Show your work and box answers. This pdf should be printed and your solution should be handwritten on the printout. Once complete, please staple in upper left corner. Thanks.

Suggested Reading You may find the following helpful resources beyond lecture,

(a.) Chapter 1 of my lecture notes for Math 221

(b.) Reduced Row Echelon Web Calculator
(see http://www.math.odu.edu/ bogacki/cgi-bin/lat.cgi?c=rref
please consider using this to check your answers, or better yet, use Matlab, but the answers and work must be shown in handwritten detailed work here. To be clear, you cannot just use Matlab and printout the result. Not yet anyway, all in good time.)

Problem 1: Solve the following system by back-substitution. In particular, solve the last equation and use the solution to simplify the preceding equation, rinse and repeat.

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 &= 55 \quad (1)\\ 2x_2 + 3x_3 + 4x_4 + 5x_5 &= 54 \quad (2)\\ 3x_3 + 4x_4 + 5x_5 &= 50 \quad (3)\\ 4x_4 + 5x_5 &= 41 \quad (4)\\ 5x_5 &= 25 \quad (5)\end{aligned}$$

Solve (5) to find $X_5 = \frac{25}{5} = 5 \therefore \underline{X_5 = 5}$.

Then (4) yields $4X_4 + 25 = 41 \Rightarrow 4X_4 = 16 \Rightarrow \underline{X_4 = 4}$.

And (3) gives $3X_3 + 4X_4 + 5X_5 = 3X_3 + 16 + 25 = 50$
hence $3X_3 = 9 \Rightarrow \underline{X_3 = 3}$.

Thus (2) gives $2X_2 + 9 + 16 + 25 = 54 \Rightarrow 2X_2 = 4 \therefore \underline{X_2 = 2}$.

Finally, (1) provides $X_1 + 4 + 9 + 16 + 25 = 55 \Rightarrow \underline{X_1 = 1}$.

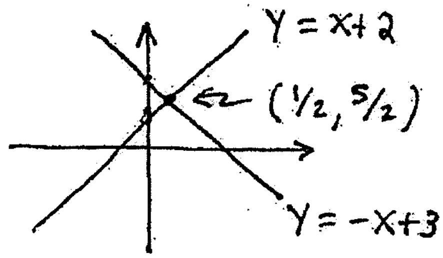
We find solution $\boxed{(1, 2, 3, 4, 5)}$

Problem 2: Solve each system and graph each equation. Explain why the system is inconsistent for the inconsistent systems.

(a.) $y = x + 2$ and $y = -x + 3$ $\Rightarrow$ $X + 2 = -X + 3$

$\Rightarrow 2X = 1$

$\Rightarrow$ $\boxed{X = 1/2 \text{ and } y = 5/2}$

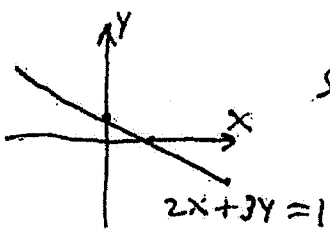


(b.) $2x + 3y = 1$ and $\underbrace{4x + 6y = 2}$

$2X + 3Y = 1$ (divide by 2)

Solution $= \{(x, y) \mid 2x + 3y = 1\} \rightarrow y = \frac{1-2x}{3}$

$= \{(x, \frac{1}{3}(1-2x)) \mid x \in \mathbb{R}\}$

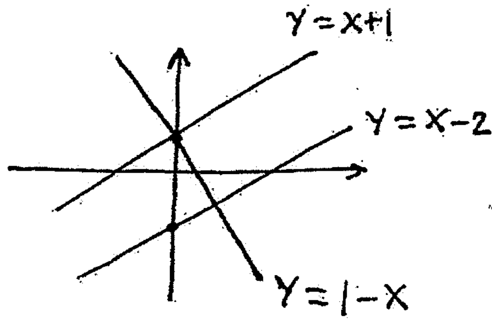


(c.) $y = x + 1$ and $y = 1 - x$ and $\underbrace{2x - 2y = 4}$

$2y = 2X - 4$

$\underline{y = X - 2}$.

(w/o row reduction $\exists$ many ways to see $\rightarrow\leftarrow$ here)



No solution, inconsistent system.

Algebraically, $X + 1 = 1 - X \Rightarrow X = 0$ then $-2y = 4 \Rightarrow y = -2$, but $y = X + 1 \Rightarrow -2 = 1$ $\rightarrow\leftarrow$.

