n$ the subspace W_1 is **orthogonal** to the subspace W_2 iff $w_i \cdot v_j = 0$ for all i, j relative to a pair of bases $\{w_i\}$ for W_1 and $\{v_j\}$ for W_2.

Proof: Suppose $\{w_i\}_{i=1}^r$ is a basis for $W_1 \leq \mathbb{R}^n$ and $\{v_j\}_{j=1}^s$ for $W_2 \leq \mathbb{R}^n$. If $W_1 \perp W_2$ then clearly $\{w_i\}_{i=1}^r$ is orthogonal to $\{v_j\}_{j=1}^s$. Conversely, suppose $\{w_i\}_{i=1}^r$ is orthogonal to $\{v_j\}_{j=1}^s$ then let $x \in W_1$ and $y \in W_2$:

$$x \cdot y = \left(\sum_{i=1}^r x_i w_i \right) \cdot \left(\sum_{i=1}^s y_j w_j \right) = \sum_{i=1}^r \sum_{j=1}^s x_i y_j (w_i \cdot v_j) = 0. \qquad \square$$

Given a subspace W which lives in $\mathbb{R}^n$ we might wonder what is the largest subspace which is orthogonal to W? In $\mathbb{R}^3$ it is clear that the xy-plane is the largest subspace which is orthogonal to the z-axis, however, if the xy-plane was viewed as a subset of $\mathbb{R}^4$ we could actually find a volume which was orthogonal to the z-axis (in particular $span\{e_1, e_2, e_4\} \perp span\{e_3\}$).

Definition 9.3.4.

Let $W \subseteq \mathbb{R}^n$ then $W^\perp$ is defined as follows:

$$W^\perp = \{v \in \mathbb{R}^n | v \bullet w = 0 \ for \ all \ w \in W\}$$

It is clear that $W^\perp$ is the largest subset in $\mathbb{R}^n$ which is orthogonal to W. Better than just that, it's the largest subspace orthogonal to W.

Proposition 9.3.5.

Let $S \subset \mathbb{R}^n$ then $S^\perp \leq \mathbb{R}^n$.

Proof: Let $x, y \in S^\perp$ and let $c \in \mathbb{R}$. Furthermore, suppose $s \in S$ and note

$$(x + cy) \cdot s = x \cdot s + c(y \cdot s) = 0 + c(0) = 0.$$

Thus an aribtrary linear combination of elements of $S^\perp$ are again in $S^\perp$ which is nonempty as $0 \in S^\perp$ hence by the subspace test $S^\perp \leq \mathbb{R}^n$. It is interesting that S need not be a subspace for this argument to hold. $\square$

Example 9.3.6. *Let $v_1 = (1, 1, 0, 0), v_2 = (0, 1, 0, 2)$. Find the orthogonal complement to $W = span\{v_1, v_2\}$. Let's treat this as a matrix problem. We wish to describe a typical vector in $W^\perp$. Towards that goal, let $r = (x, y, z, w) \in W^\perp$ then the conditions that r must satisfy are $v_1 \bullet r = v_1^T r = 0$ and $v_2 \bullet r = v_2^T r = 0$. But this is equivalent to the single matrix equation below:*

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad r = \begin{bmatrix} 2w \\ -2w \\ z \\ w \end{bmatrix} = z \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} 2 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

Thus, $W^{\perp} = span\{(0,0,1,0),(2,-2,0,1)\}$.

If you study the preceding example it becomes clear that finding the orthogonal complement of a set of vectors is equivalent to calculating the null space of a particular matrix. We have considerable experience in such calculations so this is a welcome observation.

Proposition 9.3.7.

If $S = \{v_1, v_2, \dots, v_k\} \subseteq \mathbb{R}^n$ and $A = [v_1|v_2|\cdots|v_k]$ then $S^{\perp} = Null(A^T)$

Proof: Denote $A = [v_1|v_2|\cdots|v_k] \in \mathbb{R}^{\,n\times k}$ and $x = (x_1, x_2, \dots, x_k)$. Observe that:

$$\begin{aligned} x \in Null(A^T) &\Leftrightarrow A^T x = 0 \\ &\Leftrightarrow [row_1(A^T)x, row_2(A^T)x, \cdots, row_k(A^T)x] = 0 \\ &\Leftrightarrow [(col_1(A))^T x, (col_2(A))^T x, \cdots, (col_k(A))^T x] = 0 \\ &\Leftrightarrow [v_1 \cdot x, v_2 \cdot x, \cdots, v_k \cdot x] = 0 \\ &\Leftrightarrow v_j \cdot x = 0 \text{ for } j = 1, 2, \dots, k \\ &\Leftrightarrow x \in S^{\perp} \end{aligned}$$

Therefore, $Null(A^T) = S^{\perp}$. $\square$

Given the correspondence above we should be interested in statements which can be made about the row and column space of a matrix. It turns out there are two simple statements to be made in general:

Proposition 9.3.8.

Let $A \in \mathbb{R}^{m\times n}$ then

1. $Null(A^T) \perp Col(A)$.
2. $Null(A) \perp Row(A)$.

Proof: Let $S = \{col_1(A), col_2(A), \dots, col_n(A)\}$ and use Proposition 9.3.7 to deduce $S^{\perp} = Null(A^T)$. Therefore, each column of A is orthogonal to all vectors in $Null(A^T)$, in particular each column is orthgonal to the basis for $Null(A^T)$. Since the pivot columns are a basis for $Col(A)$ we can use Proposition 9.3.3 to conclude $Null(A^T) \perp Col(A)$.

To prove of (2.) apply (1.) to $B = A^T$ to deduce $Null(B^T) \perp Col(B)$. Hence, $Null((A^T)^T) \perp Col(A^T)$ and we find $Null(A) \perp Col(A^T)$. But, $Col(A^T) = Row(A)$ thus $Null(A) \perp Row(A)$. $\square$

The proof above makes ample use of previous work. I encourage the reader to try to prove this proposition from scratch. I don't think it's that hard and you might learn something. Just take an arbitrary element of each subspace and argue why the dot-product is zero.

Proposition 9.3.9.

Let $W_1, W_2 \leq \mathbb{R}^n$, if $W_1 \perp W_2$ then $W_1 \cap W_2 = \{0\}$

Proof: let $z \in W_1 \cap W_2$ then $z \in W_1$ and $z \in W_2$ and since $W_1 \perp W_2$ it follows $z \cdot z = 0$ hence $z = 0$ and $W_1 \cap W_2 \subseteq \{0\}$. The reverse inclusion $\{0\} \subseteq W_1 \cap W_2$ is clearly true since 0 is in every subspace. Therefore, $W_1 \cap W_2 = \{0\}$ $\square$

We defined the direct sum of two subspaces in the eigenvector chapter where we learned that the eigenspaces of a A decompose at least part of $\mathbb{R}^n$ into a direct sum of invariant subspaces of L_A. If A was diagonalizable then the direct sum of the e-spaces covered all of $\mathbb{R}^n$. Just a reminder, now let's see how the direct sum is also of importance here:

Proposition 9.3.10.

Let $W \leq \mathbb{R}^n$ then

1. $\mathbb{R}^n = W \oplus W^{\perp}$.
2. $dim(W) + dim(W^{\perp}) = n$,
3. $(W^{\perp})^{\perp} = W$,

Proof: Let $W \leq \mathbb{R}^n$ and choose an orthonormal basis $\beta = \{v_1, v_2, \dots v_k\}$ for S. Let $z \in \mathbb{R}^n$ and define

$$Proj_W(z) = \sum_{i=1}^{k} (z \bullet v_i) v_i \qquad \text{and} \qquad Orth_W(z) = z - Proj_W(z).$$

Observe that $z = Proj_W(z) + Orth_W(z)$ and clearly $Proj_W(z) \in S$. We now seek to argue that $Orth_W(z) \in S^{\perp}$. Let $v_j \in \beta$ then

$$\begin{aligned} v_j \bullet Orth_W(z) &= v_j \bullet (z - Proj_W(z)) \\ &= v_j \bullet z - v_j \bullet \left(\sum_{i=1}^{k} (z \bullet v_i) v_i \right) \\ &= v_j \bullet z - \sum_{i=1}^{k} (z \bullet v_i)(v_j \bullet v_i) \\ &= v_j \bullet z - \sum_{i=1}^{k} (z \bullet v_i) \delta_{ij} \\ &= v_j \bullet z - z \bullet v_j \\ &= 0 \end{aligned}$$

Therefore, $\mathbb{R}^n = W \oplus W^{\perp}$. To prove (2.) notice we know by Proposition 9.3.5 that $W^{\perp} \leq \mathbb{R}^n$ and consequently there exists an orthonormal basis $\Gamma = \{w_1, w_2, \dots, w_l\}$ for $W^{\perp}$. Furthermore, by Proposition 9.3.9 we find $\beta \cap \Gamma = \emptyset$ since 0 is not in either basis. We argue that $\beta \cup \Gamma$ is a basis for $\mathbb{R}^n$. Observe that $\beta \cup \Gamma$ clearly spans $\mathbb{R}^n$ since $z = Proj_W(z) + Orth_W(z)$ for each $z \in \mathbb{R}^n$ and $Proj_W(z) \in span(\beta)$ while $Orth_W(z) \in span(\Gamma)$. Furthermore, I argue that $\beta \cup \Gamma$ is an orthonormal set. By construction β and Γ are orthonormal, so all we need prove is that the dot-product of vectors from β and Γ is zero, but that is immediate from the construction of Γ. We learned in Proposition 9.2.9 that orthogonality for set of nonzero vectors implies linearly independence. Hence, $\beta \cup \Gamma$ is a linearly independent spanning set for $\mathbb{R}^n$. By the dimension theorem we deduce that there must be n-vectors in $\beta \cup \Gamma$ since it must have the same number of vectors as any other basis for $\mathbb{R}^n$ (the standard basis obviously has n-vectors). Therefore,

$$dim(W) + dim(W^\perp) = n.$$

in particular, we count $dim(W^\perp) = n - k$ in my current notation. Now turn to ponder the proof of (3.). Let $z \in (W^\perp)^\perp$ and expand z in the basis $\beta \cup \Gamma$ to gain further insight, $z = z_1v_1 + z_2v_2 + \cdots z_kv_k + z_{k+1}w_1 + z_{k+2}w_2 + \cdots z_nw_{n-k}$. Since $z \in (W^\perp)^\perp$ then $z \cdot w_\perp = 0$ for all $w_\perp \in W^\perp$, in particular $z \cdot w_j = 0$ for all $j = 1, 2, \dots, n-k$. But, this implies $z_{k+1} = z_{k+2} = \cdots = z_n = 0$ since Proposition 9.2.26 showed the coordinates w.r.t. an orthonormal basis are given by dot-products. Therefore, $z \in span(\beta) = W$ and we have shown $(W^\perp)^\perp \subseteq W$. In invite the reader to prove the reverse inclusion to complete this proof. □

Two items I defined for the purposes of the proof above have application far beyond the proof. Let's state them again for future reference. I give two equivalent definitions, technically we should prove that the second basis dependent statement follows from the first basis-independent statement. Primary definitions are as a point of mathematical elegance stated in a coordinate free langauge in as much as possible, however the second statement is far more useful.

Definition 9.3.11.

Let $W \leq \mathbb{R}^n$ if $z \in \mathbb{R}^n$ and $z = u + w$ for some $u \in W$ and $w \in W^\perp$ then we define $u = Proj_W(z)$ and $w = Orth_W(z)$. Equivalently, choose an orthonormal basis $\beta = \{v_1, v_2, \dots v_k\}$ for W then if $z \in \mathbb{R}^n$ we define

$$Proj_W(z) = \sum_{i=1}^{k} (z \cdot v_i)v_i \qquad \text{and} \qquad Orth_W(z) = z - Proj_W(z).$$

Example 9.3.12. *Let $W = span\{e_1 + e_2, e_3\}$ and $x = (1, 2, 3)$ calculate $Proj_W(x)$. To begin I note that the given spanning set is orthogonal and hence linear indpendent. We need only orthonormalize to obtain an orthonormal basis β for W*

$$\beta = \{v_1, v_2\} \qquad with \qquad v_1 = \tfrac{1}{\sqrt{2}}(1, 1, 0),\ v_2 = (0, 0, 1)$$

Calculate, $v_1 \bullet x = \frac{3}{\sqrt{2}}$ and $v_2 \bullet x = 3$. Thus,

$$Proj_W((1, 2, 3)) = (v_1 \bullet x)v_1 + (v_2 \bullet x)v_2 = \tfrac{3}{\sqrt{2}}v_1 + 3v_2 = \left(\tfrac{3}{2}, \tfrac{3}{2}, 3\right)$$

Then it's easy to calculate the orthogonal part,

$$Orth_W((1,2,3)) = (1,2,3) - (\tfrac{3}{2}, \tfrac{3}{2}, 3) = (-\tfrac{1}{2}, \tfrac{1}{2}, 0).$$

As a check on the calculation note that $Proj_W(x) + Orth_W(x) = x$ *and* $Proj_W(x) \bullet Orth_W(x) = 0$.

Example 9.3.13. *Let* $W = span\{u_1, u_2, u_3\} \leq \mathbb{R}^4$ *where*

$$u_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} -1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$$

calculate $Proj_W((0,6,0,6))$ [2]. *Notice that the given spanning set appears to be linearly independent but it is not orthogonal. Apply Gram-Schmidt to fix it:*

$$\begin{aligned} v_1 &= u_1 = (2,1,2,0) \\ v_2 &= u_2 - \tfrac{u_2 \bullet v_1}{v_1 \bullet v_1} v_1 = u_2 = (0,-2,1,1) \\ v_3 &= u_3 - \tfrac{u_3 \bullet v_1}{v_1 \bullet v_1} v_1 - \tfrac{u_3 \bullet v_2}{v_2 \bullet v_2} v_2 = u_3 + \tfrac{5}{6} v_2 = (-1,2,0,-1) + (0, -\tfrac{10}{6}, \tfrac{5}{6}, \tfrac{5}{6}) \end{aligned}$$

We calculate,

$$v_3 = (-1,\ 2 - \tfrac{5}{3},\ \tfrac{5}{6},\ -1 + \tfrac{5}{6}\ ) = (-1,\ \tfrac{1}{3},\ \tfrac{5}{6},\ -\tfrac{1}{6}\ ) = \tfrac{1}{6}(-6,2,5,-1)$$

The normalized basis follows easily,

$$v_1' = \tfrac{1}{3}(2,1,2,0) \qquad v_2' = \tfrac{1}{\sqrt{6}}(0,-2,1,1) \qquad v_3' = \tfrac{1}{\sqrt{66}}(-6,2,5,-1).$$

Calculate dot-products in preparation for the projection calculation,

$$v_1' \bullet x = \tfrac{1}{3}(2,1,2,0) \bullet (0,6,0,6) = 2$$

$$v_2' \bullet x = \tfrac{1}{\sqrt{6}}(0,-2,1,1) \bullet (0,6,0,6) = \tfrac{1}{\sqrt{6}}(-12+6) = -\sqrt{6}$$

$$v_3' \bullet x = \tfrac{1}{\sqrt{66}}(-6,2,5,-1) \bullet (0,6,0,6) = \tfrac{1}{\sqrt{66}}(12-6) = \tfrac{6}{\sqrt{66}}$$

Now we calculate the projection of $x = (0,6,0,6)$ *onto* W *with ease:*

$$\begin{aligned} Proj_W(x) &= (x \bullet v_1')v_1' + (x \bullet v_2')v_2' + (x \bullet v_3')v_3' \\ &= (2)\tfrac{1}{3}(2,1,2,0) - (\sqrt{6})\tfrac{1}{\sqrt{6}}(0,-2,1,1) + (\tfrac{6}{\sqrt{66}})\tfrac{1}{\sqrt{66}}(-6,2,5,-1) \\ &= (\tfrac{4}{3}, \tfrac{2}{3}, \tfrac{4}{3}, 0) + (0,2,-1,-1) + (\tfrac{-6}{11}, \tfrac{2}{11}, \tfrac{5}{11}, \tfrac{-1}{11}) \\ &= (\tfrac{26}{33},\ \tfrac{94}{33},\ \tfrac{26}{33},\ \tfrac{-36}{33}\ ) \end{aligned}$$

and,

$$Orth_W(x) = (\tfrac{-26}{33},\ \tfrac{104}{33},\ \tfrac{-26}{33},\ \tfrac{234}{33}\ ).$$

[2] this problem is inspired from Anton & Rorres' §6.4 homework problem 3 part d. from the 9th. ed.

9.4 the closest vector problem

Suppose we are given a subspace and a vector not in the subspace, which vector in the subspace is closest to the external vector ? Naturally the projection answers this question. The projection of the external vector onto the subspace will be closest. Let me be a bit more precise:

Proposition 9.4.1. *Closest vector inequality.*

If $S \leq \mathbb{R}^n$ and $b \in \mathbb{R}^n$ such that $b \notin S$ then for all $u \in S$ with $u \neq Proj_S(b)$,

$$||b - Proj_S(b)|| < ||b - u||.$$

This means $Proj_S(b)$ is the closest vector to b in S.

Proof: Noice that $b - u = b - Proj_S(b) + Proj_S(b) - u$. Furthermore note that $b - Proj_S(b) = Orth_S(b) \in S^\perp$ whereas $Proj_S(b) - u \in S$ hence these are orthogonal vectors and we can apply the Pythagorean Theorem,

$$||b - u||^2 = ||b - Proj_S(b)||^2 + ||Proj_S(b) - u||^2$$

Notice that $u \neq Proj_S(b)$ implies $Proj_S(b) - u \neq 0$ hence $||Proj_S(b) - u||^2 > 0$. It follows that $||b - Proj_S(b)||^2 < ||b - u||^2$. And as the $|| \cdot ||$ is nonnegative[3] we can take the squareroot to obtain $||b - Proj_S(b)|| < ||b - u||$. $\square$

Remark 9.4.2.

In calculus III I show at least three distinct methods to find the point off a plane which is closest to the plane. We can minimize the distance function via the 2nd derivative test for two variables, or use Lagrange Multipliers or use the geometric solution which invokes the projection operator. It's nice that we have an explicit proof that the geometric solution is valid. We had argued on the basis of geometric intuition that $Orth_S(b)$ is the shortest vector from the plane S to the point b off the plane[4] Now we have proof. Better yet, our proof equally well applies to subspaces of $\mathbb{R}^n$. In fact, this discussion extends to the context of inner product spaces.

Example 9.4.3. *Consider* $\mathbb{R}^2$ *let* $S = span\{(1,1)\}$. *Find the point on the line* S *closest to the point* $(4,0)$.

$$Proj_S((4,0)) = \tfrac{1}{2}((1,1) \bullet (4,0)(1,1) = (2,2).$$

Thus, $(2,2) \in S$ *is the closest point to* $(4,0)$. *Geometrically, this is something you should have been able to derive for a few years now. The points* $(2,2)$ *and* $(4,0)$ *are on the perpendicular bisector of* $y = x$ *(the set* S *is nothing more than the line* $y = x$ *making the usual identification of points and vectors)*

[3]notice $a^2 < b^2$ need not imply $a < b$ in general. For example, $(5)^2 < (-7)^2$ yet $5 \not< -7$. Generally, $a^2 < b^2$ together with the added condition $a, b > 0$ implies $a < b$.

