Show your work and box answers. This pdf should be printed and your solution should be handwritten on the printout. Once complete, please staple in upper left corner. Thanks.

Suggested Reading You may find the following helpful resources beyond lecture,

(a.) Chapter 2 of my lecture notes for Math 221

Problem 16: Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$. Calculate $AB - BA$. Do A and B commute ?

$$AB = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix} = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 3 & 6 \end{bmatrix}$$

$$AB - BA = \begin{bmatrix} 6 & 1 \\ 3 & 2 \end{bmatrix} - \begin{bmatrix} 2 & 1 \\ 3 & 6 \end{bmatrix} = \boxed{\begin{bmatrix} 4 & 0 \\ 0 & -4 \end{bmatrix}}$$

No, $AB \neq BA$, so A & B do not commute.

Problem 17: Let $v = [1, 2, 3, 4] \in \mathbb{R}^{1\times 4}$ and $w = (5, 6, 7, 8) = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} \in \mathbb{R}^{4\times 1}$. Calculate vw and wv.

$$vw = [1, 2, 3, 4]\begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} = 5 + 12 + 21 + 32 = \boxed{70}$$

$$wv = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}[1, 2, 3, 4] = \boxed{\begin{bmatrix} 5 & 10 & 15 & 20 \\ 6 & 12 & 18 & 24 \\ 7 & 14 & 21 & 28 \\ 8 & 16 & 24 & 32 \end{bmatrix}}$$

Problem 18: A matrix is said to be **antisymmetric** if $A^T = -A$. In terms of components, that means $A_{ji} = -A_{ij}$ for all i, j. Consider the 3×3 matrix $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$. If $\mathrm{Col}_1(A) = \begin{bmatrix} 0 \\ 2 \\ 7 \end{bmatrix}$ and $\mathrm{Row}_3(A) = [7, 3, 0]$ then find A.

$$A = \begin{bmatrix} 0 & A_{12} & A_{13} \\ 2 & A_{22} & A_{23} \\ 7 & 3 & 0 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0 & 2 & 7 \\ A_{12} & A_{22} & 3 \\ A_{13} & A_{23} & 0 \end{bmatrix} = -\begin{bmatrix} 0 & A_{12} & A_{13} \\ 2 & A_{22} & A_{23} \\ 7 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -A_{12} & -A_{13} \\ -2 & -A_{22} & -A_{23} \\ -7 & -3 & 0 \end{bmatrix}$$

We find $A_{12} = -2$ and $A_{13} = -7$ and $A_{23} = -3$ and of course $A_{22} = -A_{22} \Rightarrow A_{22} = 0$

$$\therefore \boxed{A = \begin{bmatrix} 0 & -2 & -7 \\ 2 & 0 & -3 \\ 7 & 3 & 0 \end{bmatrix}}$$

Problem 19: Let $N = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Calculate N^2 and N^3 and show $N^k = 0$ for $k \geq 4$.

$$N^2 = NN = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

$$N^3 = NN^2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}}$$

$$N^4 = NN^3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = 0.$$

For $k \geq 4$, $N^k = N^4 N^{k-4} = 0 \cdot N^{k-4} = \boxed{0}$

Problem 20: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix}$ and $M = \begin{bmatrix} 1 & 0 \\ 3 & -1 \\ 4 & 1 \\ 5 & 2 \end{bmatrix}$. If possible calculate the matrix quantities below. If not, explain why such a calculation does not fall under our definition of matrix addition or multiplication.

(a) $A + M$

(2×3) (4×2) : cannot add, dimension mismatch.

(b) A^2 = AA

need to match.

$(2\times3)(2\times3)$: cannot multiply $3 \neq 2$.

(c) AM : $(2\times3)(4\times2)$

do not match, cannot multiply.

(d) MA $= \begin{bmatrix} 1 & 0 \\ 3 & -1 \\ 4 & 1 \\ 5 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & 2 \\ 9 & 14 & 19 \\ 15 & 22 & 29 \end{bmatrix}}$

$(4\times2)(2\times3) : (4\times3)$

(e) $AA^T + 2MM^T$ ← would be $(2\times2) + (4\times4)$ cannot add, different sizes.