Problem 3: Let a, b be constants. Solve $\left\{ \begin{array}{ll} x + 2y & = a \\ 3x + 4y & = b \end{array} \right\}$. Your answer will involve a and b.

$$\left[\begin{array}{cc|c} 1 & 2 & a \\ 3 & 4 & b \end{array}\right] \xrightarrow{r_2 - 3r_1} \left[\begin{array}{cc|c} 1 & 2 & a \\ 0 & -2 & b - 3a \end{array}\right] \xrightarrow{r_1 + r_2} \left[\begin{array}{cc|c} 1 & 0 & b - 2a \\ 0 & -2 & b - 3a \end{array}\right]$$

$$\xrightarrow{r_2/-2} \left[\begin{array}{cc|c} 1 & 0 & b - 2a \\ 0 & 1 & \frac{3}{2}a - \frac{1}{2}b \end{array}\right] \Rightarrow \boxed{x = b - 2a, \ y = \frac{3a - b}{2}}$$

a.k.a. $\underline{(b - 2a, \frac{1}{2}(3a - b))}$.

$\underline{\text{Remark}}$: $\frac{1}{4-6}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \frac{-1}{2}\begin{bmatrix} 4a - 2b \\ -3a + b \end{bmatrix} = \begin{bmatrix} -2a + b \\ (3a - b)/2 \end{bmatrix}$.

(look at this Remark in about 10 days)

Problem 4: Solve the following system of equations via row-reduction of the augmented coefficient matrix.

$$\left\{\begin{array}{rl} x+2y &= 3z-w \\ 2x-y+z &= w+2 \\ y+z+w &= 6 \\ 3z+x &= 12+y-w \end{array}\right\}$$

$$\begin{aligned} x+2y-3z+w &= 0 \\ 2x-y+z-w &= 2 \\ y+z+w &= 6 \\ x-y+3z+w &= 12 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 0 \\ 2 & -1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 1 & 6 \\ 1 & -1 & 3 & 1 & 12 \end{array}\right] \xrightarrow[r_4 - r_1]{r_2 - 2r_1} \left[\begin{array}{cccc|c} 1 & 2 & -3 & 1 & 0 \\ 0 & -5 & 7 & -3 & 2 \\ 0 & 1 & 1 & 1 & 6 \\ 0 & -3 & 6 & 0 & 12 \end{array}\right] \xrightarrow[r_4+3r_3]{r_1-2r_3,\ r_2+5r_3} \left[\begin{array}{cccc|c} 1 & 0 & -5 & -1 & -12 \\ 0 & 0 & 12 & 2 & 32 \\ 0 & 1 & 1 & 1 & 6 \\ 0 & 0 & 9 & 3 & 30 \end{array}\right]$$

$$\left[\begin{array}{cccc|c} 1 & 0 & -5 & -1 & -12 \\ 0 & 0 & 108 & 18 & 288 \\ 0 & 108 & 108 & 108 & 648 \\ 0 & 0 & 108 & 36 & 360 \end{array}\right] \xrightarrow[r_3 - r_4]{r_2 - r_4} \left[\begin{array}{cccc|c} 1 & 0 & -5 & -1 & -12 \\ 0 & 0 & 0 & -18 & -72 \\ 0 & 108 & 0 & 72 & 288 \\ 0 & 0 & 108 & 36 & 360 \end{array}\right]$$

$$\xrightarrow{r_2/-18} \left[\begin{array}{cccc|c} 1 & 0 & -5 & -1 & -12 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 108 & 0 & 72 & 288 \\ 0 & 0 & 108 & 36 & 360 \end{array}\right] \xrightarrow[r_3-72r_2,\ r_4-36r_2]{r_1+r_2} \left[\begin{array}{cccc|c} 1 & 0 & -5 & 0 & -8 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 108 & 0 & 0 & 0 \\ 0 & 0 & 108 & 0 & 216 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -5 & 0 & -8 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 & 4 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & 4 \end{array}\right]$$

$$\boxed{\text{Solution} = (2, 0, 2, 4)}$$

Problem 5: Write down the augmented coefficient matrix $[A|b]$ for each system given below:

(a.)
$$\left\{ \begin{aligned} x_1 + x_2 + 3x_3 + 4x_4 &= 12 \\ 2x_2 + 4x_3 - 2x_4 &= 16 \\ 3x_1 - x_2 + 6x_3 + x_4 &= 19 \end{aligned} \right\}$$

$$\left[\begin{array}{cccc|c} 1 & 1 & 3 & 4 & 12 \\ 0 & 2 & 4 & -2 & 16 \\ 3 & -1 & 6 & 1 & 19 \end{array}\right],$$

(b.)
$$\left\{ \begin{aligned} 2s + t &= 1 \\ 3s - t &= 3 \\ s + 4t &= 0 \end{aligned} \right\}$$

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 3 & -1 & 3 \\ 1 & 4 & 0 \end{array}\right],$$

(c.)
$$\left\{ \begin{aligned} 2x_1 + x_2 + x_3 &= 0 \\ 3x_1 - x_2 + 3x_3 &= 0 \\ x_1 + 4x_2 &= 0 \end{aligned} \right\}$$

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 3 & -1 & 3 & 0 \\ 1 & 4 & 0 & 0 \end{array}\right],$$