Example 9.4.4. *In Example 9.3.13 we found that* $W = span\{u_1, u_2, u_3\} \leq \mathbb{R}^4$ *where*

$$u_1 = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} -1 \\ 2 \\ 0 \\ -1 \end{bmatrix}$$

has $Proj_W((0,6,0,6)) = \left(\frac{26}{33}, \frac{94}{33}, \frac{26}{33}, \frac{-36}{33}\right)$. *We can calculate that*

$$rref \left[\begin{array}{ccc|c} 2 & 0 & -1 & 0 \\ 1 & -2 & 2 & 6 \\ 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & 6 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

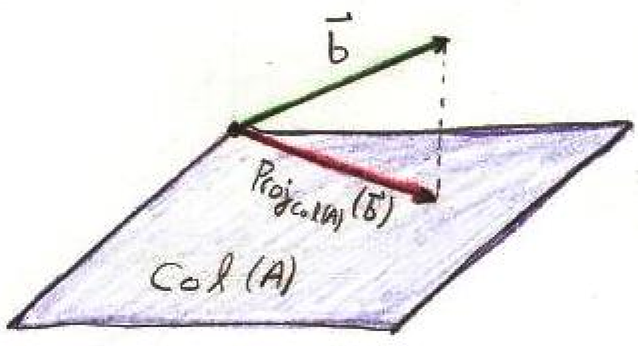
This means that $(0,6,0,6) \notin W$. *However, we learned in Proposition 9.4.1 that* $Proj_W((0,6,0,6))$ *is the vector in* W *which is closest to* $(0,6,0,6)$. *Notice that we can deduce that the orthogonal basis from Example 9.3.13 unioned with* $Orth_W((0,6,0,6))$ *will form an orthogonal basis for* $\mathbb{R}^4$.

Example 9.4.5. *Example 9.3.12 shows that* $W = span\{e_1 + e_2, e_3\}$ *and* $x = (1,2,3)$ *yields* $Proj_W(x) = \left(\frac{3}{2}, \frac{3}{2}, 3\right)$. *Again we can argue that* $x \notin Col[e_1 + e_2|e_3] = W$ *but* $Proj_W(x)$ *is in fact in* W. *Moreover,* $Proj_W(x)$ *is the closest vector to* x *which is in* W. *In this case, the geometry is that* $Orth_W(x) = \left(-\frac{1}{2}, \frac{1}{2}, 0\right)$ *is the precisely the normal vector to the plane* W.

The examples above are somewhat special in that the subspaces considered have only one dimension less than the total vector space. This means that the orthogonal projection of any vector outside the subspace will return the same vector modulo a nonzero constant. In other words, the orthogonal complement is selecting the normal vector to our subspace. In general if we had a subspace which was two or more dimensions smaller than the total vector space then there would be more variety in the output of the orthogonal projection with respect to the subspace. For example, if we consider a plane inside $\mathbb{R}^4$ then there is more than just one direction which is orthogonal to the plane, the orthogonal projection would itself fill out a plane in $\mathbb{R}^4$.

9.5 inconsistent equations

We've spent considerable time solving systems of equations which were *consistent*. What if a system of equations $Ax = b$ is inconsistent? What if anything can we say? Let $A \in \mathbb{R}^{m \times n}$ then we found in Proposition 3.6.3 $\boxed{Ax = b \text{ is consistent iff } b \in Col(A).}$ In other words, the system has a solution iff there is some linear combination of the columns of A such that we obtain b. Here the columns of A and b are both m-dimensional vectors. If $rank(A) = dim(Col(A)) = m$ then the system is consistent no matter which choice for b is made. However, if $rank(A) < m$ then there are some vectors in $\mathbb{R}^m$ which are not in the column space of A and if $b \notin Col(A)$ then there will be no $x \in \mathbb{R}^n$ such that $Ax = b$. We can picture it as follows: the $Col(A)$ is a subspace of $\mathbb{R}^m$ and $b \in \mathbb{R}^m$ is a vector pointing out of the subspace. The shadow of b onto the subspace $Col(A)$ is given by $Proj_{Col(A)}(b)$.



Notice that $Proj_{Col(A)}(b) \in Col(A)$ thus the system $Ax = Proj_{Col(A)}(b)$ has a solution for any $b \in \mathbb{R}^m$. In fact, we can argue that x which solves $Ax = Proj_{Col(A)}(b)$ is the solution which comes closest to solving $Ax = b$. Closest in the sense that $||Ax - b||^2$ is minimized. We call such x the least squares solution to $Ax = b$ (*which is kind-of funny terminology since x is not actually a solution, perhaps we should really call it the "least squares approximation"*).

Proposition 9.5.1.

If $Ax = b$ is inconsistent then the solution of $Au = Proj_{col(A)}(b)$ minimizes $||Ax - b||^2$.

Proof: We can break-up the vector b into a vector $Proj_{Col(A)}(b) \in Col(A)$ and $Orth_{col(A)}(b) \in Col(A)^\perp$ where

$$b = Proj_{Col(A)}(b) + Orth_{Col(A)}(b).$$

Since $Ax = b$ is inconsistent it follows that $b \notin Col(A)$ thus $Orth_{Col(A)}(b) \neq 0$. Observe that:

$$\begin{aligned} ||Ax - b||^2 &= ||Ax - Proj_{Col(A)}(b) - Orth_{Col(A)}(b)||^2 \\ &= ||Ax - Proj_{Col(A)}(b)||^2 + ||Orth_{Col(A)}(b)||^2 \end{aligned}$$

Therefore, the solution of $Ax = Proj_{Col(A)}(b)$ minimizes $||Ax - b||^2$ since any other vector will make $||Ax - Proj_{Col(A)}(b)||^2 > 0$. $\square$

Admittably, there could be more than one solution of $Ax = Proj_{Col(A)}(b)$, however it is usually the case that this system has a unique solution. Especially for expermentally determined data sets.

We already have a technique to calculate projections and of course we can solve systems but it is exceedingly tedious to use the proposition above from scratch. Fortunately there is no need:

Proposition 9.5.2.

If $Ax = b$ is inconsistent then the solution(s) of $Au = Proj_{Col(A)}(b)$ are solutions of the so-called **normal equations** $A^TAu = A^Tb$.

Proof: Observe that,

$$\begin{aligned} Au = Proj_{Col(A)}(b) &\Leftrightarrow b - Au = b - Proj_{Col(A)}(b) = Orth_{Col(A)}(b) \\ &\Leftrightarrow b - Au \in Col(A)^{\perp} \\ &\Leftrightarrow b - Au \in Null(A^T) \\ &\Leftrightarrow A^T(b - Au) = 0 \\ &\Leftrightarrow A^T Au = A^T b, \end{aligned}$$

where we used Proposition 9.3.8 in the third step. □

The proposition below follows immediately from the preceding proposition.

Proposition 9.5.3.

If $det(A^T A) \neq 0$ then there is a unique solution of $Au = Proj_{Col(A)}(b)$.

9.6 least squares analysis

In experimental studies we often have some model with coefficients which appear linearly. We perform an experiment, collect data, then our goal is to find coefficients which make the model fit the collected data. Usually the data will be inconsistent with the model, however we'll be able to use the idea of the last section to find the so-called *best-fit* curve. I'll begin with a simple linear model. This linear example contains all the essential features of the least-squares analysis.

9.6.1 linear least squares problem

Problem: find values of c_1, c_2 such that $y = c_1 x + c_2$ most closely models a given data set: $\{(x_1, y_1), (x_2, y_2), \dots, (x_k, y_k)\}$

Solution: Plug the data into the model and see what equations result:

$$y_1 = c_1 x_1 + c_2, \quad y_2 = c_1 x_2 + c_2, \quad \dots \quad y_k = c_1 x_k + c_2$$

arrange these as a matrix equation,

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix} = \begin{bmatrix} x_1 & 1 \\ x_2 & 1 \\ \vdots & \vdots \\ x_k & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad \Rightarrow \vec{y} = M\vec{v}$$

where $\vec{y} = (y_1, y_2, \dots, y_k), v = (c_1, c_2)$ and M is defined in the obvious way. The system $\vec{y} = M\vec{v}$ will be inconsistent due to the fact that error in the data collection will[5] make the results bounce

[5] almost always

above and below the true solution. We can solve the normal equations $M^T\vec{y} = M^TM\vec{v}$ to find c_1, c_2 which give the best-fit curve[6].

Example 9.6.1. *Find the best fit line through the points* $(0,2),(1,1),(2,4),(3,3)$. *Our model is* $y = c_1 + c_2x$. *Assemble* M *and* $\vec{y}$ *as in the discussion preceding this example:*

$$\vec{y} = \begin{bmatrix} 2 \\ 1 \\ 4 \\ 3 \end{bmatrix} \qquad M = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \qquad \Rightarrow \qquad M^TM = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 14 & 6 \\ 6 & 4 \end{bmatrix}$$

$$\text{and we calculate:} \qquad M^Ty = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ 4 \\ 3 \end{bmatrix} = \begin{bmatrix} 18 \\ 10 \end{bmatrix}$$

The normal equations[7] *are* $M^TM\vec{v} = M^T\vec{y}$. *Note that* $(M^TM)^{-1} = \frac{1}{20}\begin{bmatrix} 4 & -6 \\ -6 & 14 \end{bmatrix}$ *thus the solution of the normal equations is simply,*

$$\vec{v} = (M^TM)^{-1}M^T\vec{y} = \frac{1}{20}\begin{bmatrix} 4 & -6 \\ -6 & 14 \end{bmatrix}\begin{bmatrix} 18 \\ 10 \end{bmatrix} = \begin{bmatrix} \frac{3}{5} \\ \frac{8}{5} \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

Thus, $\boxed{y = 0.6x + 1.6}$ *is the best-fit line. This solution minimizes the vertical distances squared between the data and the model.*

It's really nice that the order of the normal equations is only as large as the number of coefficients in the model. If the order depended on the size of the data set this could be much less fun for real-world examples. Let me set-up the linear least squares problem for 3-coefficients and data from $\mathbb{R}^3$, the set-up for more coefficients and higher-dimensional data is similar. We already proved this in general in the last section, the proposition simply applies mathematics we already derived. I state it for your convenience.

Proposition 9.6.2.

Given data $\{\vec{r}_1, \vec{r}_2, \dots, \vec{r}_n\} \subset \mathbb{R}^3$, with $\vec{r}_k = (x_k, y_k, z_k)$, the best-fit of the linear model $z = c_1x + c_2y + c_3$ is obtained by solving the normal equations $M^TM\vec{v} = M^T\vec{z}$ where

$$\vec{z} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \qquad M = \begin{bmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ \vdots & \vdots & \vdots \\ x_n & y_n & 1 \end{bmatrix} \qquad \vec{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}.$$

[6] notice that if x_i are not all the same then it is possible to show $det(M^TM) \neq 0$ and then the solution to the normal equations is unique

[7] notice my choice to solve this system of 2 equations and 2 unknowns is just a choice, You can solve it a dozen different ways, you do it the way which works best for you.

Example 9.6.3. *Find the plane which is closest to the points* $(0,0,0),(1,2,3),(4,0,1),(0,3,0),(1,1,1)$. *An arbitrary*[8] *plane has the form* $z = c_1x + c_2y + c_3$. *Work on the normal equations,*

$$M = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 4 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \vec{z} = \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \quad \Rightarrow \quad M^TM = \begin{bmatrix} 0 & 1 & 4 & 0 & 1 \\ 0 & 2 & 0 & 3 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 2 & 1 \\ 4 & 0 & 1 \\ 0 & 3 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 18 & 3 & 6 \\ 3 & 14 & 6 \\ 6 & 6 & 5 \end{bmatrix}$$

$$also, \quad M^T\vec{z} = \begin{bmatrix} 0 & 1 & 4 & 0 & 1 \\ 0 & 2 & 0 & 3 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 7 \\ 5 \end{bmatrix}$$

We solve $M^TM\vec{v} = M^T\vec{z}$ *by row operations, after some calculation we find:*

$$rref[M^TM|M^T\vec{z}] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & 89/279 \\ 0 & 1 & 1 & 32/93 \\ 0 & 0 & 1 & 19/93 \end{array}\right] \quad \Rightarrow \quad \begin{array}{l} c_1 = 89/279 \\ c_2 = 32/93 \\ c_3 = 19/93 \end{array}$$

Therefore, $z = \frac{89}{293}x + \frac{32}{93}y + \frac{19}{93}$ *is the plane which is "closest" to the given points. Technically, I'm not certain that is is the absolute closest. We used the vertical distance squared as a measure of distance from the point. Distance from a point to the plane is measured along the normal direction, so there is no garauntee this is really the absolute "best" fit. For the purposes of this course we will ignore this subtle and annoying point. When I say "best-fit" I mean the least squares fit of the model.*

9.6.2 nonlinear least squares

Problem: find values of c_1, c_2 **such that** $y = c_1f_1(x)x + c_2f_2(x) + \cdots + c_nf_n(x)$ **most closely models a given data set:** $\{(x_1,y_1),(x_2,y_2),\ldots,(x_k,y_k)\}$**. We assume the coefficients** c_1, c_2 **appear linearly on (possibly nonlinear) functions** $f_1, f_2, \ldots f_n$**.**

Solution: Plug the data into the model and see what equations result:

$$\begin{aligned} y_1 &= c_1f_1(x_1) + c_2f_2(x_1) + \cdots + c_nf_n(x_1), \\ y_2 &= c_1f_1(x_2) + c_2f_2(x_2) + \cdots + c_nf_n(x_2), \\ &\vdots \quad \vdots \qquad\quad \vdots \\ y_k &= c_1f_1(x_k) + c_2f_2(x_k) + \cdots + c_nf_n(x_k) \end{aligned}$$

[8]technically, the general form for a plane is $ax + by + cz = d$, if $c = 0$ for the best solution then our model misses it. In such a case we could let x or y play the role that z plays in our set-up.

arrange these as a matrix equation,

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_k \end{bmatrix} = \begin{bmatrix} f_1(x_1) & f_2(x_1) & \cdots & f_n(x_1) \\ f_1(x_1) & f_2(x_1) & \cdots & f_n(x_1) \\ \vdots & \vdots & \vdots & \vdots \\ f_1(x_k) & f_2(x_k) & \cdots & f_n(x_k) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix} \Rightarrow \vec{y} = M\vec{v}$$

where $\vec{y} = (y_1, y_2, \dots, y_k), v = (c_1, c_2, \dots, c_n)$ and M is defined in the obvious way. The system $\vec{y} = M\vec{v}$ will be inconsistent due to the fact that error in the data collection will[9] make the results bounce above and below the true solution. We can solve the normal equations $M^T\vec{y} = M^T M\vec{v}$ to find $c_1, c_2, \dots, c_n$ which give the best-fit curve[10].

Remark 9.6.4.

Nonlinear least squares includes the linear case as a subcase, take $f_1(x) = x$ and $f_2(x) = 1$ and we return to the linear least squares examples. We will use data sets from $\mathbb{R}^2$ in this subsection. These techniques do extend to data sets with more variables as I demonstrated in the simple case of a plane.

Example 9.6.5. *Find the best-fit parabola through the data* $(0,0), (1,3), (4,4), (3,6), (2,2)$. *Our model has the form* $y = c_1x^2 + c_2x + c_3$. *Identify that* $f_1(x) = x^2, f_2(x) = x$ *and* $f_3(x) = 1$ *thus we should study the normal equations:* $M^T M\vec{v} = M^T\vec{y}$ *where:*

$$M = \begin{bmatrix} f_1(0) & f_2(0) & f_3(0) \\ f_1(1) & f_2(1) & f_3(1) \\ f_1(4) & f_2(4) & f_3(4) \\ f_1(3) & f_2(3) & f_3(3) \\ f_1(2) & f_2(2) & f_3(2) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 16 & 4 & 1 \\ 9 & 3 & 1 \\ 4 & 2 & 1 \end{bmatrix} \quad \textit{and} \quad \vec{y} = \begin{bmatrix} 0 \\ 3 \\ 4 \\ 6 \\ 2 \end{bmatrix}.$$

Hence, calculate

$$M^T M = \begin{bmatrix} 0 & 1 & 16 & 9 & 4 \\ 0 & 1 & 4 & 3 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 16 & 4 & 1 \\ 9 & 3 & 1 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 354 & 100 & 30 \\ 100 & 30 & 10 \\ 30 & 10 & 5 \end{bmatrix}$$

and,

$$M^T\vec{y} = \begin{bmatrix} 0 & 1 & 16 & 9 & 4 \\ 0 & 1 & 4 & 3 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 3 \\ 4 \\ 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 129 \\ 41 \\ 15 \end{bmatrix}$$

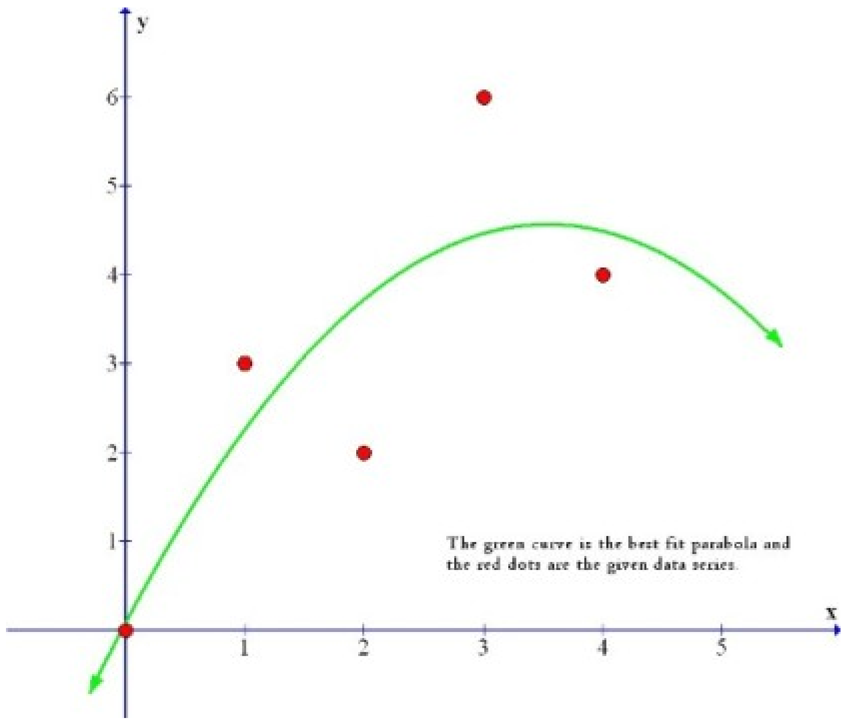
[9] almost always

[10] notice that if $f_j(x_i)$ are not all the same then it is possible to show $det(M^T M) \neq 0$ and then the solution to the normal equations is unique

After a few row operations we can deduce,

$$rref[M^T M | M^T \vec{y}] = \left[\begin{array}{ccc|c} 1 & 0 & 1 & -5/14 \\ 0 & 1 & 1 & 177/70 \\ 0 & 0 & 1 & 3/35 \end{array}\right] \quad \Rightarrow \quad \begin{array}{l} c_1 = -5/14 \approxeq -0.357 \\ c_2 = 177/70 \approxeq 2.529 \\ c_3 = 3/35 = 0.086 \end{array}$$

We find the best-fit parabola is $\boxed{y = -0.357x^2 + 2.529x + 0.086}$



Yes..., but what's this for?

Example 9.6.6. *Suppose you land on a mysterious planet. You find that if you throw a ball it's height above the ground y at time t is measured at times $t = 0, 1, 2, 3, 4$ seconds to be $y = 0, 2, 3, 6, 4$ meters respective. Assume that Newton's Law of gravity holds and determine the gravitational acceleration from the data. We already did the math in the last example. Newton's law approximated for heights near the surface of the planet simply says $y'' = -g$ which integrates twice to yield $y(t) = -gt^2/2 + v_o t + y_0$ where v_o is the initial velocity in the vertical direction. We find the best-fit parabola through the data set $\{(0,0), (1,3), (4,4), (3,6), (2,2)\}$ by the math in the last example,*

$$y(t) = -0.357t^2 + 2.529 + 0.086$$

we deduce that $g = 2(0.357)m/s^2 = 0.714m/s^2$. Apparently the planet is smaller than Earth's moon (which has $g_{moon} \approx \frac{1}{6} 9.8m/s^2 = 1.63m/s^2$.