(d.)
$$\left\{ \begin{aligned} x_1 - 2x_2 + 3x_3 - 6x_4 &= 1 \\ 2x_1 - 7x_2 + x_3 + x_4 &= 2 \end{aligned} \right\}$$

$$\left[\begin{array}{cccc|c} 1 & -2 & 3 & -6 & 1 \\ 2 & -7 & 1 & 1 & 2 \end{array}\right].$$

(e.)
$$\left\{ \begin{aligned} 3x_4 + x_6 - x_7 &= 1 \\ 2x_1 - 2x_2 - 3x_3 - 4x_4 - 5x_5 - 6x_6 - 7x_7 &= 0 \end{aligned} \right\}$$

$$\left[\begin{array}{ccccccc|c} 0 & 0 & 0 & 3 & 0 & 1 & -1 & 1 \\ 2 & -2 & -3 & -4 & -5 & -6 & -7 & 0 \end{array}\right].$$

Problem 6: Use the row-reduction technique to calculate rref$[A|b]$ for each system given in the previous problem and write down the solution set in standard form for each system. (attach the solutions in order, clearly labeled after this page)

$\boxed{P6}$

(a.)

$$\left[\begin{array}{cccc|c} 1 & 1 & 3 & 4 & 12 \\ 0 & 2 & 4 & -2 & 16 \\ 3 & -1 & 6 & 1 & 19 \end{array}\right] \xrightarrow{r_3 - 3r_1} \left[\begin{array}{cccc|c} 1 & 1 & 3 & 4 & 12 \\ 0 & 2 & 4 & -2 & 16 \\ 0 & -4 & -3 & -11 & -17 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 3 & 4 & 12 \\ 0 & 1 & 2 & -1 & 8 \\ 0 & -4 & -3 & -11 & -17 \end{array}\right]$$

$$\xrightarrow[r_3 + 4r_2]{r_1 - r_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 5 & 4 \\ 0 & 1 & 2 & -1 & 8 \\ 0 & 0 & 5 & -15 & 15 \end{array}\right] \xrightarrow{r_3/5} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 5 & 4 \\ 0 & 1 & 2 & -1 & 8 \\ 0 & 0 & 1 & -3 & 3 \end{array}\right]$$

$$\xrightarrow[r_2 - 2r_3]{r_1 - r_3} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 8 & 1 \\ 0 & 1 & 0 & 5 & 2 \\ 0 & 0 & 1 & -3 & 3 \end{array}\right] \qquad \begin{aligned} x_1 + 8x_4 &= 1 \\ x_2 + 5x_4 &= 2 \\ x_3 - 3x_4 &= 3 \end{aligned}$$

Hence $\boxed{\text{Solution Set} = \{ (1 - 8x_4,\ 2 - 5x_4,\ 3 + 3x_4,\ x_4) \mid x_4 \in \mathbb{R} \}}$

(b.)

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 3 & -1 & 3 \\ 1 & 4 & 0 \end{array}\right] \xrightarrow[r_2 - 3r_3]{r_1 - 2r_3} \left[\begin{array}{cc|c} 0 & -7 & 1 \\ 0 & -9 & 3 \\ 1 & 4 & 0 \end{array}\right] \xrightarrow[r_3 + \frac{4}{7}r_1]{r_2 - \frac{9}{7}r_1} \left[\begin{array}{cc|c} 0 & -7 & 1 \\ 0 & 0 & 12/7 \\ 1 & 0 & 4/7 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right]$$

$\boxed{\text{Solution Set} = \emptyset}$ (inconsistent system) (no solution)

(c.)

$$\left[\begin{array}{ccc|c} 2 & 1 & 1 & 0 \\ 3 & -1 & 3 & 0 \\ 1 & 4 & 0 & 0 \end{array}\right] \xrightarrow[r_2 - 3r_3]{r_1 - 2r_3} \left[\begin{array}{ccc|c} 0 & -7 & 1 & 0 \\ 0 & -13 & 3 & 0 \\ 1 & 4 & 0 & 0 \end{array}\right] \xrightarrow[7r_2]{13r_1} \left[\begin{array}{ccc|c} 0 & -91 & 13 & 0 \\ 0 & -91 & 21 & 0 \\ 1 & 4 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{r_2 - r_1} \left[\begin{array}{ccc|c} 0 & -91 & 13 & 0 \\ 0 & 0 & 8 & 0 \\ 1 & 4 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & -91 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 4 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

$\boxed{\text{Solution} = (0,0,0)}$

(d.)

$$\left[\begin{array}{cccc|c} 1 & -2 & 3 & -6 & 1 \\ 2 & -7 & 1 & 1 & 2 \end{array}\right] \xrightarrow{r_2 - 2r_1} \left[\begin{array}{cccc|c} 1 & -2 & 3 & -6 & 1 \\ 0 & -3 & -5 & 13 & 0 \end{array}\right] \xrightarrow{r_2/-3} \left[\begin{array}{cccc|c} 1 & -2 & 3 & -6 & 1 \\ 0 & 1 & 5/3 & -13/3 & 0 \end{array}\right]$$

$$\xrightarrow{r_1 - \frac{2}{3}r_2} \left[\begin{array}{cccc|c} 1 & 0 & -1/3 & -44/3 & 1 \\ 0 & -3 & -5 & 13 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & -1/3 & -44/3 & 1 \\ 0 & 1 & 5/3 & -13/3 & 0 \end{array}\right]$$

$$\xrightarrow{r_1 + 2r_2} \left[\begin{array}{cccc|c} 1 & 0 & 19/3 & -44/3 & 1 \\ 0 & 1 & 5/3 & -13/3 & 0 \end{array}\right] \Rightarrow \begin{aligned} x_1 &= 1 - \tfrac{19}{3}x_3 + \tfrac{44}{3}x_4 \\ x_2 &= -\tfrac{5}{3}x_3 + \tfrac{13}{3}x_4 \end{aligned}$$

$\boxed{\text{Solution Set} = \{ (1 - \tfrac{19}{3}x_3 + \tfrac{44}{3}x_4,\ -\tfrac{5}{3}x_3 + \tfrac{13}{3}x_4,\ x_3,\ x_4) \mid x_3, x_4 \in \mathbb{R} \}}$

$\boxed{P6}$ continued,

$$\left[\begin{array}{ccccccc|c} 0 & 0 & 0 & 3 & 0 & 1 & -1 & 1 \\ 2 & -2 & -3 & -4 & -5 & -6 & -7 & 0 \end{array}\right]$$

$$\begin{array}{c} \xrightarrow{r_1/3} \\ \xrightarrow{r_2/2} \end{array} \left[\begin{array}{ccccccc|c} 0 & 0 & 0 & 1 & 0 & 1/3 & -1/3 & 1/3 \\ 1 & -1 & -\frac{3}{2} & -2 & -\frac{5}{2} & -3 & -\frac{7}{2} & 0 \end{array}\right]$$

$$\xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{ccccccc|c} 1 & -1 & -3/2 & -2 & -5/2 & -3 & -7/2 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1/3 & -1/3 & 1/3 \end{array}\right]$$

$$\xrightarrow{r_1 + 2r_2} \left[\begin{array}{ccccccc|c} 1 & -1 & -3/2 & 0 & -5/2 & -7/3 & -25/6 & 2/3 \\ 0 & 0 & 0 & 1 & 0 & 1/3 & -1/3 & 1/3 \end{array}\right]$$

$$\text{Solution} = \left(\frac{2}{3} + x_2 + \frac{3}{2}x_3 + \frac{5}{2}x_4 + \frac{7}{3}x_5 + \frac{25}{6}x_6,\ x_2,\ x_3,\ \frac{1}{3} - \frac{1}{3}x_6 + \frac{1}{3}x_7,\ x_5,\ x_6,\ x_7\right)$$

where $x_2, x_3, x_5, x_6, x_7 \in \mathbb{R}$ are free parameters.

Problem 7: Consider the following systems:

$$\text{(I.)}\ \left\{\begin{aligned} x_1 + x_3 &= 1 \\ 2x_1 + 3x_2 + 8x_3 &= 1 \\ 3x_1 - x_2 + x_3 &= -4 \end{aligned}\right\}$$

$$\text{(II.)}\ \left\{\begin{aligned} x_1 + x_3 &= 1 \\ 2x_1 + 3x_2 + 8x_3 &= 5 \\ 3x_1 - x_2 + x_3 &= 2 \end{aligned}\right\}$$

These systems share the same matrix of coefficients A whereas the inhomogeneous terms differ. If $[A|b_I]$ is the augmented coefficient matrix for (I.) and $[A|b_{II}]$ is the augmented coefficient matrix for (II.) then we may solve both systems by row-reducing $[A|b_I|b_{II}]$.

(a.) write down $[A|b_I|b_{II}]$

$$\left[\begin{array}{ccc|c|c} 1 & 0 & 1 & 1 & 1 \\ 2 & 3 & 8 & 1 & 5 \\ 3 & -1 & 1 & -4 & 2 \end{array}\right]$$

(b.) use row-reduction to calculate rref$[A|b_I|b_{II}]$

$$\left[\begin{array}{ccc|cc} 1 & 0 & 1 & 1 & 1 \\ 2 & 3 & 8 & 1 & 5 \\ 3 & -1 & 1 & -4 & 2 \end{array}\right] \xrightarrow[r_3 - 3r_1]{r_2 - 2r_1} \left[\begin{array}{ccc|c|c} 1 & 0 & 1 & 1 & 1 \\ 0 & 3 & 6 & -1 & 3 \\ 0 & -1 & -2 & -7 & -1 \end{array}\right] \xrightarrow{r_2 + 3r_3} \left[\begin{array}{ccc|cc} 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & -22 & 0 \\ 0 & -1 & -2 & -7 & -1 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|cc} 1 & 0 & 1 & 1 & 1 \\ 0 & 1 & 2 & 7 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] \sim \underline{\left[\begin{array}{ccc|c|c} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right] = \text{rref}[A|b_I|b_{II}]}$$

(c.) write down the standard solution to system (I.) and (II.), or state no solution.