Remark 9.6.7.

If I know for certain that the ball is at $y = 0$ at $t = 0$ would it be equally reasonable to assume y_o in our model? If we do it simplifies the math. The normal equations would only be order 2 in that case.

Example 9.6.8. *Find the best-fit parabola that passes through the origin and the points $(1,3),(4,4),(3,6),(2,2)$. To begin we should state our model: since the parabola goes through the origin we know the y-intercept is zero hence $y = c_1x^2 + c_2x$. Identify $f_1(x) = x^2$ and $f_2(x) = x$. As usual set-up the M and $\vec{y}$,*

$$M = \begin{bmatrix} f_1(1) & f_2(1) \\ f_1(4) & f_2(4) \\ f_1(3) & f_2(3) \\ f_1(2) & f_2(2) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 16 & 4 \\ 9 & 3 \\ 4 & 2 \end{bmatrix} \qquad \text{and} \qquad \vec{y} = \begin{bmatrix} 3 \\ 4 \\ 6 \\ 2 \end{bmatrix}.$$

Calculate,

$$M^TM = \begin{bmatrix} 1 & 16 & 9 & 4 \\ 1 & 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 16 & 4 \\ 9 & 3 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 354 & 100 \\ 100 & 30 \end{bmatrix} \Rightarrow (M^TM)^{-1} = \frac{1}{620}\begin{bmatrix} 30 & -100 \\ -100 & 354 \end{bmatrix}$$

and,

$$M^T\vec{y} = \begin{bmatrix} 1 & 16 & 9 & 4 \\ 1 & 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 6 \\ 2 \end{bmatrix} = \begin{bmatrix} 129 \\ 41 \end{bmatrix}$$

We solve $M^TM\vec{v} = M^T\vec{y}$ by multiplying both sides by $(M^TM)^{-1}$ which yeilds,

$$\vec{v} = (M^TM)^{-1}M^T\vec{y} = \frac{1}{620}\begin{bmatrix} 30 & -100 \\ -100 & 354 \end{bmatrix}\begin{bmatrix} 129 \\ 41 \end{bmatrix} = \begin{bmatrix} -23/62 \\ 807/310 \end{bmatrix} \Rightarrow \begin{array}{l} c_1 = -23/62 \approxeq -0.371 \\ c_2 = 807/310 \approxeq 2.603 \end{array}$$

Thus the best-fit parabola through the origin is $\boxed{y = -0.371x^2 + 2.603x}$

Sometimes an application may not allow for direct implementation of the least squares method, however a rewrite of the equations makes the unknown coefficients appear linearly in the model.

Example 9.6.9. *Newton's Law of Cooling states that an object changes temperature T at a rate proportional to the difference between T and the room-temperature. Suppose room temperature is known to be 70^o then $dT/dt = -k(T-70) = -kT + 70k$. Calculus reveals solutions have the form $T(t) = c_0e^{-kt} + 70$. Notice this is very intuitive since $T(t) \to 70$ for $t >> 0$. Suppose we measure the temperature at successive times and we wish to find the best model for the temperature at time t. In particular we measure: $T(0) = 100$, $T(1) = 90$, $T(2) = 85$, $T(3) = 83$, $T(4) = 82$. One unknown coefficient is k and the other is c_1. Clearly k does not appear linearly. We can remedy* *this by working out the model for the natural log of $T - 70$. Properties of logarithms will give us a model with linearly appearing unknowns:*

$$\ln(T(t) - 70) = \ln(c_0 e^{-kt}) = \ln(c_0) + \ln(e^{-kt}) = \ln(c_0) - kt$$

Let $c_1 = \ln(c_0), c_2 = -k$ then identify $f_1(t) = 1$ while $f_2(t) = t$ and $y = \ln(T(t) - 70$. Our model is $y = c_1 f_1(t) + c_2 f_2(t)$ and the data can be generated from the given data for $T(t)$:

$$\begin{aligned}
t_1 = 0 : y_1 &= \ln(T(0) - 70) = \ln(100 - 70) = \ln(30) \\
t_2 = 1 : y_2 &= \ln(T(1) - 90) = \ln(90 - 70) = \ln(20) \\
t_3 = 2 : y_3 &= \ln(T(2) - 85) = \ln(85 - 70) = \ln(15) \\
t_4 = 3 : y_4 &= \ln(T(2) - 83) = \ln(83 - 70) = \ln(13) \\
t_5 = 4 : y_5 &= \ln(T(2) - 82) = \ln(82 - 70) = \ln(12)
\end{aligned}$$

Our data for (t, y) is $(0, \ln 30), (1, \ln 20), (2, \ln 15), (3, \ln 13), (4, \ln 12)$. We should solve normal equations $M^T M\vec{v} = M^T\vec{y}$ where

$$M = \begin{bmatrix} f_1(0) & f_2(0) \\ f_1(1) & f_2(1) \\ f_1(2) & f_2(2) \\ f_1(3) & f_2(3) \\ f_1(4) & f_2(4) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \\ 1 & 4 \end{bmatrix} \qquad \text{and} \qquad \vec{y} = \begin{bmatrix} \ln 30 \\ \ln 20 \\ \ln 15 \\ \ln 13 \\ \ln 12 \end{bmatrix}.$$

We can calculate $M^T M = \begin{bmatrix} 5 & 10 \\ 10 & 30 \end{bmatrix}$ and $M^T\vec{y} \approxeq \begin{bmatrix} 14.15 \\ 26.05 \end{bmatrix}$. Solve $M^T M\vec{v} = M^T\vec{y}$ by multiplication by inverse of $M^T M$:

$$\vec{y} = (M^T M)^{-1} M^T\vec{y} = \begin{bmatrix} 3.284 \\ -0.2263 \end{bmatrix} \quad \Rightarrow \quad \begin{matrix} c_1 \approxeq 3.284 \\ c_2 \approxeq -0.2263 \end{matrix}.$$

Therefore, $y(t) = \ln(T(t) - 70) = 3.284 - 0.2263$ we identify that $k = 0.2263$ and $ln(c_0) = 3.284$ which yields $c_0 = e^{3.284} = 26.68$. We find the best-fit temperature function is

$$\boxed{T(t) = 26.68e^{-0.2263t} + 70.}$$

Now we could give good estimates for the temperature $T(t)$ for other times. If Newton's Law of cooling is an accurate model and our data was collected carefully then we ought to be able to make accurate predictions with our model.

Remark 9.6.10.

The accurate analysis of data is more involved than my silly examples reveal here. Each experimental fact comes with an error which must be accounted for. A real experimentalist never gives just a number as the answer. Rather, one must give a number and an uncertainty or error. There are ways of accounting for the error of various data. Our approach here takes all data as equally valid. There are weighted best-fits which minimize a weighted least squares. Technically, this takes us into the realm of math of inner-product spaces. Finite dimensional inner-product spaces also allows for least-norm analysis. The same philosophy guides the analysis: the square of the norm measures the sum of the squares of the errors in the data. The collected data usually does not precisely fit the model, thus the equations are inconsistent. However, we project the data onto the plane representative of model solutions and this gives us the best model for our data. Generally we would like to minimize χ^2, this is the notation for the sum of the squares of the error often used in applications. In statistics finding the best-fit line is called doing "linear regression".

9.7 orthogonal transformations and geometry

If we begin with an orthogonal subset of $\mathbb{R}^n$ and we preform a linear transformation then will the image of the set still be orthogonal? We would like to characterize linear transformations which maintain orthogonality. These transformations should take an orthogonal basis to a new basis which is still orthogonal.

Definition 9.7.1.

If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation such that $T(x) \cdot T(y) = x \cdot y$ for all $x, y \in \mathbb{R}^n$ then we say that T is an **orthogonal transformation**

Example 9.7.2. *Let $\{e_1, e_2\}$ be the standard basis for $\mathbb{R}^2$ and let $R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ be a rotation of the coordinates by angle θ in the clockwise direction,*

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta + y\sin\theta \\ -x\sin\theta + y\cos\theta \end{bmatrix}$$

As a check on my sign conventions, consider rotating $(1, 0)$ by $R(\pi/2)$, we obtain $(x', y') = (0, 1)$. See the picture for how to derive these transformations from trigonometry. Intuitively, a rotation should not change the length of a vector, let's check the math: let $v, w \in \mathbb{R}^2$,

$$\begin{aligned} R(\theta)v \bullet R(\theta)w &= [R(\theta)v]^T R(\theta)w \\ &= v^T R(\theta)^T R(\theta)w \end{aligned}$$

Now calculate $R(\theta)^T R(\theta)$,

$$R(\theta)^T R(\theta) = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos^2\theta + \sin^2\theta & 0 \\ 0 & \sin^2\theta + \cos^2\theta \end{bmatrix} = I$$

Therefore, $R(\theta)v \bullet R(\theta) = v^T I w = v^T w = v \bullet w$ *for all* $v, w \in \mathbb{R}^2$ *and we find* $L_{R(\theta)}$ *is an orthogonal transformation.*

This shows the matrix of a rotation L_R satisfies $R^T R = I$. Is this always true or was this just a special formula for rotations? Or is this just a two-dimensional thing? What if we look at orthhogonal transformations on $\mathbb{R}^n$ what general condition is there on the matrix of the transformation?

Definition 9.7.3.

Let $A \in \mathbb{R}^{\,n\times n}$ then we say that A is an **orthogonal matrix** iff $A^T A = I$. Moreover, we say A is a **reflection matrix** if A is orthogonal and $det(A) = -1$ whereas we say A is a **rotation matrix** if A is orthogonal with $det(A) = 1$. The set of all orthogonal $n \times n$ matrices is denoted $O(n)$ and the set of all $n \times n$ rotation matrices is denoted $SO(n)$.

Proposition 9.7.4. matrix of an orthogonal transformation is orthogonal

If A is the matrix of an orthogonal transformation on $\mathbb{R}^n$ then $A^T A = I$ and either A is a rotation matrix or A is a reflection matrix.

Proof: Suppose $L(x) = Ax$ and L is an orthogonal transformation on $\mathbb{R}^n$. Notice that

$$L(e_i) \cdot L(e_j) = [Ae_i]^T A e_j = e_i^T [A^T A] e_j$$

and

$$e_i \cdot e_j = e_i^T e_j = e_i^T I e_j$$

hence $e_i^T [A^T A - I] e_j = 0$ for all i, j thus $A^T A - I = 0$ by Example 2.5.11 and we find $A^T A = I$. Hence,

$$det(A^T A) = det(I) \quad \Leftrightarrow \quad det(A)det(A) = 1 \quad \Leftrightarrow det(A) = \pm 1$$

Thus $A \in SO(n)$ or A is a reflection matrix. $\square$

The proposition below is immediate from the definitions of length, angle and linear transformation.

Proposition 9.7.5. orthogonal transformations preserve lengths and angles

If $v, w \in \mathbb{R}^n$ and L is an orthogonal transformation such that $v' = L(v)$ and $w' = L(w)$ then the angle between v' and w' is the same as the angle between v and w, in addition the length of v' is the same as v.

Remark 9.7.6.

Reflections, unlike rotations, will spoil the "handedness" of a coordinate system. If we take a right-handed coordinate system and perform a reflection we will obtain a new coordinate system which is left-handed. If you'd like to know more just ask me sometime.

If orthogonal transformations preserve the geometry of $\mathbb{R}^n$ you might wonder if there are other non-linear transformations which also preserve distance and angle. The answer is yes, but we need to be careful to distinguish between the length of a vector and the distance bewtween points. It turns out that the translation defined below will preserve the distance, but not the norm or length of a vector.

Definition 9.7.7.

Fix $b \in \mathbb{R}^n$ then a translation by b is the mapping $T_b(x) = x + b$ for all $x \in \mathbb{R}^n$.

This is known as an **affine transformation**, it is not linear since $T(0) = b \neq 0$ in general. (if $b = 0$ then the translation is both affine and linear). Anyhow, affine transformations should be familar to you: $y = mx + b$ is an affine transformation on $\mathbb{R}$.

Proposition 9.7.8. translations preserve geometry

Suppose $T_b : \mathbb{R}^n \to \mathbb{R}^n$ is a translation then

1. If $\angle(xyz)$ denotes the angle formed by line segments $\bar{xy}, \bar{yz}$ which have endpoints x, y and y, z respectively then $\angle(T_b(x)T_b(y)T_b(z)) = \angle(xyz)$
2. The distance from x to y is the equal to the distance from $T_b(x)$ to $T_b(y)$.

Proof: I'll begin with (2.) since it's easy:

$$d(T_b(x), T_b(y)) = ||T_b(y) - T_b(x)|| = ||y + b - (x + b)|| = ||y - x|| = d(x, y).$$

Next, the angle $\angle(xyz)$ is the angle between $x - y$ and $z - y$. Likewise the angle $\angle T_b(x)T_b(y)T_b(z)$ is the angle between $T_b(x) - T_b(y)$ and $T_b(z) - T_b(y)$. But, these are the same vectors since $T_b(x) - T_b(y) = x + b - (y + b) = x - y$ and $T_b(z) - T_b(y) = z + b - (y + b) = z - y$. $\square$

Definition 9.7.9.

Suppose $T(x) = Ax + b$ where $A \in SO(n)$ and $b \in \mathbb{R}^n$ for all $x \in \mathbb{R}^n$ then we say T is a **rigid motion**.

In high-school geometry you studied the concept of *congruence*. To objects were congruent if they had the same size and shape. From the viewpoint of analytic geometry we can say two objects are congruent iff one is the image of the other with respect to some rigid motion. We leave further discussion of such matters to the modern geometry course where you study these concepts in depth.

Remark 9.7.10.

In Chapter 6 of my *Mathematical Models in Physics* notes I describe how Euclidean geometry is implicit and foundational in classical Newtonian Mechanics. The concept of a rigid motion is used to define what is meant by an *intertial frame*. I have these notes posted on my website, ask if your interested. Chapter 7 of the same notes describes how Special Relativity has hyperbolic geometry as its core. The dot-product is replaced with a Minkowski-product which yields all manner of curious results like time-dilation, length contraction, and the constant speed of light. If your interested in hearing a lecture or two on the geometry of Special Relativity please ask and I'll try to find a time and a place.

9.8 eigenvectors and orthogonality

We can apply the Gram-Schmidt process to orthogonalize the set of e-vectors. If the resulting set of orthogonal vectors is still an eigenbasis then we can prove the matrix formed from e-vectors is an orthogonal matrix.

Proposition 9.8.1.

If $A \in \mathbb{R}^{n\times n}$ has e-values $\lambda_1, \lambda_2, \dots, \lambda_n$ with orthonormal e-vectors $v_1, v_2, \dots, v_n$ and if we define $V = [v_1|v_2|\cdots|v_n]$ then $V^{-1} = V^T$ and $D = V^T AV$ where D is a diagonal matrix with the eigenvalues down the diagonal: $D = [\lambda_1 e_1|\lambda_2 e_2|\cdots|\lambda_n e_n]$.

Proof: Orthonormality implies $v_i^T v_j = \delta_{ij}$. Observe that

$$V^T V = \left[\begin{array}{c} v_1^T \\ \hline v_2^T \\ \hline \vdots \\ \hline v_n^T \end{array}\right] [v_1|v_2|\cdots|v_n] = \begin{bmatrix} v_1^T v_1 & v_1^T v_2 & \cdots & v_1^T v_n \\ v_1^T v_1 & v_1^T v_2 & \cdots & v_1^T v_n \\ \vdots & \vdots & \cdots & \vdots \\ v_n^T v_1 & v_n^T v_2 & \cdots & v_n^T v_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

Thus $V^{-1} = V^T$. The proposition follows from Proposition 7.5.2. $\square$

This is great news. We now have hope of finding the diagonalization of a matrix without going to the trouble of inverting the e-vector matrix. Notice that there is no gaurantee that we can find n-orthonormal e-vectors. Even in the case we have n-linearly independent e-vectors it could happen that when we do the Gram-Schmidt process the resulting vectors are not e-vectors. That said, there is one important, and common, type of example where we are in fact gauranteed the existence of an orthonormal eigenbases for A.

Theorem 9.8.2.

A matrix $A \in \mathbb{R}^{n\times n}$ is symmetric iff there exists an orthonormal eigenbasis for A.

Proof: I'll prove the reverse implication in these notes. Your text has a complete proof of the forward implication in Appendix C, it's very neat, but we don't have that much time. Assume there exists and orthonormal eigenbasis $\{v_1, v_2, \dots, v_n\}$ for A. Let $V = [v_1|v_2|\cdots|v_n]$ and use Proposition 9.8.1, $V^T AV = D$ where D is a diagonal matrix with the e-values down the diagonal. Clearly $D^T = D$. Transposing the equation yields $(V^T AV)^T = D$. Use the socks-shoes property for transpose to see $(V^T AV)^T = V^T A^T (V^T)^T = V^T A^T V$. We find that $V^T A^T V = V^T AV$. Multiply on the left by V and on the right by V^T and we find $A^T = A$ thus A is symmetric. $\square$.

This theorem is a useful bit of trivia to know. But, be careful not to overstate the result. This theorem does not state that all diagonalizable matrices are symmetric.

Example 9.8.3. *In Example 7.2.13 we found the e-values and e-vectors of* $A = \begin{bmatrix} 0 & 0 & -4 \\ 2 & 4 & 2 \\ 2 & 0 & 6 \end{bmatrix}$ *were* $\lambda_1 = \lambda_2 = 4$ *and* $\lambda_3 = 2$ *with e-vectors*

$$u_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad u_3 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

We argued in Example 7.4.4 that $\{u_1, u_2, u_3\}$ *is an eigenbasis. In view of the Theorem above we know there is no way to perform the Gram-Schmidt process and get and orthonormal set of e-vectors for* A*. We could orthonormalize the basis, but it would not result in a set of e-vectors. We can be certain of this since* A *is not symmetric. I invite you to try Gram-Schmidt and see how the process spoils the e-values. The principle calculational observation is simply that when you add e-vectors with different e-values there is no reason to expect the sum is again an e-vector. There is an exception to my last observation, what is it?*

Example 9.8.4. *Let* $A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix}$. *Observe that* $det(A - \lambda I) = -\lambda(\lambda + 1)(\lambda - 3)$ *thus* $\lambda_1 = 0, \lambda_2 = -1, \lambda_3 = 3$. *We can calculate orthonormal e-vectors of* $v_1 = (1, 0, 0)$, $v_2 = \frac{1}{\sqrt{2}}(0, 1, -1)$ *and* $v_3 = \frac{1}{\sqrt{2}}(0, 1, 1)$. *I invite the reader to check the validity of the following equation:*

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Its really neat that to find the inverse of a matrix of orthonormal e-vectors we need only take the transpose; note $\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

9.9 orthogonal matrices and the QR factorization

Suppose we have an orthogonal basis $\beta = \{v_1, v_2, \dots, v_n\}$ for $\mathbb{R}^n$. Let's investigate the properties of the matrix of this basis. Note that $||v_j|| \neq 0$ for each j since β is linearly independent set of vectors. Moreover, if we denote $||v_j|| = l_j$ then we can compactly summarize orthogonality of β with the following relation:

$$v_j \cdot v_k = l_j^2 \delta_{jk}.$$

As a matrix equation we recognize that $[v_j]^T v_k$ is also the $jk-th$ component of the product of $[\beta]^T$ and $[\beta]$. Let me expand on this in matrix notation:

$$[\beta]^T[\beta] = \begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix} [v_1|v_2|\cdots|v_n] = \begin{bmatrix} v_1^T v_1 & v_1^T v_2 & \cdots & v_1^T v_n \\ v_2^T v_1 & v_2^T v_2 & \cdots & v_2^T v_n \\ \vdots & \vdots & \cdots & \vdots \\ v_n^T v_1 & v_n^T v_2 & \cdots & v_n^T v_n \end{bmatrix} = \begin{bmatrix} l_1^2 & 0 & \cdots & 0 \\ 0 & l_2^2 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & l_n^2 \end{bmatrix}$$

This means that $[\beta]^T$ is almost the inverse of $[\beta]$. Observe if we had $l_j = 1$ for $j = 1, 2, \dots, n$ then $[\beta]^T = [\beta]^{-1}$. In other words, if we use an orthonormal basis then the inverse of the basis matrix is obtained by transposition. In fact, matrices with this property have a name:

Definition 9.9.1.