I.) $\text{rref}\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \longrightarrow \boxed{\text{No Solution}}$

II.) $\text{rref}\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \longrightarrow \boxed{\begin{aligned} x_1 &= 1 - x_3 \\ x_2 &= 1 - 2x_3 \end{aligned}}, \ \underline{x_3 \in \mathbb{R}}.$

$$\boxed{\text{Solution Set} = \{(1 - x_3, 1 - 2x_3, x_3) \mid x_3 \in \mathbb{R}\}}$$

Problem 8: Let a, b, c be constants. Consider $\left\{\begin{aligned} x_1 + x_3 &= a \\ 2x_1 + 3x_2 + 8x_3 &= b \\ 3x_1 - x_2 + x_3 &= c \end{aligned}\right\}$. What condition must be given for a, b, c in order that this system of equations be consistent ? Comment on how this coincides with your work in the preceding problem.

Can use same row reduction since coeff. matrix is same.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 2 & 3 & 8 & b \\ 3 & -1 & 1 & c \end{array}\right] \xrightarrow[r_3 - 3r_1]{r_2 - 2r_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 3 & 6 & b - 2a \\ 0 & -1 & -2 & c - 3a \end{array}\right]$$

$$\xrightarrow{3r_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 3 & 6 & b - 2a \\ 0 & -3 & -6 & 3c - 9a \end{array}\right] \xrightarrow{r_3 + r_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 3 & 6 & b - 2a \\ 0 & 0 & 0 & -11a + b + 3c \end{array}\right]$$

$$\xrightarrow{r_2/3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & 2 & (b - 2a)/3 \\ 0 & 0 & 0 & -11a + b + 3c \end{array}\right]$$

The needed condition is $\boxed{-11a + b + 3c = 0}$

Problem 9: (Lay, §1.1#28) question about condition needed on coefficient matrix in order for for the system to be consistent for arbtrary inhomogeneous term.

Consider constants a, b, c, d with $a \neq 0$. If $\left\{\begin{array}{l} ax_1 + bx_2 = f \\ cx_1 + dx_2 = g \end{array}\right\}$ is consistent $\forall f, g$ then find necessary condition on a, b, c, d.

$$\left[\begin{array}{cc|c} a & b & f \\ c & d & g \end{array}\right] \xrightarrow{r_1/a} \left[\begin{array}{cc|c} 1 & b/a & f/a \\ c & d & g \end{array}\right]$$

$$\xrightarrow{r_2 - cr_1} \left[\begin{array}{cc|c} 1 & b/a & f/a \\ 0 & d - \frac{cb}{a} & g - \frac{cf}{a} \end{array}\right]$$

Hence $\left(d - \frac{cb}{a}\right) x_2 = g - \frac{cf}{a}$

$$\Rightarrow (ad - bc) x_2 = ag - cf$$

If $ad - bc = 0 \Rightarrow ag - cf = 0$, but this is not true for arbitrary f, g, thus $\boxed{ad - bc \neq 0}$

When $ad - bc \neq 0$ we calculate

$$x_2 = \frac{ag - cf}{ad - bc}$$

Also, $ax_1 + bx_2 = f$

$$x_1 = \frac{f - bx_2}{a} = \frac{1}{a}\left(f - b\left(\frac{ag - cf}{ad - bc}\right)\right)$$

$$= \frac{1}{a}\left[\frac{f(ad - \cancel{bc}) - b(ag - \cancel{cf})}{ad - bc}\right]$$

$$= \frac{fd - bg}{ad - bc}$$

Remark:

$x_1 = \dfrac{\det \begin{bmatrix} f & b \\ g & d \end{bmatrix}}{\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}}$ and also we see that $x_2 = \dfrac{\det \begin{bmatrix} a & f \\ c & g \end{bmatrix}}{\det \begin{bmatrix} a & b \\ c & d \end{bmatrix}}$

(this is an example of Cramer's Rule which we learn later

Problem 10: You are given the following row-reduction:

$$\text{rref}\begin{bmatrix} 1 & 2 & 7 & 7 & 3 \\ 2 & 4 & 1 & 0 & 6 \\ 3 & 6 & 3 & 13 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}.$$

In view of the given reduction,

(a.) provide a consistent system of three equations and three unknowns x, y, z and provide its solution.

$$\underbrace{\begin{aligned} x + 2y + 7z &= 3 \\ 2x + 4y + z &= 6 \\ 3x + 6y + 3z &= 9 \end{aligned}}_{\text{Columns } 1,2,3,5}$$

$$\boxed{\begin{aligned} x &= 3 - 2y \\ z &= 0 \\ y &\in \mathbb{R} \end{aligned}}$$

$$\boxed{\text{Solution Set} = \{(3-2y, y, 0) \mid y \in \mathbb{R}\}}$$

(b.) provide an inconsistent system of three equations and three unknowns x, y, z and provide its solution.

$$\underbrace{\begin{aligned} x + 2y + 7z &= 7 \\ 2x + 4y + z &= 0 \\ 3x + 6y + 3z &= 13 \end{aligned}}_{\text{columns } 1,2,3,4}$$

inconsistent, see the 4th column in the rref.