Let $A \in \mathbb{R}^{n \times n}$ then we say that A is an **orthogonal matrix** iff $A^T A = I$. The set of all orthogonal $n \times n$ matrices is denoted $O(n)$.

The discussion preceding the definition provides a proof for the following proposition:

Proposition 9.9.2. matrix of an orthonormal basis is an orthogonal matrix

If β is an orthonormal basis then $[\beta]^T[\beta] = I$ or equivalently $[\beta]^T = [\beta]^{-1}$.

So far we have considered only bases for all of $\mathbb{R}^n$ but we can also find similar results for a subspace $W \leq \mathbb{R}^n$. Suppose $dim(W) < n$. If β is an orthonormal basis for W then it is still true that $[\beta]^T[\beta] = I_{dim(W)}$ however since $[\beta]$ is not a square matrix it does not make sense to say that $[\beta]^T = [\beta]^{-1}$. The QR-factorization of a matrix is tied to this discussion.

Proposition 9.9.3. QR factorization of a full-rank matrix

If $A \in \mathbb{R}^{m \times n}$ is a matrix with linearly independent columns then there exists a matrix $Q \in \mathbb{R}^{m \times n}$ whose columns form an orthonormal basis for $Col(A)$ and square matrix $R \in \mathbb{R}^{n \times n}$ which is upper triangular and has $R_{ii} > 0$ for $i = 1, 2, \dots, n$.

Proof: begin by performing the Gram-Schmidt procedure on the columns of A. Next, normalize that orthogonal basis to obtain an orthonormal basis $\beta = \{u_1, u_2, \dots, u_n\}$ for $Col(A)$. Note that since each column in A is in $Col(A)$ it follows that some linear combination of the vectors in β will produce that column;

$$col_j(A) = R_{1j}u_1 + R_{2j}u_2 + \cdots + R_{nj}u_n = [u_1|u_2|\cdots|u_n][R_{1j}, R_{2j}, \cdots, R_{nj}]^T$$

for some constants $R_{1j}, R_{2j}, \cdots, R_{nj} \in \mathbb{R}$. Let R be the matrix formed from the coefficients of the linear combinations that link columns of A and the orthonormal basis; in particular define R such that $col_j(R) = (R_{1j}, R_{2j}, \cdots, R_{nj})$. It follows that if we denote $[\beta] = Q$ we have for each $j = 1, 2, \dots, n$ the relation

$$col_j(A) = Qcol_j(R)$$

Hence,

$$A = [col_1(A)|col_2(A)|\cdots|col_n(A)] = [Qcol_1(R)|Qcol_2(R)|\cdots|Qcol_n(R)]$$

and we find by the concatenation proposition

$$A = Q[col_1(R)|col_2(R)|\cdots|col_n(R)] = QR$$

where $R \in \mathbb{R}^{\ n\times n}$ as we wished. It remains to show that R is upper triangular with positive diagonal entries. Recall how Gram-Schmidt is accomplished (I'll do normalization along side the orthogonalization for the purposes of this argument). We began by defining $u_1 = \frac{1}{||col_1(A)||}col_1(A)$ hence $col_1(A) = ||col_1(A)||u_1$ and we identify that $col_1(R) = (||col_1(A)||, 0, \dots, 0)$. The next step in the algorithm is to define u_2 by calculating v_2 (since we normalized $u_1 \cdot u_1 = 1$)

$$v_2 = col_2(A) - (col_2(A) \cdot u_1)u_1$$

and normalizing (I define l_2 in the last equality below)

$$u_2 = \frac{1}{||col_2(A) - (col_2(A) \cdot u_1)u_1||}v_2 = \frac{1}{l_2}v_2$$

In other words, $l_2u_2 = v_2 = col_2(A) - (col_2(A) \cdot u_1)u_1$ hence

$$col_2(A) = l_2u_2 - (col_2(A) \cdot u_1)u_1$$

From which we can read the second column of R as

$$col_2(R) = (-(col_2(A) \cdot u_1), l_2, 0, \dots, 0).$$

Continuing in this fashion, if we define l_j to be the length of the orthogonalization of $col_j(A)$ with respect to the preceding $\{u_1, u_2, \dots, u_{j-1}\}$ orthonormal vectors then a calculation similar to the one just performed will reveal that

$$col_j(R) = (\star, \dots, \star, l_j, 0, \dots, 0)$$

and $\star$ are possibly nonzero components in rows 1 through $j - 1$ of the column vector and l_j is the j-th component which is necessarily posititive since it is the length of some nonzero vector. Put all of this together and we find that R is upper triangular with positive diagonal entries[11]. $\square$

Very well, we now know that a QR-factorization exists for a matrix with LI columns. This leaves us with two natural questions:

1. how do we calculate the factorization of a given matrix A ?
2. what is the use of the QR factorization ?

[11] see Lay pg. 405-406 if you don't like my proof

We will answer (1.) with an example or two and I will merely scratch the surface for question (2.). If you took a serious numerical linear algebra course then it is likely you would delve deeper.

Example 9.9.4. *I make use of Example 8.2.25 to illustrate how to find the QR-factorization of a matrix. Basically once you find the Gram-Schmidt then it is as simple as multiplying the orthonormalized column vectors and the matrix since $A = QR$ implies $R = Q^T A$.*

Using Ex. 8.2.25 &

$$A = \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 3 & 3 \end{bmatrix}$$

Illustrating How to calculate QR-factorization of A

Gram.-Schmidt

$$Q = \begin{bmatrix} 1/\sqrt{3} & 0 & -5/\sqrt{42} & 1/\sqrt{14} \\ 0 & 1 & 0 & 0 \\ 1/\sqrt{3} & 0 & 1/\sqrt{42} & -3/\sqrt{14} \\ 1/\sqrt{3} & 0 & 4/\sqrt{42} & 2/\sqrt{14} \end{bmatrix}$$

Trying to find R such that $A = QR$

Instead, solve $Q^T A = Q^T Q R = R$

$$R = Q^T A = \begin{bmatrix} 1/\sqrt{3} & 0 & 1/\sqrt{3} & 1/\sqrt{3} \\ 0 & 1 & 0 & 0 \\ -5/\sqrt{42} & 0 & 1/\sqrt{42} & 4/\sqrt{42} \\ 1/\sqrt{14} & 0 & -3/\sqrt{14} & 2/\sqrt{14} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 3 \\ 0 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \\ 1 & 1 & 3 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3/\sqrt{3} & 3/\sqrt{3} & 5/\sqrt{3} & 6/\sqrt{3} \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 14/\sqrt{42} & -3/\sqrt{42} \\ 0 & 0 & 0 & 9/\sqrt{14} \end{bmatrix}$$

as advertised, upper Δ with positive diagonals!

Finally, returning to (2.). One nice use of the QR-factorization is to simplify calculation of the normal equations. We sought to solve $A^T Au = A^T b$. Suppose that $A = QR$ to obtain:

$$(QR)^T (QR)u = (QR)^T b \quad \Rightarrow \quad R^T Q^T QRu = R^T Q^T b \quad \Rightarrow \quad \boxed{Ru = Q^T b}.$$

This problem is easily solved by back-substitution since R is upper-triangular. I may ask you a homework to examine this in more detail for a specific example.

Chapter 10

quadratic forms

10.1 conic sections and quadric surfaces

Some of you have taken calculus III others have not, but most of you still have much to learn about level curves and surfaces. Let me give two examples to get us started:

$$x^2 + y^2 = 4 \qquad \text{level curve; generally has form } f(x,y) = k$$

$$x^2 + 4y^2 + z^2 = 1 \qquad \text{level surface; generally has form } F(x,y,z) = k$$

Alternatively, some special surfaces can be written as a graph. The top half of the ellipsoid $F(x,y,z) = x^2 + 4y^2 + z^2 = 1$ is the $graph(f)$ where $f(x,y) = \sqrt{1 - x^2 - 4y^2}$ and $graph(f) = \{x, y, f(x,y) \,|(x,y) \in dom(f)\}$. Of course there is a great variety of examples to offer here and I only intend to touch on a few standard examples in this section. Our goal is to see what linear algebra has to say about conic sections and quadric surfaces.

10.1.1 quadratic forms and their matrix

Definition 10.1.1.

Generally, a **quadratic form** Q is a function $Q : \mathbb{R}^n \to \mathbb{R}$ whose formula can be written $Q(\vec{x}) = \vec{x}^T A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$ where $A \in \mathbb{R}^{\ n\times n}$ such that $A^T = A$. In particular, if $\vec{x} = (x, y)$ and $A = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$ then

$$Q(\vec{x}) = \vec{x}^T A\vec{x} = ax^2 + bxy + byx + cy^2 = ax^2 + 2bxy + y^2.$$

The $n = 3$ case is similar,denote $A = [A_{ij}]$ and $\vec{x} = (x, y, z)$ so that

$$Q(\vec{x}) = \vec{x}^T A\vec{x} = A_{11}x^2 + 2A_{12}xy + 2A_{13}xz + A_{22}y^2 + 2A_{23}yz + A_{33}z^2.$$

Generally, if $[A_{ij}] \in \mathbb{R}^{\ n\times n}$ and $\vec{x} = [x_i]^T$ then the associated quadratic form is

$$Q(\vec{x}) = \vec{x}^T A\vec{x} = \sum_{i,j} A_{ij}x_ix_j = \sum_{i=1}^{n} A_{ii}x_i^2 + \sum_{i<j} 2A_{ij}x_ix_j.$$

In case you wondering, yes you could write a given quadratic form with a different matrix which is not symmetric, but we will find it convenient to insist that our matrix is symmetric since that choice is always possible for a given quadratic form.

Also, you may recall (from the future) I said a **bilinear form** was a mapping from $V \times V \to \mathbb{R}$ which is linear in each slot. For example, an inner-product as defined in Definition 11.1.1 is a symmetric, positive definite bilinear form. When we discussed $< x, y >$ we allowed $x \neq y$, in contrast a quadratic form is more like $< x, x >$. Of course the dot-product is also an inner product and we can write a given quadratic form in terms of a dot-product:

$$\vec{x}^T A\vec{x} = \vec{x} \cdot (A\vec{x}) = (A\vec{x}) \cdot \vec{x} = \vec{x}^T A^T \vec{x}$$

Some texts actually use the middle equality above to define a symmetric matrix.

Example 10.1.2.

$$2x^2 + 2xy + 2y^2 = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Example 10.1.3.

$$2x^2 + 2xy + 3xz - 2y^2 - z^2 = \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} 2 & 1 & 3/2 \\ 1 & -2 & 0 \\ 3/2 & 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Proposition 10.1.4.

The values of a quadratic form on $\mathbb{R}^n - \{0\}$ is completely determined by it's values on the $(n-1)$-sphere $S_{n-1} = \{\vec{x} \in \mathbb{R}^n \mid ||\vec{x}|| = 1\}$. In particular, $Q(\vec{x}) = ||\vec{x}||^2 Q(\hat{x})$ where $\hat{x} = \frac{1}{||\vec{x}||}\vec{x}$.

Proof: Let $Q(\vec{x}) = \vec{x}^T A\vec{x}$. Notice that we can write any nonzero vector as the product of its magnitude $||x||$ and its direction $\hat{x} = \frac{1}{||\vec{x}||}\vec{x}$,

$$Q(\vec{x}) = Q(||\vec{x}||\hat{x}) = (||\vec{x}||\hat{x})^T A||\vec{x}||\hat{x} = ||\vec{x}||^2\hat{x}^T A\hat{x} = ||x||^2 Q(\hat{x}).$$

Therefore $Q(\vec{x})$ is simply proportional to $Q(\hat{x})$ with proportionality constant $||\vec{x}||^2$. $\square$

The proposition above is very interesting. It says that if we know how Q works on unit-vectors then we can extrapolate its action on the remainder of $\mathbb{R}^n$. If $f : S \to \mathbb{R}$ then we could say $f(S) > 0$ iff $f(s) > 0$ for all $s \in S$. Likewise, $f(S) < 0$ iff $f(s) < 0$ for all $s \in S$. The proposition below follows from the proposition above since $||\vec{x}||^2$ ranges over all nonzero positive real numbers in the equations above.

Proposition 10.1.5.

If Q is a quadratic form on $\mathbb{R}^n$ and we denote $\mathbb{R}^n_* = \mathbb{R}^n - \{0\}$

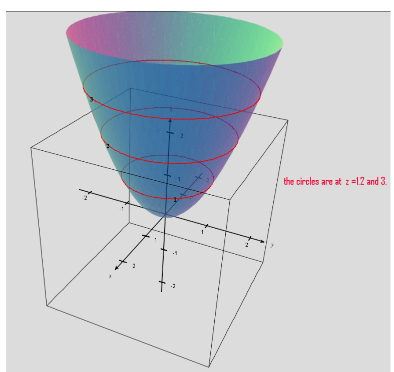
1.(negative definite) $Q(\mathbb{R}^n_*) < 0$ iff $Q(S_{n-1}) < 0$

2.(positive definite) $Q(\mathbb{R}^n_*) > 0$ iff $Q(S_{n-1}) > 0$

3.(non-definite) $Q(\mathbb{R}^n_*) = \mathbb{R} - \{0\}$ iff $Q(S_{n-1})$ has both positive and negative values.

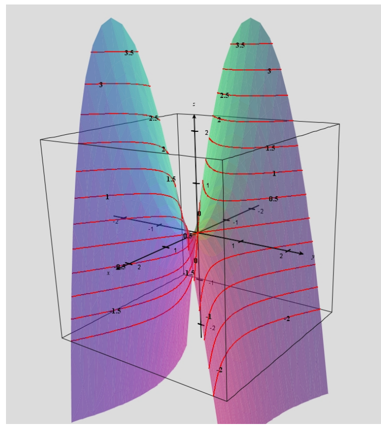
Before I get too carried away with the theory let's look at a couple examples.

Example 10.1.6. *Consider the quadric form $Q(x, y) = x^2 + y^2$. You can check for yourself that $z = Q(x, y)$ is a cone and Q has positive outputs for all inputs except $(0, 0)$. Notice that $Q(v) = ||v||^2$ so it is clear that $Q(S_1) = 1$. We find agreement with the preceding proposition. Next, think about the application of $Q(x, y)$ to level curves; $x^2 + y^2 = k$ is simply a circle of radius $\sqrt{k}$ or just the origin. Here's a graph of $z = Q(x, y)$:*



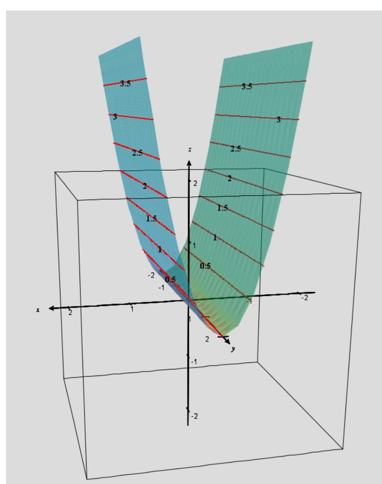
Notice that $Q(0, 0) = 0$ is the absolute minimum for Q. Finally, let's take a moment to write $Q(x, y) = [x, y]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ in this case the matrix is diagonal and we note that the e-values are $\lambda_1 = \lambda_2 = 1$.

Example 10.1.7. *Consider the quadric form $Q(x,y) = x^2 - 2y^2$. You can check for yourself that $z = Q(x,y)$ is a hyperboloid and Q has non-definite outputs since sometimes the x^2 term dominates whereas other points have $-2y^2$ as the dominent term. Notice that $Q(1,0) = 1$ whereas $Q(0,1) = -2$ hence we find $Q(S_1)$ contains both positive and negative values and consequently we find agreement with the preceding proposition. Next, think about the application of $Q(x,y)$ to level curves; $x^2 - 2y^2 = k$ yields either hyperbolas which open vertically ($k > 0$) or horizontally ($k < 0$) or a pair of lines $y = \pm\frac{x}{2}$ in the $k = 0$ case. Here's a graph of $z = Q(x,y)$:*



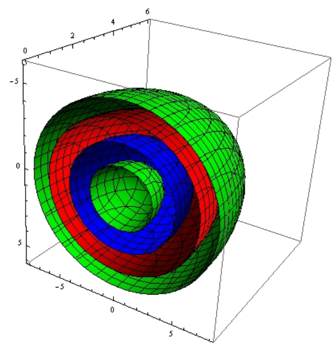
The origin is a **saddle point**. *Finally, let's take a moment to write* $Q(x,y) = [x,y]\begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ *in this case the matrix is diagonal and we note that the e-values are $\lambda_1 = 1$ and $\lambda_2 = -2$.*

Example 10.1.8. *Consider the quadric form $Q(x,y) = 3x^2$. You can check for yourself that $z = Q(x,y)$ is parabola-shaped trough along the y-axis. In this case Q has positive outputs for all inputs except $(0,y)$, we would call this form* **positive semi-definite**. *A short calculation reveals that $Q(S_1) = [0,3]$ thus we again find agreement with the preceding proposition (case 3). Next, think about the application of $Q(x,y)$ to level curves; $3x^2 = k$ is a pair of vertical lines: $x = \pm\sqrt{k/3}$ or just the y-axis. Here's a graph of $z = Q(x,y)$:*



Finally, let's take a moment to write $Q(x,y) = [x,y]\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ *in this case the matrix is diagonal and we note that the e-values are* $\lambda_1 = 3$ *and* $\lambda_2 = 0$.

Example 10.1.9. *Consider the quadric form* $Q(x,y,z) = x^2 + 2y^2 + 3z^2$. *Think about the application of* $Q(x,y,z)$ *to level surfaces;* $x^2 + 2y^2 + 3z^2 = k$ *is an ellipsoid. I can't graph a function of three variables, however, we can look at level surfaces of the function. I use Mathematica to plot several below:*



Finally, let's take a moment to write $Q(x,y,z) = [x,y,z]\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$ *in this case the matrix is diagonal and we note that the e-values are* $\lambda_1 = 1$ *and* $\lambda_2 = 2$ *and* $\lambda_3 = 3$.

The examples given thus far are the simplest cases. We don't really need linear algebra to understand them. In contrast, e-vectors and e-values will prove a useful tool to unravel the later examples.

Proposition 10.1.10.

If Q is a quadratic form on $\mathbb{R}^n$ with matrix A and e-values $\lambda_1, \lambda_2, \dots, \lambda_n$ with orthonormal e-vectors $v_1, v_2, \dots, v_n$ then

$$Q(v_i) = {\lambda_i}^2$$

for $i = 1, 2, \dots, n$. Moreover, if $P = [v_1|v_2|\cdots|v_n]$ then

$$Q(\vec{x}) = (P^T\vec{x})^T P^T A P P^T \vec{x} = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_n y_n^2$$

where we defined $\vec{y} = P^T\vec{x}$.