(c.) write down a system of three equations in x_1, x_2, x_3, x_4 and provide its solution.

$$\begin{aligned} x_1 + 2x_2 + 7x_3 + 7x_4 &= 3 \\ 2x_1 + 4x_2 + x_3 &= 6 \\ 3x_1 + 6x_2 + 3x_3 + 13x_4 &= 9 \end{aligned}$$

$$\boxed{\begin{aligned} x_1 &= 3 - 2x_2 \\ x_3 &= 0 \\ x_4 &= 0 \\ x_2 &\in \mathbb{R} \end{aligned}}$$

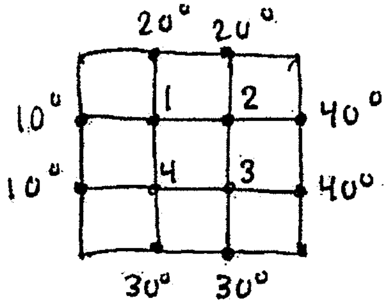
(d.) write down a homogeneous system of three equations in x_1, x_2, x_3, x_4, x_5 and provide its solution.

$$\begin{aligned} x_1 + 2x_2 + 7x_3 + 7x_4 + 3x_5 &= 0 \\ 2x_1 + 4x_2 + x_3 + 6x_5 &= 0 \\ 3x_1 + 6x_2 + 3x_3 + 13x_4 + 9x_5 &= 0 \end{aligned}$$

$$\text{rref}\left[\begin{array}{ccccc|c} 1 & 2 & 7 & 7 & 3 & 0 \\ 2 & 4 & 1 & 0 & 6 & 0 \\ 3 & 6 & 3 & 13 & 9 & 0 \end{array}\right] = \left[\begin{array}{ccccc|c} 1 & 2 & 0 & 0 & 3 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

$$\therefore \boxed{\text{Solution Set} = \{(-2x_2 - 3x_5, x_2, 0, 0, x_5) \mid x_2, x_5 \in \mathbb{R}\}}$$

Problem 11: (Lay, §1.1#34) an applied linear system based on temperature distribution.



$$T_1 = \frac{10+20+T_2+T_4}{4}$$

$$T_2 = \frac{20+40+T_3+T_1}{4}$$

$$T_3 = \frac{40+30+T_4+T_2}{4}$$

$$T_4 = \frac{10+30+T_3+T_1}{4}$$

I'll work through #33 to set-up temp. eq^n based on temp. given by average of nearest neighbor temps.

$$4T_1 - T_2 - T_4 = 30$$
$$4T_2 - T_1 - T_3 = 60$$
$$4T_3 - T_2 - T_4 = 70$$
$$4T_4 - T_1 - T_3 = 40$$

$$\left[\begin{array}{cccc|c} 4 & -1 & 0 & -1 & 30 \\ -1 & 4 & -1 & 0 & 60 \\ 0 & -1 & 4 & -1 & 70 \\ -1 & 0 & -1 & 4 & 40 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_4} \left[\begin{array}{cccc|c} -1 & 0 & -1 & 4 & 40 \\ -1 & 4 & -1 & 0 & 60 \\ 0 & -1 & 4 & -1 & 70 \\ 4 & -1 & 0 & -1 & 30 \end{array}\right]$$

$$\xrightarrow[r_4+4r_1]{r_2-r_1} \left[\begin{array}{cccc|c} -1 & 0 & -1 & 4 & 40 \\ 0 & 4 & 0 & -4 & 20 \\ 0 & -1 & 4 & -1 & 70 \\ 0 & -1 & -4 & 15 & 190 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & -4 & -40 \\ 0 & 1 & 0 & -1 & 5 \\ 0 & -1 & 4 & -1 & 70 \\ 0 & -1 & -4 & 15 & 190 \end{array}\right]$$

$$\xrightarrow[r_4+r_2]{r_3+r_2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & -4 & -40 \\ 0 & 1 & 0 & -1 & 5 \\ 0 & 0 & 4 & -2 & 75 \\ 0 & 0 & -4 & +14 & 195 \end{array}\right] \xrightarrow{r_4+r_3} \left[\begin{array}{cccc|c} 1 & 0 & 1 & -4 & -40 \\ 0 & 1 & 0 & -1 & 5 \\ 0 & 0 & 4 & -2 & 75 \\ 0 & 0 & 0 & 12 & 270 \end{array}\right]$$