Let me restate the proposition above in simple terms: we can transform a given quadratic form to a diagonal form by finding orthonormalized e-vectors and performing the appropriate coordinate transformation. Since P is formed from orthonormal e-vectors we know that P will be either a rotation or reflection. This proposition says we can remove "cross-terms" by transforming the quadratic forms with an appropriate rotation.

Example 10.1.11. *Consider the quadric form $Q(x,y) = 2x^2 + 2xy + 2y^2$. It's not immediately obvious (to me) what the level curves $Q(x,y) = k$ look like. We'll make use of the preceding proposition to understand those graphs. Notice $Q(x,y) = [x,y]\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. Denote the matrix of the form by A and calculate the e-values/vectors:*

$$det(A - \lambda I) = det\begin{bmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{bmatrix} = (\lambda-2)^2 - 1 = \lambda^2 - 4\lambda + 3 = (\lambda-1)(\lambda-3) = 0$$

Therefore, the e-values are $\lambda_1 = 1$ and $\lambda_2 = 3$.

$$(A - I)\vec{u}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

I just solved $u + v = 0$ to give $v = -u$ choose $u = 1$ then normalize to get the vector above. Next,

$$(A - 3I)\vec{u}_2 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

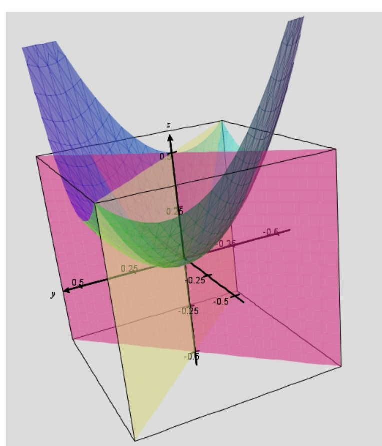
I just solved $u - v = 0$ to give $v = u$ choose $u = 1$ then normalize to get the vector above. Let $P = [\vec{u}_1|\vec{u}_2]$ and introduce new coordinates $\vec{y} = (\bar{x}, \bar{y})$ defined by $\vec{y} = P^T\vec{x}$. Note these can be inverted by multiplication by P to give $\vec{x} = P\vec{y}$. Observe that

$$P = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{ll} x & = \frac{1}{2}(\bar{x} + \bar{y}) \\ y & = \frac{1}{2}(-\bar{x} + \bar{y}) \end{array} \quad or \quad \begin{array}{ll} \bar{x} & = \frac{1}{2}(x - y) \\ \bar{y} & = \frac{1}{2}(x + y) \end{array}$$

The proposition preceding this example shows that substitution of the formulas above into Q yield[1]:

$$\tilde{Q}(\bar{x}, \bar{y}) = \bar{x}^2 + 3\bar{y}^2$$

It is clear that in the barred coordinate system the level curve $Q(x,y) = k$ is an ellipse. If we draw the barred coordinate system superposed over the xy-coordinate system then you'll see that the graph of $Q(x,y) = 2x^2 + 2xy + 2y^2 = k$ is an ellipse rotated by 45 degrees. Or, if you like, we can plot $z = Q(x,y)$:



[1]technically $\tilde{Q}(\bar{x}, \bar{y})$ is $Q(x(\bar{x}, \bar{y}), y(\bar{x}, \bar{y}))$

Example 10.1.12. *Consider the quadric form $Q(x,y) = x^2+2xy+y^2$. It's not immediately obvious (to me) what the level curves $Q(x,y)=k$ look like. We'll make use of the preceding proposition to understand those graphs. Notice $Q(x,y) = [x,y]\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. Denote the matrix of the form by A and calculate the e-values/vectors:*

$$det(A-\lambda I) = det\begin{bmatrix} 1-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = (\lambda-1)^2 - 1 = \lambda^2 - 2\lambda = \lambda(\lambda-2) = 0$$

Therefore, the e-values are $\lambda_1 = 0$ and $\lambda_2 = 2$.

$$(A-0)\vec{u}_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

I just solved $u+v=0$ to give $v=-u$ choose $u=1$ then normalize to get the vector above. Next,

$$(A-2I)\vec{u}_2 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

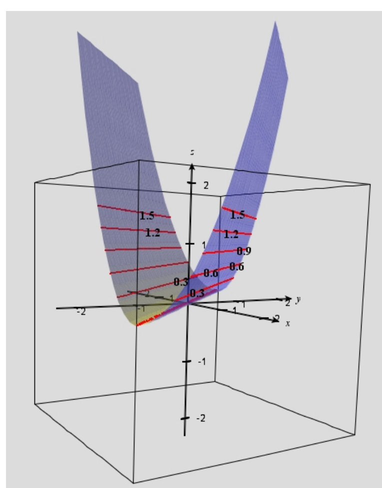
I just solved $u-v=0$ to give $v=u$ choose $u=1$ then normalize to get the vector above. Let $P = [\vec{u}_1|\vec{u}_2]$ and introduce new coordinates $\vec{y} = (\bar{x},\bar{y})$ defined by $\vec{y} = P^T\vec{x}$. Note these can be inverted by multiplication by P to give $\vec{x} = P\vec{y}$. Observe that

$$P = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{ll} x & = \frac{1}{2}(\bar{x}+\bar{y}) \\ y & = \frac{1}{2}(-\bar{x}+\bar{y}) \end{array} \quad or \quad \begin{array}{ll} \bar{x} & = \frac{1}{2}(x-y) \\ \bar{y} & = \frac{1}{2}(x+y) \end{array}$$

The proposition preceding this example shows that substitution of the formulas above into Q yield:

$$\tilde{Q}(\bar{x},\bar{y}) = 2\bar{y}^2$$

It is clear that in the barred coordinate system the level curve $Q(x,y)=k$ is a pair of paralell lines. If we draw the barred coordinate system superposed over the xy-coordinate system then you'll see that the graph of $Q(x,y) = x^2+2xy+y^2 = k$ is a line with slope -1. Indeed, with a little algebraic insight we could have anticipated this result since $Q(x,y) = (x+y)^2$ so $Q(x,y)=k$ implies $x+y=\sqrt{k}$ thus $y=\sqrt{k}-x$. Here's a plot which again verifies what we've already found:



Example 10.1.13. *Consider the quadric form $Q(x,y) = 4xy$. It's not immediately obvious (to me) what the level curves $Q(x,y) = k$ look like. We'll make use of the preceding proposition to understand those graphs. Notice $Q(x,y) = [x,y]\begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$. Denote the matrix of the form by A and calculate the e-values/vectors:*

$$det(A - \lambda I) = det\begin{bmatrix} -\lambda & 2 \\ 2 & -\lambda \end{bmatrix} = \lambda^2 - 4 = (\lambda+2)(\lambda-2) = 0$$

Therefore, the e-values are $\lambda_1 = -2$ and $\lambda_2 = 2$.

$$(A+2I)\vec{u}_1 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

I just solved $u + v = 0$ to give $v = -u$ choose $u = 1$ then normalize to get the vector above. Next,

$$(A-2I)\vec{u}_2 = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Rightarrow \quad \vec{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

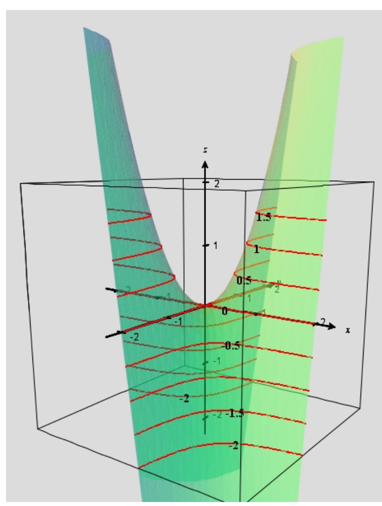
I just solved $u - v = 0$ to give $v = u$ choose $u = 1$ then normalize to get the vector above. Let $P = [\vec{u}_1|\vec{u}_2]$ and introduce new coordinates $\vec{y} = (\bar{x},\bar{y})$ defined by $\vec{y} = P^T\vec{x}$. Note these can be inverted by multiplication by P to give $\vec{x} = P\vec{y}$. Observe that

$$P = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \quad \Rightarrow \quad \begin{array}{ll} x & = \frac{1}{2}(\bar{x}+\bar{y}) \\ y & = \frac{1}{2}(-\bar{x}+\bar{y}) \end{array} \quad or \quad \begin{array}{ll} \bar{x} & = \frac{1}{2}(x-y) \\ \bar{y} & = \frac{1}{2}(x+y) \end{array}$$

The proposition preceding this example shows that substitution of the formulas above into Q yield:

$$\tilde{Q}(\bar{x},\bar{y}) = -2\bar{x}^2 + 2\bar{y}^2$$

It is clear that in the barred coordinate system the level curve $Q(x,y) = k$ is a hyperbola. If we draw the barred coordinate system superposed over the xy-coordinate system then you'll see that the graph of $Q(x,y) = 4xy = k$ is a hyperbola rotated by 45 degrees. The graph $z = 4xy$ is thus a hyperbolic paraboloid:



The fascinating thing about the mathematics here is that if you don't want to graph $z = Q(x, y)$, but you do want to know the general shape then you can determine which type of quadraic surface you're dealing with by simply calculating the eigenvalues of the form.

Remark 10.1.14.

I made the preceding triple of examples all involved the same rotation. This is purely for my lecturing convenience. In practice the rotation could be by all sorts of angles. In addition, you might notice that a different ordering of the e-values would result in a redefinition of the barred coordinates. [2]

We ought to do at least one 3-dimensional example.

Example 10.1.15. *Consider the quadric form Q defined below:*

$$Q(x, y, z) = [x, y, z] \begin{bmatrix} 6 & -2 & 0 \\ -2 & 6 & 0 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Denote the matrix of the form by A and calculate the e-values/vectors:

$$\begin{aligned} det(A - \lambda I) &= det \begin{bmatrix} 6-\lambda & -2 & 0 \\ -2 & 6-\lambda & 0 \\ 0 & 0 & 5-\lambda \end{bmatrix} \\ &= [(\lambda - 6)^2 - 4](5 - \lambda) \\ &= (5 - \lambda)[\lambda^2 - 12\lambda + 32](5 - \lambda) \\ &= (\lambda - 4)(\lambda - 8)(5 - \lambda) \end{aligned}$$

Therefore, the e-values are $\lambda_1 = 4$, $\lambda_2 = 8$ and $\lambda_3 = 5$. After some calculation we find the following orthonormal e-vectors for A:

$$\vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \vec{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad \vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

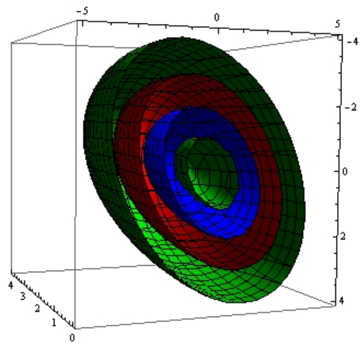
Let $P = [\vec{u}_1|\vec{u}_2|\vec{u}_3]$ and introduce new coordinates $\vec{y} = (\bar{x}, \bar{y}, \bar{z})$ defined by $\vec{y} = P^T\vec{x}$. Note these can be inverted by multiplication by P to give $\vec{x} = P\vec{y}$. Observe that

$$P = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & \sqrt{2} \end{bmatrix} \Rightarrow \begin{array}{rcl} x &=& \frac{1}{2}(\bar{x} + \bar{y}) \\ y &=& \frac{1}{2}(-\bar{x} + \bar{y}) \\ z &=& \bar{z} \end{array} \quad or \quad \begin{array}{rcl} \bar{x} &=& \frac{1}{2}(x - y) \\ \bar{y} &=& \frac{1}{2}(x + y) \\ \bar{z} &=& z \end{array}$$

The proposition preceding this example shows that substitution of the formulas above into Q yield:

$$\tilde{Q}(\bar{x}, \bar{y}, \bar{z}) = 4\bar{x}^2 + 8\bar{y}^2 + 5\bar{z}^2$$

It is clear that in the barred coordinate system the level surface $Q(x, y, z) = k$ is an ellipsoid. If we draw the barred coordinate system superposed over the xyz-coordinate system then you'll see that the graph of $Q(x, y, z) = k$ is an ellipsoid rotated by 45 degrees around the $z - axis$. Plotted below are a few representative ellipsoids:



Remark 10.1.16.

If you would like to read more about conic sections or quadric surfaces and their connection to e-values/vectors I reccommend sections 9.6 and 9.7 of Anton's 9th. ed. text. I have yet to add examples on how to include translations in the analysis. It's not much more trouble but I decided it would just be an unecessary complication this semester. Also, section 7.1,7.2 and 7.3 in Lay's text show a bit more about how to use this math to solve concrete applied problems. You might also take a look in Strang's text, his discussion of tests for positive-definite matrices is much more complete than I will give here.

10.1.2 summary of quadratic form analysis

There is a connection between the shape of level curves $Q(x_1, x_2, \dots, x_n) = k$ and the graph $x_{n+1} = f(x_1, x_2, \dots, x_n)$ of f. I'll discuss $n = 2$ but these comments equally well apply to $w = f(x, y, z)$ or higher dimensional examples. Consider a critical point (a, b) for $f(x, y)$ then the Taylor expansion about (a, b) has the form

$$f(a + h, b + k) = f(a, b) + Q(h, k)$$

where $Q(h, k) = \frac{1}{2}h^2 f_{xx}(a, b) + hkf_{xy}(a, b) + \frac{1}{2}h^2 f_{yy}(a, b) = [h, k][Q](h, k)$. Since $[Q]^T = [Q]$ we can find orthonormal e-vectors $\vec{u}_1, \vec{u}_2$ for $[Q]$ with e-values λ_1 and λ_2 respective. Using $U = [\vec{u}_1|\vec{u}_2]$ we can introduce rotated coordinates $(\bar{h}, \bar{k}) = U(h, k)$. These will give

$$Q(\bar{h}, \bar{k}) = \lambda_1 \bar{h}^2 + \lambda_2 \bar{k}^2$$

Clearly if $\lambda_1 > 0$ and $\lambda_2 > 0$ then $f(a, b)$ yields the local minimum whereas if $\lambda_1 < 0$ and $\lambda_2 < 0$ then $f(a, b)$ yields the local maximum. Edwards discusses these matters on pgs. 148-153. In short, supposing $f \approx f(p) + Q$, if all the e-values of Q are positive then f has a local minimum of $f(p)$ at p whereas if all the e-values of Q are negative then f reaches a local maximum of $f(p)$ at p.

Otherwise Q has both positive and negative e-values and we say Q is non-definite and the function has a saddle point. If all the e-values of Q are positive then Q is said to be **positive-definite** whereas if all the e-values of Q are negative then Q is said to be **negative-definite**. Edwards gives a few nice tests for ascertaining if a matrix is positive definite without explicit computation of e-values. Finally, if one of the e-values is zero then the graph will be like a trough.

Remark 10.1.17. *summary of the summary.*

In short, the behaviour of a quadratic form $Q(x) = x^T Ax$ is governed by it's spectrum $\{\lambda_1, \lambda_2, \dots, \lambda_k\}$. Moreover, the form can be written as $Q(y) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \cdots + \lambda_k y_k^2$ by choosing the coordinate system which is built from the orthonormal eigenbasis of $col(A)$. In this coordinate system questions of optimization become trivial (see section 7.3 of Lay for applied problems)

10.2 Taylor series for functions of two or more variables

It turns out that linear algebra and e-vectors can give us great insight into locating local extrema for a function of several variables. To summarize, we can calculate the multivariate Taylor series and we'll find that the quadratic terms correspond to a *quadratic form*. In fact, each quadratic form has a symmetric matrix representative. We know that symmetric matrices are diagonalizable hence the e-values of a symmetric matrix will be real. Moreover, the eigenvalues tell you what the min/max value of the function is at a critical point (usually). This is the n-dimensional generalization of the 2nd-derivative test from calculus. I'll only study the $n = 2$ and $n = 3$ case in this course. If you'd like to see these claims explained in more depth feel free ask me about offering Advanced Calculus.

Our goal here is to find an analogue for Taylor's Theorem for function from $\mathbb{R}^n$ to $\mathbb{R}$. Recall that if $g : U \subseteq \mathbb{R} \to \mathbb{R}$ is *smooth* at $a \in \mathbb{R}$ then we can compute as many derivatives as we wish, moreover we can generate the Taylor's series for g centered at a:

$$g(a+h) = g(a) + g'(a)h + \frac{1}{2}g''(a)h^2 + \frac{1}{3!}g''(a)h^3 + \cdots = \sum_{n=0}^{\infty} \frac{g^{(n)}(a)}{n!}h^n$$

The equation above assumes that g is analytic at a. In other words, the function actually matches it's Taylor series near a. This concept can be made rigorous by discussing the remainder. If one can show the remainder goes to zero then that proves the function is analytic. (*read p117-127 of Edwards for more on these concepts, I did cover some of that in class this semester, Theorem 6.3 is particularly interesting*).

10.2.1 deriving the two-dimensional Taylor formula

The idea is fairly simple: create a function on $\mathbb{R}$ with which we can apply the ordinary Taylor series result. Much like our discussion of directional derivatives we compose a function of two variables with linear path in the domain. Let $f : U \subseteq \mathbb{R}^2 \to \mathbb{R}$ be smooth with smooth partial derivatives of all orders. Furthermore, let $(a, b) \in U$ and construct a line through (a, b) with direction vector (h_1, h_2) as usual:

$$\phi(t) = (a, b) + t(h_1, h_2) = (a + th_1, b + th_2)$$

for $t \in \mathbb{R}$. Note $\phi(0) = (a, b)$ and $\phi'(t) = (h_1, h_2) = \phi'(0)$. Construct $g = f \circ \phi : \mathbb{R} \to \mathbb{R}$ and differentiate, note we use the chain rule for functions of several variables in what follows:

$$\begin{aligned} g'(t) = (f \circ \phi)'(t) &= f'(\phi(t))\phi'(t) \\ &= \nabla f(\phi(t)) \cdot (h_1, h_2) \\ &= h_1 f_x(a + th_1, b + th_2) + h_2 f_y(a + th_1, b + th_2) \end{aligned}$$

Note $g'(0) = h_1 f_x(a, b) + h_2 f_y(a, b)$. Differentiate again (I omit $(\phi(t))$ dependence in the last steps),

$$\begin{aligned} g''(t) &= h_1 f_x'(a + th_1, b + th_2) + h_2 f_y'(a + th_1, b + th_2) \\ &= h_1 \nabla f_x(\phi(t)) \cdot (h_1, h_2) + h_2 \nabla f_y(\phi(t)) \cdot (h_1, h_2) \\ &= h_1^2 f_{xx} + h_1 h_2 f_{yx} + h_2 h_1 f_{xy} + h_2^2 f_{yy} \\ &= h_1^2 f_{xx} + 2h_1 h_2 f_{xy} + h_2^2 f_{yy} \end{aligned}$$

Thus, making explicit the point dependence, $g''(0) = h_1^2 f_{xx}(a, b) + 2h_1 h_2 f_{xy}(a, b) + h_2^2 f_{yy}(a, b)$. We may construct the Taylor series for g up to quadratic terms:

$$\begin{aligned} g(0 + t) &= g(0) + tg'(0) + \frac{1}{2} g''(0) + \cdots \\ &= f(a, b) + t[h_1 f_x(a, b) + h_2 f_y(a, b)] + \frac{t^2}{2}\left[h_1^2 f_{xx}(a, b) + 2h_1 h_2 f_{xy}(a, b) + h_2^2 f_{yy}(a, b)\right] + \cdots \end{aligned}$$

Note that $g(t) = f(a + th_1, b + th_2)$ hence $g(1) = f(a + h_1, b + h_2)$ and consequently,

$$\begin{aligned} f(a + h_1, b + h_2) = f(a, b) &+ h_1 f_x(a, b) + h_2 f_y(a, b) + \\ &+ \frac{1}{2}\left[h_1^2 f_{xx}(a, b) + 2h_1 h_2 f_{xy}(a, b) + h_2^2 f_{yy}(a, b)\right] + \cdots \end{aligned}$$

Omitting point dependence on the 2^{nd} derivatives,

$$\boxed{f(a + h_1, b + h_2) = f(a, b) + h_1 f_x(a, b) + h_2 f_y(a, b) + \tfrac{1}{2}\left[h_1^2 f_{xx} + 2h_1 h_2 f_{xy} + h_2^2 f_{yy}\right] + \cdots}$$

Sometimes we'd rather have an expansion about (x, y). To obtain that formula simply substitute $x - a = h_1$ and $y - b = h_2$. Note that the point (a, b) is fixed in this discussion so the derivatives are not modified in this substitution,

$$\begin{aligned} f(x, y) = f(a, b) &+ (x - a) f_x(a, b) + (y - b) f_y(a, b) + \\ &+ \frac{1}{2}\left[(x - a)^2 f_{xx}(a, b) + 2(x - a)(y - b) f_{xy}(a, b) + (y - b)^2 f_{yy}(a, b)\right] + \cdots \end{aligned}$$

At this point we ought to recognize the first three terms give the tangent plane to $z = f(z, y)$ at $(a, b, f(a, b))$. The higher order terms are nonlinear corrections to the linearization, these quadratic terms form a *quadratic form*. If we computed third, fourth or higher order terms we'd find that, using $a = a_1$ and $b = a_2$ as well as $x = x_1$ and $y = x_2$,

$$\boxed{f(x,y) = \sum_{n=0}^{\infty}\sum_{i_1=0}^{n}\sum_{i_2=0}^{n}\cdots\sum_{i_n=0}^{n}\frac{1}{n!}\frac{\partial^{(n)}f(a_1,a_2)}{\partial x_{i_1}\partial x_{i_2}\cdots\partial x_{i_n}}(x_{i_1}-a_{i_1})(x_{i_2}-a_{i_2})\cdots(x_{i_n}-a_{i_n})}$$

The multivariate Taylor formula for a function of j-variables for $j > 2$ is very similar. Rather than even state the formula I will show a few examples in the subsection that follows.