$$\xrightarrow[r_3/2]{r_4/12} \left[\begin{array}{cccc|c} 1 & 0 & 1 & -4 & -40 \\ 0 & 1 & 0 & -1 & 5 \\ 0 & 0 & 2 & -1 & 75/2 \\ 0 & 0 & 0 & 1 & 45/2 \end{array}\right] \xrightarrow[\substack{r_2+r_4 \\ r_3+r_4}]{r_1+4r_4} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 50 \\ 0 & 1 & 0 & 0 & 55/2 \\ 0 & 0 & 2 & 0 & 60 \\ 0 & 0 & 0 & 1 & 45/2 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 20 \\ 0 & 1 & 0 & 0 & 55/2 \\ 0 & 0 & 1 & 0 & 30 \\ 0 & 0 & 0 & 1 & 45/2 \end{array}\right]$$

$$\boxed{T_1 = 20^\circ,\ T_2 = \frac{55}{2}^\circ,\ T_3 = 30^\circ,\ T_4 = \frac{45^\circ}{2}}$$

Problem 12: Techniques for solving linear equations also apply to nonlinear equations if they allow a nice substitution. Make a $x_1 = x^2$ and $x_2 = y^2$ substitution to solve:

$$\begin{aligned} x^2 + y^2 &= 4 \\ x^2 - y^2 &= 1 \end{aligned}$$

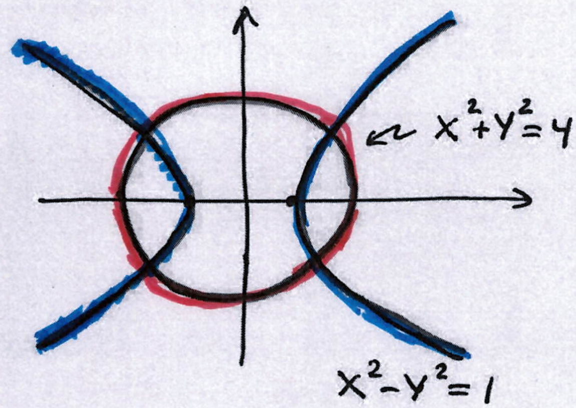
← horizontally opening hyperbol

How many solutions are in the solution set ? Does this make sense graphically ? Make a quick sketch of what is going on here.

$$\pm\begin{pmatrix} \boxed{x^2 + y^2 = 4} \\ \boxed{x^2 - y^2 = 1} \end{pmatrix}$$

$(+)\quad 2x^2 = 5 \longrightarrow x = \pm\sqrt{\frac{5}{2}}$

$(-)\quad 2y^2 = 3 \longrightarrow y = \pm\sqrt{\frac{3}{2}}$



There are 4 solutions corresponding to the 4 points of intersection

$$\left(\sqrt{\tfrac{5}{2}}, \sqrt{\tfrac{3}{2}}\right), \left(\sqrt{\tfrac{5}{2}}, -\sqrt{\tfrac{3}{2}}\right), \left(-\sqrt{\tfrac{5}{2}}, \sqrt{\tfrac{3}{2}}\right), \left(-\sqrt{\tfrac{5}{2}}, -\sqrt{\tfrac{3}{2}}\right)$$

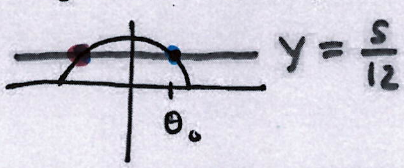
Problem 13: Solve $\cos\theta - \sin\beta = 1/2$ and $\cos\theta + \sin\beta = 1/3$ given that $-\pi/2 \le \theta, \beta \le \pi/2$

$$\pm\begin{pmatrix} \cos\theta - \sin\beta = 1/2 \\ \cos\theta + \sin\beta = 1/3 \end{pmatrix}$$

$(+)\quad 2\cos\theta = 5/6$

$\cos\theta = \frac{5}{12}$

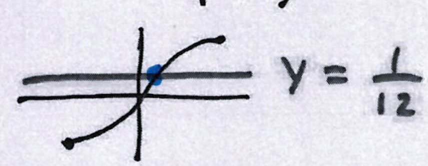
$\theta_0 = \cos^{-1}\left(\frac{5}{12}\right)$



$(-)\quad -2\sin\beta = -1/6$

$\sin\beta = \frac{1}{12}$

$\beta = \sin^{-1}\left(\frac{1}{12}\right)$



$$\boxed{\theta = \pm\cos^{-1}\left(\tfrac{5}{12}\right) \text{ and } \beta = \sin^{-1}\left(\tfrac{1}{12}\right)}$$ (two solutions)

Problem 14: Interpolation problem:

(a.) Find $f(x) = Ax^3 + Bx^2 + Cx + D$ for which $(0,4), (1,8), (2,24), (-1,0) \in \text{graph}(f)$.