10.2.2 examples

Example 10.2.1. *Suppose $f(x, y) = exp(-x^2 - y^2 + 2y - 1)$ expand f about the point $(0, 1)$:*

$$f(x,y) = exp(-x^2)exp(-y^2 + 2y - 1) = exp(-x^2)exp(-(y-1)^2)$$

expanding,

$$f(x,y) = (1 - x^2 + \cdots)(1 - (y-1)^2 + \cdots) = 1 - x^2 - (y-1)^2 + \cdots$$

Recenter about the point $(0, 1)$ by setting $x = h$ and $y = 1 + k$ so

$$f(h, 1+k) = 1 - h^2 - k^2 + \cdots$$

If (h, k) is near $(0, 0)$ then the dominant terms are simply those we've written above hence the graph is like that of a quadraic surface with a pair of negative e-values. It follows that $f(0, 1)$ is a local maximum. In fact, it happens to be a global maximum for this function.

Example 10.2.2. *Suppose $f(x, y) = 4-(x-1)^2+(y-2)^2+Aexp(-(x-1)^2-(y-2)^2)+2B(x-1)(y-2)$ for some constants A, B. Analyze what values for A, B will make $(1, 2)$ a local maximum, minimum or neither. Expanding about $(1, 2)$ we set $x = 1 + h$ and $y = 2 + k$ in order to see clearly the local behaviour of f at $(1, 2)$,*

$$\begin{aligned} f(1+h, 2+k) &= 4 - h^2 - k^2 + Aexp(-h^2 - k^2) + 2Bhk \\ &= 4 - h^2 - k^2 + A(1 - h^2 - k^2) + 2Bhk\cdots \\ &= 4 + A - (A+1)h^2 + 2Bhk - (A+1)k^2 + \cdots \end{aligned}$$

There is no nonzero linear term in the expansion at $(1, 2)$ which indicates that $f(1, 2) = 4 + A$ may be a local extremum. In this case the quadratic terms are nontrivial which means the graph of this function is well-approximated by a quadraic surface near $(1, 2)$. The quadratic form $Q(h, k) = -(A+1)h^2 + 2Bhk - (A+1)k^2$ has matrix

$$[Q] = \begin{bmatrix} -(A+1) & B \\ B & -(A+1)^2 \end{bmatrix}.$$

The characteristic equation for Q is

$$det([Q]-\lambda I) = det\begin{bmatrix} -(A+1)-\lambda & B \\ B & -(A+1)^2-\lambda \end{bmatrix} = (\lambda+A+1)^2 - B^2 = 0$$

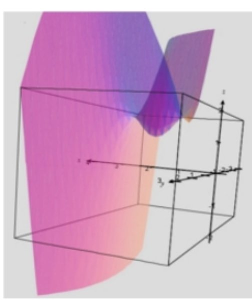
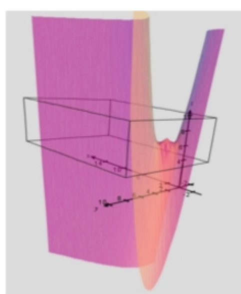
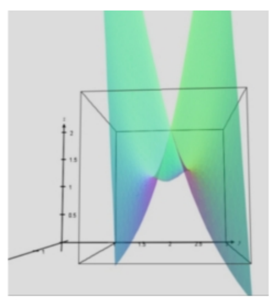
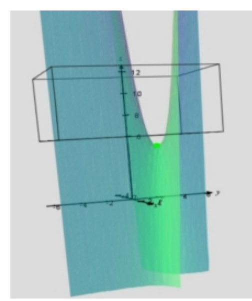
We find solutions $\lambda_1 = -A-1+B$ and $\lambda_2 = -A-1-B$. The possibilities break down as follows:

1. *if $\lambda_1, \lambda_2 > 0$ then $f(1,2)$ is local minimum.*
2. *if $\lambda_1, \lambda_2 < 0$ then $f(1,2)$ is local maximum.*
3. *if just one of λ_1, λ_2 is zero then f is constant along one direction and min/max along another so technically it is a local extremum.*
4. *if $\lambda_1\lambda_2 < 0$ then $f(1,2)$ is not a local etremum, however it is a saddle point.*

In particular, the following choices for A, B will match the choices above

1. *Let $A = -3$ and $B = 1$ so $\lambda_1 = 3$ and $\lambda_2 = 1$;*
2. *Let $A = 3$ and $B = 1$ so $\lambda_1 = -3$ and $\lambda_2 = -5$*
3. *Let $A = -3$ and $B = -2$ so $\lambda_1 = 0$ and $\lambda_2 = 4$*
4. *Let $A = 1$ and $B = 3$ so $\lambda_1 = 1$ and $\lambda_2 = -5$*

Here are the graphs of the cases above, note the analysis for case 3 is more subtle for Taylor approximations as opposed to simple quadraic surfaces. In this example, case 3 was also a local minimum. In contrast, in Example 10.1.12 the graph was like a trough. The behaviour of f away from the critical point includes higher order terms whose influence turns the trough into a local minimum.

Example 10.2.3. *Suppose $f(x,y) = \sin(x)\cos(y)$ to find the Taylor series centered at $(0,0)$ we can simply multiply the one-dimensional result $\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots$ and $\cos(y) = 1 - \frac{1}{2!}y^2 + \frac{1}{4!}y^4 + \cdots$ as follows:*

$$\begin{aligned} f(x,y) &= (x - \tfrac{1}{3!}x^3 + \tfrac{1}{5!}x^5 + \cdots)(1 - \tfrac{1}{2!}y^2 + \tfrac{1}{4!}y^4 + \cdots) \\ &= x - \tfrac{1}{2}xy^2 + \tfrac{1}{24}xy^4 - \tfrac{1}{6}x^3 - \tfrac{1}{12}x^3y^2 + \cdots \\ &= x + \cdots \end{aligned}$$

The origin $(0,0)$ is a critical point since $f_x(0,0) = 0$ and $f_y(0,0) = 0$, however, this particular critical point escapes the analysis via the quadratic form term since $Q = 0$ in the Taylor series for this function at $(0,0)$. This is analogous to the inconclusive case of the 2nd derivative test in calculus III.

Example 10.2.4. *Suppose $f(x,y,z) = xyz$. Calculate the multivariate Taylor expansion about the point $(1,2,3)$. I'll actually calculate this one via differentiation, I have used tricks and/or calculus II results to shortcut any differentiation in the previous examples. Calculate first derivatives*

$$f_x = yz \qquad f_y = xz \qquad f_z = xy,$$

and second derivatives,

$$\begin{array}{ccc} f_{xx} = 0 & f_{xy} = z & f_{xz} = y \\ f_{yx} = z & f_{yy} = 0 & f_{yz} = x \\ f_{zx} = y & f_{zy} = x & f_{zz} = 0, \end{array}$$

and the nonzero third derivatives,

$$f_{xyz} = f_{yzx} = f_{zxy} = f_{zyx} = f_{yxz} = f_{xzy} = 1.$$

It follows,

$$\begin{aligned} &f(a+h, b+k, c+l) = \\ &\quad = f(a,b,c) \ + \ f_x(a,b,c)h \ + \ f_y(a,b,c)k \ + \ f_z(a,b,c)l \ + \\ &\qquad \tfrac{1}{2}(\ f_{xx}hh + f_{xy}hk + f_{xz}hl + f_{yx}kh + f_{yy}kk + f_{yz}kl + f_{zx}lh + f_{zy}lk + f_{zz}ll\ ) + \cdots \end{aligned}$$

Of course certain terms can be combined since $f_{xy} = f_{yx}$ etc... for smooth functions (we assume smooth in this section, moreover the given function here is clearly smooth). In total,

$$f(1+h, 2+k, 3+l) = 6 + 6h + 3k + 2l + \frac{1}{2}\big(3hk + 2hl + 3kh + kl + 2lh + lk\big) + \frac{1}{3!}(6)hkl$$

Of course, we could also obtain this from simple algebra:

$$f(1+h, 2+k, 3+l) = (1+h)(2+k)(3+l) = 6 + 6h + 3k + l + 3hk + 2hl + kl + hkl.$$

Remark 10.2.5.

One very interesting application of the orthogonal complement theorem is to the method of Lagrange multipliers. The problem is to maximize an objective function $f(x_1, x_2, \dots, x_n)$ with respect to a set of constraint functions $g_1(x_1, x_2, \dots, x_n) = 0$, $g_2(x_1, x_2, \dots, x_n) = 0$ and $g_k(x_1, x_2, \dots, x_n) = 0$. One can argue that extreme values for f must satisfy

$$\nabla f = \lambda_1 \nabla g_1 + \lambda_2 \nabla g_2 + \cdots + \lambda_k \nabla g_k$$

for a particular set of Lagrange multipliers $\lambda_1, \lambda_2, \dots, \lambda_k$. The crucial step in the analysis relies on the orthogonal decomposition theorem. It is the fact that forces the gradient of the objective function to reside in the span of the gradients of the constraints. See my Advanced Calculus 2010 notes Chapter 8, or consult many advanced calculus texts.

10.3 intertia tensor, an application of quadratic forms

We can use quadratic forms to elegantly state a number of interesting quantities in classical mechanics. For example, the translational kinetic energy of a mass m with velocity v is

$$T_{trans}(v) = \frac{m}{2} v^T v = [v_1, v_2, v_3] \begin{bmatrix} m/2 & 0 & 0 \\ 0 & m/2 & 0 \\ 0 & 0 & m/2 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}.$$

On the other hand, the rotational kinetic energy of an object with moment of intertia I and angular velocity ω with respect to a particular axis of rotation is

$$T_{rot}(v) = \frac{I}{2} \omega^T \omega.$$

In addition you might recall that the force F applied at radial arm r gave rise to a torque of $\tau = r \times F$ which made the angular momentum $L = I\omega$ have the time-rate of change $\tau = \frac{dL}{dt}$. In the first semester of physics this is primarily all we discuss. We are usually careful to limit the discussion to rotations which happen to occur with respect to a particular axis. But, what about other rotations? What about rotations with respect to less natural axes of rotation? How should we describe the rotational physics of a rigid body which spins around some axis which doesn't happen to line up with one of the nice examples you find in an introductory physics text?

The answer is found in extending the idea of the moment of intertia to what is called the inertia tensor I_{ij} (in this section I is not the identity). To begin I'll provide a calculation which motivates the definition for the inertia tensor.

Consider a rigid mass with density $\rho = dm/dV$ which is a function of position $r = (x_1, x_2, x_3)$. Suppose the body rotates with angular velocity ω about some axis through the origin, however it is otherwise not in motion. This means all of the energy is rotational. Suppose that dm is at r then we define $v = (\dot{x_1}, \dot{x_2}, \dot{x_3}) = dr/dt$. In this context, the velocity v of dm is also given by the cross-product with the angular velocity; $v = \omega \times r$. Using the einstein repeated summation notation the k-th component of the cross-product is nicely expressed via the Levi-Civita symbol; $(\omega \times r)_k = \epsilon_{klm}\omega_l x_m$. Therefore, $v_k = \epsilon_{klm}\omega_l x_m$. The infinitesimal kinetic energy due to this little bit of rotating mass dm is hence

$$\begin{aligned}
dT &= \frac{dm}{2} v_k v_k \\
&= \frac{dm}{2}(\epsilon_{klm}\omega_l x_m)(\epsilon_{kij}\omega_i x_j) \\
&= \frac{dm}{2}\epsilon_{klm}\epsilon_{kij}\omega_l\omega_i x_m x_j \\
&= \frac{dm}{2}(\delta_{li}\delta_{mj} - \delta_{lj}\delta_{mi})\omega_l\omega_i x_m x_j \\
&= \frac{dm}{2}(\delta_{li}\delta_{mj}\omega_l\omega_i x_m x_j - \delta_{lj}\delta_{mi}\omega_l\omega_i x_m x_j) \\
&= \omega_l \frac{dm}{2}(\delta_{li}\delta_{mj} x_m x_j - \delta_{lj}\delta_{mi} x_m x_j)\omega_i \\
&= \omega_l \left[\frac{dm}{2}(\delta_{li}||r||^2 - x_l x_i)\right]\omega_i.
\end{aligned}$$

Integrating over the mass, if we add up all the little bits of kinetic energy we obtain the total kinetic energy for this rotating body: we replace dm with $\rho(r)dV$ and the integration is over the volume of the body,

$$T = \int \omega_l \left[\frac{1}{2}(\delta_{li}||r||^2 - x_l x_i)\right]\omega_i \rho(r) dV$$

However, the body is rigid so the angular velocity is the same for each dm and we can pull the components of the angular velocity out of the integration[3] to give:

$$T = \frac{1}{2}\omega_j \underbrace{\left[\int (\delta_{jk}||r||^2 - x_j x_k)\rho(r) dV\right]}_{I_{jk}} \omega_k$$

This integral defines the intertia tensor I_{jk} for the rotating body. Given the inertia tensor I_{lk} the kinetic energy is simply the value of the quadratic form below:

$$T(\omega) = \frac{1}{2}\omega^T \omega = [\omega_1, \omega_2, \omega_3] \begin{bmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{bmatrix} \begin{bmatrix} \omega_1 \\ \omega_2 \\ \omega_3 \end{bmatrix}.$$

The matrix above is not generally diagonal, however you can prove it is symmetric (easy). Therefore, we can find an orthonormal eigenbasis $\beta = \{u_1, u_2, u_3\}$ and if $P = [\beta]$ then it follows by orthonormality of the basis that $[I]_{\beta,\beta} = P^T[I]P$ is diagonal. The eigenvalues of the inertia tensor (the matrix $[I_{jk}]$) are called the **principle moments of inertia** and the eigenbasis $\beta = \{u_1, u_2, u_3\}$ define the **principle axes** of the body.

[3] I also relabled the indices to have nicer final formula, nothing profound here

The study of the rotational dynamics flows from analyzing the equations:

$$L_i = I_{ij}\omega_j \qquad \text{and} \qquad \tau_i = \frac{dL_i}{dt}$$

If the initial angular velocity is in the direction of a principle axis u_1 then the motion is basically described in the same way as in the introductory physics course provided that the torque is also in the direction of u_1. The moment of intertia is simply the first principle moment of inertia and $L = \lambda_1 \omega$. However, if the torque is not in the direction of a princple axis or the initial angular velocity is not along a principle axis then the motion is more complicated since the rotational motion is connected to more than one axis of rotation. Think about a spinning top which is spinning in place. There is wobbling and other more complicated motions that are covered by the mathematics described here.

Example 10.3.1. *The intertia tensor for a cube with one corner at the origin is found to be*

$$I = \frac{2}{3}Ms^2 \begin{bmatrix} 1 & -3/8 & -3/8 \\ -3/8 & 1 & -3/8 \\ -3/8 & -3/8 & 1 \end{bmatrix}$$

Introduce $m = M/8$ to remove the fractions,

$$I = \frac{2}{3}Ms^2 \begin{bmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{bmatrix}$$

You can calculate that the e-values are $\lambda_1 = 2$ and $\lambda_2 = 11 = \lambda_3$ with principle axis in the directions

$$u_1 = \frac{1}{\sqrt{3}}(1,1,1), \ u_2 = \frac{1}{\sqrt{2}}(-1,1,0), \ u_3 = \frac{1}{\sqrt{2}}(-1,0,1).$$

The choice of u_2, u_3 is not unique. We could just as well choose any other orthonormal basis for $span\{u_2, u_3\} = W_{11}$.

Finally, a word of warning, for a particular body there may be so much symmetry that no particular eigenbasis is specified. There may be many choices of an orthonormal eigenbasis for the system. Consider a sphere. Any orthonormal basis will give a set of principle axes. Or, for a right circular cylinder the axis of the cylinder is clearly a principle axis however the other two directions are arbitrarily chosen from the plane which is the orthogonal complement of the axis. I think it's fair to say that if a body has a unique (up to ordering) set of principle axes then the shape has to be somewhat ugly. Symmetry is beauty but it implies ambiguity for the choice of certain princple axes.

Chapter 11

Fourier analysis

Technically we are not going to engage in analysis in the proper sense in this work. However, the goal of this chapter is to briefly explain the more abstract concept of an inner product and then to show how the concept of least squares approximation generalizes to function space. In particular, sines and cosines are seen to be orthonormal with respect to an integration-based inner product. Then arbitrary functions can be broken into orthonormal components and reconstructed as sums of sines and cosine. To be fussy, we'd need to leave the realm of finite linear combinations and study infinite sums. Rather than completing this thought we content ourselves here to see examples of truncated Fourier series. What we do is analogous to looking at Taylor polynomials rather than Taylor series. In the Differential Equations course the concept of a Fourier series fits together with the technique of separation of variables and allows the solution of partial differential equations which are central to the study of mathematical physics; heat equations, wave equations and Laplace equations. More than simply solving a particular niche PDE, this Fourier technique and the questions it raised are largely responsible for much of the rigorization of analysis which occupied the minds of great nineteenth century mathematicians. Furthermore, Fourier analysis is a very important tool in the engineer's mathematical toolbox. It is not uncommon for students of advanced electrical engineering to take multiple courses on this topic.

11.1 inner products

We follow Chapter 6 of Anton & Rorres' *Elementary Linear Algebra* 9th. ed., this material is also § 7.5 of Spence, Insel & Friedberg's *Elementary Linear Algebra, a Matrix Approach*. The definition of an inner product is based on the idea of the dot product. Proposition 9.1.4 summarized the most important properties. We take these as the definition for an inner product. If you examine proofs in § 9.1 you'll notice most of what I argued was based on using these 4 simple facts for the dot-product.

WARNING: the next couple pages is dense. It's a reiteration of the main theoretical accomplishments of this chapter in the context of inner product spaces. If you need to see examples first then skip ahead as needed.

Definition 11.1.1.

Let V be a vector space over $\mathbb{R}$. If there is a function $< \, , \, >: V \times V \to \mathbb{R}$ such that for all $x, y, z \in V$ and $c \in \mathbb{R}$,

1. $< x, y > = < y, x >$ (**symmetric**),
2. $< x + y, z > = < x, z > + < y, z >$,
3. $< cx, y > = \ c < x, y >$,
4. $< x, x > \geq \ 0$ and $< x, x >= 0$ iff $x = 0$,

then we say $< \, , \, >$ is an **inner product** on V. In this case we say V with $< \ >$ is an inner product space. Items (1.), (2.) and (3.) together allow us to call $< \, , \, >$ a real-valued **symmetric-bilinear-form** on V. We may find it useful to use the notation $g(x, y) =< x, y >$ for some later arguments, one should keep in mind the notation $< \, , \, >$ is not the only choice.

Technically, items (2.) and (3.) give us "linearity in the first slot". To obtain bilinearity we need to have linearity in the second slot as well. This means $< x, y + z >=< x, y > + < x, z >$ and $< x, cy >= c < x, y >$ for all $x, y, z \in V$ and $c \in \mathbb{R}$. Fortunately, the symmetry property will transfer the linearity to the second slot. I leave that as an exercise for the reader.

Example 11.1.2. *Obviously $\mathbb{R}^n$ together with the dot-product forms an inner product space. Moreover, the dot-product is an inner product.*

Once we have an inner product for a vector space then we also have natural definitions for the length of a vector and the distance between two points.

Definition 11.1.3.

Let V be an inner product vector space with inner product $< \, , \, >$. The **norm** or **length** of a vector is defined by $||x|| = \sqrt{< x, x >}$ for each $x \in V$. Likewise the **distance** between $a, b \in V$ is defined by $d(a, b) = \sqrt{< b - a, b - a >} = ||b - a||$ for all $a, b \in V$. We say these are the length and distance functions induced by $< \, , \, >$. Likewise the **angle** between two nonzero vectors is defined implicitly by $< v, w >= ||v|| ||w|| \cos(\theta)$.

As before the definition above is only logical if certain properties hold for the inner product, norm and distance function. Happily we find all the same general properties for the inner product and its induced norm and distance function.

Proposition 11.1.4.

If V is an inner product space with induced norm $|| \cdot ||$ and $x, y \in V$ then $| < x, y > | \leq ||x|| \, ||y||$.

Proof: since $||x|| = \sqrt{<x,x>}$ the proof we gave for the case of the dot-product equally well applies here. You'll notice in retrospect I only used those 4 properties which we take as the defining axioms for the inner product. $\square$

In fact, all the propositions from §9.1 apply equally well to an arbitrary finite-dimensional inner product space. The proof of the proposition below is similar to those I gave in §9.1

Proposition 11.1.5. *Properties for induced norm and distance function on an inner product space.*

If V is an inner product space with inner product $<\,,\,>$ and norm $||x|| = \sqrt{x,x}$ and distance function $d(x,y) = ||y-x||$ then for all $x,y,z \in V$ and $c \in \mathbb{R}$

$(i.)\ \|\|x\|\| \geq 0$	$(v.)\ d(x,y) \geq 0$
$(ii.)\ \|\|x\|\| = 0 \Leftrightarrow x = 0$	$(vi.)\ d(x,y) = 0 \Leftrightarrow x = y$
$(iii.)\ \|\|cx\|\| = \|c\|\|\|x\|\|$	$(vii.)\ d(x,y) = d(y,x)$
$(iv.)\ \|\|x+y\|\| \leq \|\|x\|\| + \|\|y\|\|$	$(viii.)\ d(x,z) \leq d(x,y) + d(y,z)$

An norm is simply an operation which satisfies $(i.) - (iv.)$. If we are given a vector space with a norm then that is called a normed linear space. If in addition all Cauchy sequences converge in the space it is said to be a complete normed linear space. A **Banach Space** is defined to be a complete normed linear space. A distance function is simply an operation which satisfies $(v.) - (viii.)$. A set with a distance function is called a **metric space**. I'll let you ponder all these things in some other course, I mention them here merely for breadth. These topics are more interesting infinite-dimensional case.

What is truly interesting is that the orthogonal complement theorems and closest vector theory transfer over to the case of an inner product space.

Definition 11.1.6.

Let V be an inner product space with inner product $<\,,\,>$. Let $x, y \in V$ then we say x is **orthogonal** to y iff $<x,y>= 0$. A set S is said to be orthogonal iff every pair of vectors in S is orthogonal. If $W \leq V$ then the **orthogonal complement** of W is defined to be $W^\perp = \{v \in V \mid v \cdot w = 0\ \forall w \in W\}$.

Proposition 11.1.7. *Orthogonality results for inner product space.*

If V is an inner product space with inner product $<\,,\,>$ and norm $||x|| = \sqrt{x,x}$ then for all $x,y,z \in V$ and $W \leq V$,

$(i.)\ <x,y>= 0 \ \Rightarrow\ ||x+y||^2 = ||x||^2 + ||y||^2$
$(ii.)$ if $S \subset V$ is orthogonal $\Rightarrow$ S is linearly independent
$(iii.)\ S \subset V \ \Rightarrow\ S^\perp \leq V$
$(iv.)\ W^\perp \cap W = \{0\}$
$(v.)\ V = W \oplus W^\perp$

Definition 11.1.8.

Let V be an inner product space with inner product $< \, , \, >$. A basis of $< \, , \, >$-orthogonal vectors is an **orthogonal basis**. Likewise, if every vector in an orthogonal basis has length one then we call it an **orthonormal basis**.

Every finite dimensional inner product space permits a choice of an orthonormal basis. Examine my proof in the case of the dot-product. You'll find I made all arguments on the basis of the axioms for an inner-product. The Gram-Schmidt process works equally well for inner product spaces, we just need to exchange dot-products for inner-products as appropriate.

Proposition 11.1.9. *Orthonormal coordinates and projection results.*

If V is an inner product space with inner product $< \, , \, >$ and $\beta = \{v_1, v_2, \dots, v_k\}$ is a orthonormal basis for a subspace W then

$(i.)$ $w =< w, v_1 > v_1+ < w, v_2 > v_2 + \cdots + < w, v_k > v_k$ for each $w \in W$,
$(ii.)$ $Proj_W(x) \equiv< x, v_1 > v_1+ < x, v_2 > v_2 + \cdots + < x, v_k > v_k \in W$ for each $x \in V$,
$(iii.)$ $Orth_W(x) \equiv x - Proj_W(x) \in W^{\perp}$for each $x \in V$,
$(iv.)$ $x = Proj_W(x) + Orth_W(x)$ and $< Proj_W(x), Orth_W(x) >= 0$ for each $x \in V$,
$(v.)$ $||x - Proj_W(x)|| < ||x - y||$ for all $y \notin W$.

Notice that we can use the Gram-Schmidt idea to implement the least squares analysis in the context of an inner-product space. However, we cannot multiply abstract vectors by matrices so the short-cut normal equations may not make sense in this context. We have to implement the closest vector idea without the help of those normal equations. I'll demonstrate this idea in the Fourier analysis section.

11.1.1 examples of inner-products

The dot-product is just one of many inner products. We examine an assortment of other inner-products for various finite dimensional vector spaces.

Example 11.1.10. *Let* $V = \mathbb{R}^2$ *and define* $\langle v, w \rangle = v_1 w_1 + 3 v_2 w_2$ *for all* $v = (v_1, v_2)$, $w = (w_1, w_2) \in V$. *Let* $u, v, w \in V$ *and* $c \in \mathbb{R}$,

(1.) *symmetric property,*

$$\langle v, w \rangle = \ v_1 w_1 + 3 v_2 w_2 = w_1 v_1 + 3 w_2 v_2 = \langle w, v \rangle.$$

(2.) *additive property:*

$$\begin{aligned} \langle u + v, w \rangle &= (u+v)_1 w_1 + 3(u+v)_2 w_2 \\ &= (u_1 + v_1) w_1 + 3(u_2 + v_2) w_2 \\ &= u_1 w_1 + v_1 w_1 + 3 u_2 w_2 + 3 v_2 w_2 \\ &= \langle u, w \rangle + \langle v, w \rangle. \end{aligned}$$

(3.) *homogeneous property:*

$$\begin{aligned}\langle cv, w\rangle &= cv_1w_1 + 3cv_2w_2 \\ &= c(v_1w_1 + 3v_2w_2) \\ &= c\langle v, w\rangle\end{aligned}$$

(4.) *positive definite property:*

$$\langle v, v\rangle \;\; = v_1^2 + 3v_2^2 \geq 0 \text{ and } \langle v, v\rangle = 0 \;\Leftrightarrow\; v = 0.$$

Notice $e_1 = (1, 0)$ *is an orthonormalized vector with respect to* $\langle\ ,\ \rangle$ *but* $e_2 = (0, 1)$ *not unit-length. Instead,* $\langle e_2, e_2\rangle = 3$ *thus* $\|e_2\| = \sqrt{3}$ *so the unit-vector in the* e_2*-direction is* $u = \frac{1}{\sqrt{3}}(0, 1)$ *and with respect to* $\langle\ ,\ \rangle$ *we have an orthonormal basis* $\{e_1, u\}$.

Example 11.1.11. *Let* $V = \mathbb{R}^{m\times n}$ *we define the Frobenious inner-product as follows:*

$$< A, B >= \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}B_{ij}.$$

It is clear that $< A, A >\geq 0$ *since it is the sum of squares and it is also clear that* $< A, A >= 0$ *iff* $A = 0$*. Symmetry follows from the calculation*

$$< A, B >= \sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}B_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{n} B_{ij}A_{ij} =< B, A >$$

where we can commute B_{ij} *and* A_{ij} *for each pair* i, j *since the components are just real numbers. Linearity and homogeneity follow from:*

$$\begin{aligned}< \lambda A + B, C > &= \sum_{i=1}^{m}\sum_{j=1}^{n}(\lambda A + B)_{ij}C_{ij} = \sum_{i=1}^{m}\sum_{j=1}^{n}(\lambda A_{ij} + B_{ij})C_{ij} \\ &= \lambda\sum_{i=1}^{m}\sum_{j=1}^{n} A_{ij}C_{ij} + \sum_{i=1}^{m}\sum_{j=1}^{n} B_{ij}C_{ij} = \lambda < A, C > + < B, C >\end{aligned}$$

Therefore. the Frobenius inner-product is in fact an inner product. The Frobenious norm of a matrix is induced as usual:

$$||A|| = \sqrt{< A, A >}$$

as a consequence of the theory in this chapter we already know a few interesting properties form the matrix-norm, in particular $|| < A, B > || \leq ||A||||B||$*. The particular case of square matrices allows further comments. If* $A, B \in \mathbb{R}^{\,n\times n}$ *then notice*

$$< A, B >= \sum_{i,j} A_{ij}B_{ij} = \sum_{i}\sum_{j} A_ij(B^T)_{ji} = trace(AB^T) \quad\Rightarrow\quad ||A|| = trace(AA^T)$$

We find an interesting identity for any square matrix $|trace(AB^T)| \leq \sqrt{trace(AA^T)trace(BB^T)}$.

Example 11.1.12. *Let $C[a,b]$ denote the set of functions which are continuous on $[a,b]$. This is an infinite dimensional vector space. We can define an inner-product via the definite integral of the product of two functions: let $f,g \in C[a,b]$ define*

$$< f,g >= \int_a^b f(x)g(x)dx.$$

We can prove this is an inner-product. I'll just show additivity,

$$\begin{aligned} < f+g, h > &= \int_a^b (f(x)+g(x))(x)h(x)dx \\ &= \int_a^b f(x)h(x)dx + \int_a^b g(x)h(x)dx =< f,h > + < g,h > . \end{aligned}$$

I leave the proof of the other properties to the reader.

Example 11.1.13. *Consider the inner-product $< f,g >= \int_{-1}^1 f(x)g(x)dx$ for $f,g \in C[-1,1]$. Let's calculate the length squared of the standard basis:*

$$< 1,1 >= \int_{-1}^1 1 \cdot 1dx = 2, \qquad < x,x >= \int_{-1}^1 x^2dx = \frac{x^3}{3}\bigg|_{-1}^1 = \frac{2}{3}$$

$$< x^2,x^2 >= \int_{-1}^1 x^4dx = \frac{x^5}{5}\bigg|_{-1}^1 = \frac{2}{5}$$

Notice that the standard basis of P_2 are not all $< \ , \ >$-orthogonal:

$$< 1,x >= \int_{-1}^1 xdx = 0 \qquad < 1,x^2 >=< x,x >= \int_{-1}^1 x^2dx = \frac{2}{3} \qquad < x,x^2 >= \int_{-1}^1 x^3dx = 0$$

We can use the Gram-Schmidt process on $\{1,x,x^2\}$ to find an orthonormal basis for P_2 on $[-1,1]$. Let, $u_1(x) = 1$ and

$$\begin{aligned} u_2(x) &= x - \frac{< x,1 >}{< 1,1 >} = x \\ u_3(x) &= x^2 - \frac{< x^2,x >}{< x,x >}x - \frac{< x^2,1 >}{< 1,1 >} = x^2 - \frac{1}{3} \end{aligned}$$

We have an orthogonal set of functions $\{u_1,u_2,u_3\}$ we already calculated the length of u_1 and u_2 so we can immediately normalize those by dividing by their lengths; $v_1(x) = \frac{1}{\sqrt{2}}$ and $v_2(x) = \sqrt{\frac{3}{2}}x$. We need to calculate the length of u_3 so we can normalize it as well:

$$< u_3,u_3 >= \int_{-1}^1 \left(x^2 - \tfrac{1}{3}\right)^2 dx = \int_{-1}^1 \left(x^4 - \tfrac{2}{3}x^2 + \tfrac{1}{9}\right)dx = \tfrac{2}{5} - \tfrac{4}{9} + \tfrac{2}{9} = \tfrac{8}{45}$$

Thus $v_3(x) = \sqrt{\frac{8}{45}}\left(x^2 - \frac{1}{3}\right)$ *has length one. Therefore,* $\left\{\frac{1}{\sqrt{2}}, \sqrt{\frac{3}{2}}x, \sqrt{\frac{8}{45}}\left(x^2 - \frac{1}{3}\right)\right\}$ *is an orthonormal basis for* P_2 *restricted to* $[-1, 1]$*. Other intervals would not have the same basis. This construction depends both on our choice of inner-product and the interval considered. Incidentally, these are the first three* **Legendre Polynomials**. *These arise naturally as solutions to certain differential equations. The theory of* **orthogonal polynomials** *is full of such calculations. Orthogonal polynomials are quite useful as approximating functions. If we offered a second course in differential equations we could see the full function of such objects.*

Example 11.1.14. *Clearly* $f(x) = e^x \notin P_2$*. What is the least-squares approximation of* f*? Use the projection onto* P_2*:* $Proj_{\ P_2}(f) = < f, v_1 > v_1 + < f, v_2 > v_2 + < f, v_3 > v_3$*. We calculate,*

$$< f, v_1 >= \int_{-1}^{1} \tfrac{1}{\sqrt{2}} e^x dx = \tfrac{1}{\sqrt{2}}(e^1 - e^{-1}) \approxeq 1.661$$

$$< f, v_2 >= \int_{-1}^{1} \sqrt{\tfrac{3}{2}} x e^x dx = \sqrt{\tfrac{3}{2}}(xe^x - e^x)|_{-1}^{1} = \sqrt{\tfrac{3}{2}}[-(-e^{-1} - e^{-1})] = \sqrt{6}e^{-1} \approxeq 0.901$$

$$< f, v_3 >= \int_{-1}^{1} \sqrt{\tfrac{8}{45}}\left(x^2 - \tfrac{1}{3}\right) e^x dx = \tfrac{2e}{3} - \tfrac{14e^{-1}}{3} \approxeq 0.0402$$

Thus,

$$\begin{aligned} Proj_{\ P_2}(f)(x) &= 1.661 v_1(x) + 0.901 v_2(x) + 0.0402 v_3(x) \\ &= 1.03 + 1.103x + 0.017x^2 \end{aligned}$$

This is closest a quadratic can come to approximating the exponential function on the interval $[-1, 1]$*. What's the giant theoretical leap we made in this example? We wouldn't face the same leap if we tried to approximate* $f(x) = x^4$ *with* P_2*. What's the difference? Where does* e^x *live?*

Example 11.1.15. *Consider* $C[-\pi, \pi]$ *with inner product* $< f, g >= \int_{-\pi}^{\pi} f(x)g(x)dx$*. The set of sine and cosine functions* $\{1, \cos(x), \sin(x), \cos(2x), \sin(2x), \dots, \cos(kx), \sin(kx)\}$ *is an orthogonal set of functions.*

$$< \cos(mx), \cos(nx) >= \int_{-\pi}^{\pi} \cos(mx)\cos(nx)dx = \pi\delta_{mn}$$

$$< \sin(mx), \sin(nx) >= \int_{-\pi}^{\pi} \sin(mx)\sin(nx)dx = \pi\delta_{mn}$$

$$< \sin(mx), \cos(nx) >= \int_{-\pi}^{\pi} \sin(mx)\cos(nx)dx = 0$$

Thus we find the following is a set of <u>*orthonormal*</u> *functions*

$$\boxed{\beta_{trig} = \{\tfrac{1}{\sqrt{2\pi}},\ \tfrac{1}{\sqrt{\pi}}\cos(x),\ \tfrac{1}{\sqrt{\pi}}\sin(x),\ \tfrac{1}{\sqrt{\pi}}\cos(2x),\ \tfrac{1}{\sqrt{\pi}}\sin(2x),\ \dots,\ \tfrac{1}{\sqrt{\pi}}\cos(kx),\ \tfrac{1}{\sqrt{\pi}}\sin(kx)\}}$$

11.1.2 Fourier analysis

The idea of Fourier analysis is based on the least-squares approximation and the last example of the preceding section. We wish to represent a function with a sum of sines and cosines, this is called a **Fourier sum**. Much like a power series, the more terms we use to approximate the function the closer the approximating sum of functions gets to the real function. In the limit the approximation can become exact, the Fourier sum goes to a Fourier series. I do not wish to confront the analytical issues pertaining to the convergence of Fourier series. As a practical matter, it's difficult to calculate infinitely many terms so in practice we just keep the first say 10 or 20 terms and it will come very close to the real function. The advantage of a Fourier sum over a polynomial is that sums of trigonometric functions have natural periodicities. If we approximate the function over the interval $[-\pi, \pi]$ we will also find our approximation repeats itself outside the interval. This is desireable if one wishes to model a wave-form of some sort. Enough talk. Time for an example. (there also an example in your text on pages 540-542 of Spence, Insel and Friedberg)

Example 11.1.16. *Suppose* $f(t) = \begin{cases} 1 & 0 < t < \pi \\ -1 & -\pi < t < 0 \end{cases}$ *and* $f(t + 2n\pi) = f(t)$ *for all* $n \in \mathbb{Z}$. *This is called a* **square wave** *for the obvious reason (draw its graph). Find the first few terms in a Fourier sum to represent the function. We'll want to use the projection: it's convenient to bring the normalizing constants out so we can focus on the integrals without too much clutter.* [1]

$$Proj_W(f)(t) = \tfrac{1}{2\pi} < f, 1 > + \tfrac{1}{\pi} < f, \cos t > \cos t + \tfrac{1}{\pi} < f, \sin t > \sin t + \\ + \tfrac{1}{\pi} < f, \cos 2t > \cos 2t + \tfrac{1}{\pi} < f, \sin 2t > \sin 2t + \cdots$$