$$f(0) = \underline{D = 4}.$$

$$+\begin{pmatrix} f(1) = A + B + C + 4 = 8 \\ f(-1) = -A + B - C + 4 = 0 \end{pmatrix} \quad (*)$$

$$2B + 8 = 8 \Rightarrow \underline{B = 0}.$$

$$f(2) = 8A + 2C + 4 = 24 \Rightarrow \begin{pmatrix} 8A + 2C = 20 \\ -\;8A + 8C = 32 \end{pmatrix}$$

From $(*)$ $\quad A + C + 4 = 8 \Rightarrow$

$$-6C = -12$$

$$\underline{C = 2}.$$

Finally, $\underline{A + 2 + 4 = 8}$

$$\Rightarrow \underline{A = 2}.$$

$$\boxed{f(x) = 2x^3 + 0 \cdot x^2 + 2x + 4} \leftarrow \boxed{\begin{matrix} A = 2 \\ B = 0 \\ C = 2 \\ D = 4 \end{matrix}}$$

(b.) Find the set of all cubic polynomials whose graphs contain $(1,2)$ and $(-1,2)$.

$$f(x) = Ax^3 + Bx^2 + Cx + D$$
$$f(1) = A + B + C + D = 2$$
$$f(-1) = -A + B - C + D = 2$$

$$\begin{array}{cccc} A & B & C & D \end{array}$$
$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ -1 & 1 & -1 & 1 & 2 \end{array}\right] \xrightarrow{r_2 + r_1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 2 & 0 & 2 & 4 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 2 \\ 0 & 1 & 0 & 1 & 2 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 2 \end{array}\right] \longrightarrow \begin{matrix} A = -C \\ B = 2 - D \end{matrix}$$

$$\boxed{\{ -Cx^3 + (2 - D)x^2 + Cx + D \mid C, D \in \mathbb{R} \}}$$

Problem 15: I suggest using a mixture of row-reduction and back-substitution to solve the following system.

$$\left.\begin{aligned} x_1 + x_2 + x_3 &= 6 \\ x_1 + 2x_2 + 3x_3 &= 14 \\ x_1 + 2x_2 + 4x_3 &= 17 \end{aligned}\right\} \text{ solve these } 1^{st} \text{ then plug-into}$$

$$\left.\begin{aligned} 2x_3 - 4x_4 - 5x_5 - 6x_6 &= -71 \\ -x_3 + 4x_4 + 6x_5 + 7x_6 &= 85 \\ -x_3 + 4x_4 + 7x_5 + 5x_6 &= 78 \end{aligned}\right\} **$$

$$\begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 14 \\ 1 & 2 & 4 & 17 \end{array}\right] \end{array} \xrightarrow[r_3 - r_1]{r_2 - r_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 8 \\ 0 & 1 & 3 & 11 \end{array}\right] \xrightarrow[r_3 - r_2]{r_1 - r_2} \underbrace{\left[\begin{array}{ccc|c} 1 & 0 & -1 & -2 \\ 0 & 1 & 2 & 8 \\ 0 & 0 & 1 & 3 \end{array}\right]}_{*}$$

From $*$ we find $X_3 = 3$ and

$$X_2 = 8 - 2X_3 = 8 - 6 = 2$$
$$X_1 = -2 + X_3 = -2 + 3 = 1$$

Thus $\underline{X_1 = 1,\ X_2 = 2,\ X_3 = 3}$.

We find $**$ gives system,

$$-4X_4 - 5X_5 - 6X_6 = -71 - 2(3) = -77$$
$$4X_4 + 6X_5 + 7X_6 = 85 + 3 = 88$$
$$4X_4 + 7X_5 + 5X_6 = 78 + 3 = 81$$

$$\begin{array}{c} \begin{array}{ccc} (x_4 & x_5 & x_6) \end{array} \\ \left[\begin{array}{ccc|c} -4 & -5 & -6 & -77 \\ 4 & 6 & 7 & 88 \\ 4 & 7 & 5 & 81 \end{array}\right] \end{array} \xrightarrow[r_3 + r_1]{r_2 + r_1} \left[\begin{array}{ccc|c} -4 & -5 & -6 & -77 \\ 0 & 1 & 1 & 11 \\ 0 & 2 & -1 & 4 \end{array}\right]$$

$$\xrightarrow[r_3 - 2r_2]{r_1 + 5r_2} \left[\begin{array}{ccc|c} -4 & 0 & -1 & -22 \\ 0 & 1 & 1 & 11 \\ 0 & 0 & -3 & -18 \end{array}\right] \sim \left[\begin{array}{ccc|c} -4 & 0 & -1 & -22 \\ 0 & 1 & 1 & 11 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

$$\xrightarrow[r_2 - r_3]{r_1 + r_3} \left[\begin{array}{ccc|c} -4 & 0 & 0 & -16 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 5 \\ 0 & 0 & 1 & 6 \end{array}\right]$$

$$\therefore\ X_4 = 4,\ X_5 = 5,\ X_6 = 6$$

$$\boxed{\text{Solution } (1, 2, 3, 4, 5, 6)}$$