Where $W = span(\beta_{trig})$. *The square wave is constant on* $(0, \pi]$ *and* $[-\pi, 0)$ *and the value at zero is not defined (you can give it a particular value but that will not change the integrals that calculate the Fourier coefficients). Calculate,*

$$< f, 1 >= \int_{-\pi}^{\pi} f(t)dt = 0$$

$$< f, \cos t >= \int_{-\pi}^{\pi} \cos(t)f(t)dt = 0$$

Notice that $f(t)$ *and* $\cos(t)f(t)$ *are odd functions so we can conclude the integrals above are zero without further calculation. On the other hand,* $\sin(-t)f(-t) = (-\sin t)(-f(t)) = \sin t f(t)$ *thus* $\sin(t)f(t)$ *is an even function, thus:*

$$< f, \sin t >= \int_{-\pi}^{\pi} \sin(t)f(t)dt = 2\int_{0}^{\pi} \sin(t)f(t)dt = 2\int_{0}^{\pi} \sin(t)dt = 4$$

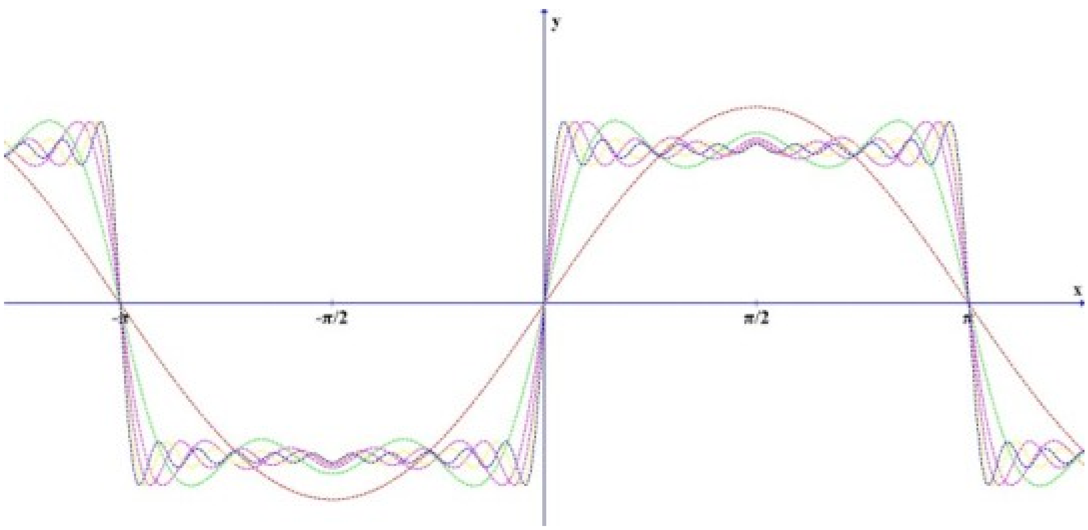
[1] In fact, various texts put these little normalization factors in different places so when you look up results on Fourier series beware conventional discrepancies

Notice that $f(t)\cos(kt)$ *is odd for all* $k \in \mathbb{N}$ *thus* $< f, \cos(kt) >= 0$. *Whereas,* $f(t)\sin(kt)$ *is even for all* $k \in \mathbb{N}$ *thus*

$$\begin{aligned} < f, \sin kt > &= \int_{-\pi}^{\pi} \sin(kt) f(t)dt = 2\int_0^{\pi} \sin(kt)f(t)dt \\ &= 2\int_0^{\pi} \sin(kt)dt = \frac{2}{k}\big[1-\cos(k\pi)\big] = \begin{cases} 0, & k \text{ even} \\ \frac{4}{k}, & k \text{ odd} \end{cases} \end{aligned}$$

Putting it all together we find (the $\sim$ *indicates the functions are nearly the same except for a finite subset of points),*

$$f(t) \sim \frac{4}{\pi}\left(\sin t + \frac{1}{3}\sin 3t + +\frac{1}{5}\sin 5t + \cdots\right) = \sum_{n=1}^{\infty} \frac{4}{(2n-1)\pi}\sin(2n-1)t$$



I have graphed the Fourier sums up the sum with 11 *terms.*

Remark 11.1.17.

The treatment of Fourier sums and series is by no means complete in these notes. There is much more to say and do. Our goal here is simply to connect Fourier analysis with the more general story of orthogonality. In the math 334 course we use Fourier series to construct solutions to partial differential equations. Those calculations are foundational to describe interesting physical examples such as the electric and magnetic fields in a waveguide, the vibrations of a drum, the flow of heat through some solid, even the vibrations of a string instrument.

Chapter 12

AI-ppendix on history

> Numbers shape the past,
> AI uncovers the truth,
> History's math speaks. EXTRANEOUS G, 2025

The ethics of using generative AI to write historical essays raises important questions about authorship, accuracy, and the potential for bias. While AI can be a useful tool for generating content quickly and assisting with research, it is crucial to recognize that historical analysis requires careful interpretation, critical thinking, and a deep understanding of context. Generative AI, though capable of producing coherent text, lacks the ability to engage in nuanced analysis or the capacity for moral and ethical reasoning, which are essential when writing about complex historical events. One of the ethical concerns is the risk of AI perpetuating inaccuracies or biased interpretations, as it can only reflect the data it has been trained on, which may include flawed or outdated perspectives. Additionally, using AI to write historical essays without acknowledging its role in the process can lead to questions of intellectual honesty and accountability. Scholars, educators, and writers must ensure that AI-generated content is thoroughly vetted and supplemented with human insight to maintain the integrity of historical discourse and ensure that it respects the nuances and complexities of the past.

12.1 history of matrices

Matrices have a long and fascinating history, tracing back to ancient civilizations. The earliest recorded use of matrices can be found in ancient China, in the form of "suanshu" (mathematical books) from the Han Dynasty (around 200 BCE). The Chinese used matrix-like structures to solve systems of linear equations, but the formalized study of matrices as we know them today didn't begin until the 19th century. Matrices were first introduced in a more structured mathematical context by the British mathematician James Joseph Sylvester and the German mathematician Carl Friedrich Gauss. Sylvester coined the term "matrix" in 1850, and around the same time, mathematicians like Arthur Cayley and Augustin-Louis Cauchy began developing matrix algebra in earnest, setting the foundation for modern linear algebra.

In the modern era, matrices are indispensable in various fields, particularly in the study of linear transformations and systems of equations. They serve as powerful tools in disciplines such as physics, computer science, economics, engineering, and statistics. In linear algebra, matrices are used to represent and solve systems of linear equations, with applications ranging from electrical circuits to computer graphics. In computer science, matrices form the backbone of algorithms in machine learning, where they are used to represent datasets and transformations. They are also essential in the analysis of networks and in cryptography, where matrix operations help in the encoding and decoding of information. The versatility and wide-ranging applications of matrices in both theoretical and practical contexts underscore their importance in modern mathematics and applied sciences.

12.2 history of vector spaces

Vector spaces, also known as linear spaces, have a rich history rooted in the development of mathematics, particularly in the study of geometry and algebra. The concept of vectors and their manipulation began with the work of mathematicians like René Descartes, who introduced the Cartesian coordinate system in the 17th century, providing a framework for geometric objects to be represented algebraically. The formalization of vector spaces came much later, in the 19th century, with the development of abstract algebra and linear algebra. The foundational work of mathematicians such as Hermann Grassmann and Giuseppe Peano in the mid-1800s helped formalize the structure of vector spaces. Grassmann, in particular, contributed to the idea of "linear independence" and "linear span," key concepts in vector space theory. Peano's work, along with that of others, further helped shape the modern understanding of vector spaces as sets equipped with operations like addition and scalar multiplication, subject to specific axioms.

The use of vector spaces spans numerous fields of mathematics and science. In linear algebra, vector spaces form the foundation for the study of linear transformations, systems of linear equations, and matrix theory. The theory of vector spaces is central in understanding various mathematical structures and phenomena, from geometry to physics. In physics, vector spaces are used to describe the state spaces of quantum mechanics, the forces acting on particles, and other physical quantities that have both magnitude and direction. In computer science, vector spaces are employed in areas like machine learning, where they represent data in high-dimensional space for tasks such as classification and clustering. Additionally, vector spaces have applications in signal processing, economics, and even linguistics, where they can be used to represent and analyze relationships between words or documents in natural language processing. The abstraction and versatility of vector spaces make them one of the most powerful and widely used tools in modern mathematics and applied sciences.

12.3 history of linear transformations

Linear transformations are a fundamental concept in linear algebra with deep historical roots in the development of mathematics. The formal theory of linear transformations emerged in the 19th century, alongside the broader development of linear algebra, which focused on solving systems of linear equations and studying vector spaces. Early contributors to the field, such as Augustin-Louis Cauchy and Karl Friedrich Gauss, laid the groundwork by studying the behavior of linear mappings between vector spaces. However, it was the work of mathematicians like Arthur Cayley and Ferdinand Frobenius who formally recognized and characterized the idea of a linear transformation as a map between two vector spaces that preserves the operations of vector addition and scalar multiplication.

The concept of a matrix as a representation of a linear transformation also developed in parallel, particularly with the work of Cayley and the study of determinants, which helped understand the properties of such transformations.

The use of linear transformations is extensive and spans many fields of mathematics, science, and engineering. In geometry, linear transformations can describe operations such as rotations, scaling, and shearing, providing a way to map points and objects from one space to another while preserving their linear structure. In computer graphics, linear transformations are crucial for manipulating images and models, enabling scaling, rotating, and translating objects in 2D or 3D space. In physics, linear transformations help model phenomena such as the behavior of physical systems under changes of reference frames or the application of certain forces. Furthermore, in computer science, particularly in machine learning, linear transformations play a significant role in algorithms for dimensionality reduction, such as principal component analysis (PCA), where data is projected into a lower-dimensional space to simplify analysis or classification tasks. In addition, linear transformations are pivotal in quantum mechanics, where they are used to describe the evolution of quantum states. Thus, linear transformations are not only a cornerstone of mathematical theory but also an essential tool across many applied disciplines.

12.4 history of Jordan Form and eigenvectors

The concept of eigenvectors and eigenvalues has its origins in the study of linear transformations and matrix theory, and it became a central part of linear algebra in the 19th century. The term "eigen" comes from the German word for "own" or "self," reflecting the idea that eigenvectors are vectors that remain in the same direction when a linear transformation is applied, though they may be scaled by an eigenvalue. The earliest work related to eigenvectors can be traced to the mathematician Augustin-Louis Cauchy in the early 19th century, who worked on the characteristic equation of matrices. The formal theory was later developed and refined by mathematicians such as Carl Friedrich Gauss

and David Hilbert, who advanced the algebraic understanding of how matrices act on vectors. The discovery and formalization of eigenvectors and eigenvalues allowed for a deeper exploration of linear transformations and their associated behaviors, especially in the context of diagonalization and the spectral theorem.

Eigenvectors and their corresponding eigenvalues have profound applications across various fields of mathematics and science. In linear algebra, they are essential for understanding matrix diagonalization, which simplifies the process of solving systems of linear equations, particularly in cases involving large or complex matrices. In physics, eigenvectors and eigenvalues are used to describe systems in equilibrium, such as the vibration modes of mechanical structures or the energy levels of quantum systems. In engineering, they are critical in fields like control theory, where they help model and analyze dynamic systems. One important concept related to eigenvectors is the Jordan Form (or Real Jordan Form in the case of real matrices), which provides a way to represent any square matrix in a canonical form. The Jordan Normal Form generalizes diagonalization by allowing matrices that cannot be fully diagonalized to be transformed into a block diagonal form with Jordan blocks. This form is especially useful when dealing with non-diagonalizable matrices, allowing for simpler computations in cases where diagonalization is not possible. The Jordan Form is invaluable in advanced mathematics and theoretical physics, as it provides a powerful tool for analyzing the structure and behavior of linear systems, making it crucial for understanding stability, oscillation, and other phenomena in both abstract and applied contexts.

The Real Jordan Form (or Jordan Normal Form over the real numbers) is a canonical form used to represent a square matrix, particularly when it cannot be fully diagonalized. Unlike the diagonalization process, which works only for matrices with a complete set of linearly independent eigenvectors, the Real Jordan Form applies to matrices with a complex or defective spectrum, where some eigenvalues may have fewer than the expected number of linearly independent eigenvectors. To calculate the Real Jordan Form, one first finds the eigenvalues of the matrix, and then computes the generalized eigenvectors associated with each eigenvalue. These generalized eigenvectors fill Jordan chains, which are used to construct Jordan blocks—a block-diagonal matrix with each block corresponding to an eigenvalue and possibly containing ones in certain positions, depending on the number of generalized eigenvectors. The Real Jordan Form involves organizing these Jordan blocks into a diagonal or block-diagonal structure, with each block representing an eigenspace or a generalized eigenspace. In cases where the matrix has complex eigenvalues, the Real Jordan Form uses 2x2 real blocks to represent the complex eigenvalues and their corresponding eigenvectors. The process typically involves solving for the null spaces of powers of the matrix subtracted by the eigenvalue times the identity matrix, a task that requires careful computation of matrix powers and eigenvectors.

12.5 history of inner product spaces and Fourier analysis

The history of inner product spaces can be traced back to the development of geometric and algebraic concepts in the 19th century. The idea of an inner product, a generalization of the dot product, was formalized as part of the broader development of functional analysis and vector spaces. Early work on geometric spaces focused on Euclidean geometry and the notion of angles and distances, which were foundational to the concept of the inner product. In the 19th century, mathematicians such as Hermann Grassmann, Karl Weierstrass, and others advanced the theory of vector spaces and linear transformations, eventually leading to the abstraction of the inner product. The formalization of inner product spaces as a generalization of Euclidean space came in the late 19th and early 20th centuries, particularly with the work of David Hilbert, who introduced Hilbert spaces. These are complete inner product spaces, where the inner product defines the geometry of the space and provides a way to measure angles, lengths, and orthogonality, essential for developing the theory of functional analysis.

Fourier analysis, which deals with the representation of functions as sums of sine and cosine waves, also emerged in the 19th century, building on earlier work in harmonic analysis. The French mathematician Jean-Baptiste Joseph Fourier is credited with developing Fourier series in the early 1800s as part of his work on heat conduction. Fourier's groundbreaking insight was that a function, even if it was not periodic, could be expressed as an infinite sum of sinusoidal components, providing a powerful tool for solving partial differential equations. Fourier's work revolutionized the study of heat transfer and wave motion, but it also laid the foundation for what would become a vast field of mathematical analysis. The mathematical rigor surrounding Fourier analysis was developed over the following decades, especially with the work of mathematicians such as Pafnuty Chebyshev and Henri Lebesgue, who further formalized the theory of convergence of Fourier series.

The connection between inner product spaces and Fourier analysis became more apparent with the rise of functional analysis and Hilbert spaces in the early 20th century. Fourier analysis, as a tool for decomposing functions into frequency components, naturally fits within the framework of inner product spaces, where functions can be treated as vectors in an infinite-dimensional space. The inner product in these spaces enables the projection of functions onto orthogonal basis functions, such as sines and cosines in the classical Fourier series or more generalized bases in modern Fourier transforms. The formal development of Fourier analysis on Hilbert spaces in the mid-20th century provided a rigorous foundation for the applications of Fourier methods in signal processing, quantum mechanics, and other areas of physics and engineering. Fourier analysis, now extended to Fourier transforms and other related methods, remains a cornerstone of mathematical analysis, with its connection to inner product spaces providing the mathematical machinery for understanding and manipulating signals, waveforms, and other phenomena in both theory and practice.

12.6 history and future of abstract linear algebra

Abstract linear algebra, as we know it today, emerged in the 19th and early 20th centuries as mathematicians sought to extend the ideas of linearity and vector spaces beyond the confines of Euclidean geometry. The roots of abstract linear algebra trace back to the work of mathematicians such as Hermann Grassmann, who in the mid-1800s introduced the concept of *exterior algebra*, which laid the groundwork for later developments in vector spaces and their algebraic structures. Around the same time, Arthur Cayley and Karl Weierstrass developed foundational ideas in matrix theory, which is now a central component of linear algebra. The real breakthrough came in the early 20th century with the work of David Hilbert, who formalized the notion of infinite-dimensional vector spaces, now known as *Hilbert spaces*. This work, along with the formalization of inner product spaces and the study of linear transformations in these spaces, marked the birth of modern abstract linear algebra, shifting the field from geometric intuition to more abstract, algebraic treatments of vector spaces and linear maps.

During the 20th century, the scope of abstract linear algebra expanded significantly as mathematicians formalized and generalized various concepts, including eigenvalues, eigenvectors, and linear transformations. The development of group theory, ring theory, and module theory further contributed to the abstraction of algebraic structures beyond the realm of matrices and finite-dimensional vector spaces. As a result, linear algebra became a critical part of abstract algebra and functional analysis, and it found applications in many branches of mathematics, including topology, number theory, and representation theory. Concepts like *Banach spaces* (complete normed vector spaces) and *Lie groups* (smooth groups with algebraic structure) further broadened the applicability of abstract linear algebra, showing its profound connection to areas of geometry and mathematical physics.

In contemporary research, abstract linear algebra continues to evolve in multiple directions.

One major area of current exploration is the theory of *noncommutative algebra*, which generalizes linear algebraic structures to settings where commutative operations (such as addition or multiplication) do not necessarily hold. This has profound applications in quantum mechanics and other areas of physics, where operators often do not commute. Additionally, the study of *tensor categories* and *category theory* has provided new insights into the relationships between different algebraic structures, offering a more unified and flexible framework for understanding linear transformations across diverse settings. In applied mathematics, abstract linear algebra is at the core of advancements in machine learning and data science, where techniques such as *principal component analysis* (PCA) and *singular value decomposition* (SVD) rely on abstract concepts of matrix decomposition and eigendecomposition. Moreover, the rise of *infinite-dimensional spaces* in modern analysis, including the study of quantum field theory and functional analysis, continues to highlight the relevance and potential of abstract linear algebra in understanding complex systems in both theoretical and applied contexts.

Remark 12.6.1. *Believing that AI-generated writing is inherently worth reading is misguided because AI, despite its impressive capabilities, lacks the ability to understand context, nuance, and the underlying meaning of human experiences. AI operates based on patterns learned from vast datasets, but it cannot grasp the complexities of human emotions, historical context, or the ethical and cultural dimensions that shape our world. This lack of genuine comprehension makes AI-generated content prone to inaccuracies, oversimplifications, and a lack of depth. Readers often turn to writing for insight, reflection, and understanding, which requires a level of critical thinking and emotional connection that AI cannot replicate. While AI can assist with generating ideas or providing structure, it cannot replace the value of authentic, human-driven narratives or analyses, which are crafted with intention, empathy, and expertise.*

Moreover, relying on AI-generated writing for consumption diminishes the importance of human creativity and the intellectual rigor that goes into producing thoughtful, meaningful content. Writing is not just about stringing words together; it's a process of engaging with ideas, questioning assumptions, and fostering a connection between the author and the audience. Human writers bring unique perspectives, personal experiences, and a sense of purpose to their work, qualities that AI simply cannot emulate. Trusting AI-generated content without scrutiny can lead to a homogenization of ideas and a loss of diversity in thought, as algorithms often reflect prevailing trends in data rather than presenting novel or critical viewpoints. Thus, while AI can be a useful tool, it is wrong to elevate its output to the level of meaningful reading, as it lacks the depth, intentionality, and originality that come from human minds.