$(2\times3)(3\times2)$

$(4\times2)(2\times4)$

(f) $A^TA + 2M^TM$

$\underbrace{(3\times2)(2\times3)}_{3\times3}$ $\underbrace{(2\times4)(4\times2)}_{2\times2}$

again, cannot add, sizes do not match.

Problem 21: Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ a & b & c & d \end{bmatrix}$ where a, b, c, d are constants. Let e_i denote the i-th standard basis vector for $\mathbb{R}^4$ where $i = 1, 2, 3, 4$. For example, $e_2 = (0, 1, 0, 0) \in \mathbb{R}^{4\times 1} = \mathbb{R}^4$. Also, let $\bar{e}_j$ denote the j-th standard basis element for $\mathbb{R}^3 = \mathbb{R}^{3\times 1}$. For example, $\bar{e}_2 = (0, 1, 0)$. With this notation in mind, calculate:

(a) $Ae_2 = \text{col}_2(A) = \underline{\begin{bmatrix} 2 \\ 6 \\ b \end{bmatrix}}.$

(b) $\bar{e}_3^T A = \text{row}_3(A) = \underline{[a, b, c, d]}.$

(c) $A[e_1|e_2] = [Ae_1 | Ae_2] = [\text{col}_1(A) | \text{col}_2(A)] = \underline{\begin{bmatrix} 1 & 2 \\ 5 & 6 \\ a & b \end{bmatrix}}.$

(d) $\left[\frac{\bar{e}_2^T}{\bar{e}_3^T}\right] A = \left[\frac{e_2^T A}{e_3^T A}\right] = \left[\frac{\text{row}_2(A)}{\text{row}_3(A)}\right] = \underline{\begin{bmatrix} 5 & 6 & 7 & 8 \\ a & b & c & d \end{bmatrix}}.$

(e) $\bar{e}_3^T A e_2 = [0, 0, 1] \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ a & b & c & d \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$

$= [0, 0, 1] \begin{bmatrix} 2 \\ 6 \\ b \end{bmatrix}$

$= \underline{b}.$ (no accident, $e_3^T A e_2 = A_{32} = b$.)

Problem 22: Let $X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ and $A = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$. Solve $AX = I$ where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the 2×2 identity matrix. Does your solution also solve $XA = I$?

$$AX = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} = \left[\begin{array}{c|c} x-2z & y-2w \\ 3x+z & 3y+w \end{array}\right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{aligned} x - 2z &= 1 \\ 3x + z &= 0 \\ y - 2w &= 0 \\ 3y + w &= 1 \end{aligned}$$

$$\begin{array}{c} \begin{array}{cccc} x & z & y & w \end{array} \\ \left[\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 1 \\ 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 3 & 1 & 1 \end{array}\right] \end{array} \xrightarrow[r_4 - 3r_3]{r_2 - 3r_1} \left[\begin{array}{cccc|c} 1 & -2 & 0 & 0 & 1 \\ 0 & 7 & 0 & 0 & -3 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 7 & 1 \end{array}\right] \begin{array}{l} \\ \longrightarrow z = -3/7 \\ \\ \longrightarrow w = 1/7 \end{array}$$

Then $x - 2z = 1 \Rightarrow x + \frac{6}{7} = 1 \Rightarrow \underline{x = 1/7}$.

and $y - 2w = 0 \Rightarrow y = 2w \Rightarrow \underline{y = 2/7}$.

Hence $\boxed{X = \begin{bmatrix} 1/7 & 2/7 \\ -3/7 & 1/7 \end{bmatrix}}$

Observe $AX = \frac{1}{7}\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = I.$

Likewise $XA = \frac{1}{7}\begin{bmatrix} 1 & 2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix} = \frac{1}{7}\begin{bmatrix} 7 & 0 \\ 0 & 7 \end{bmatrix} = I$

<u>YES, the solution X solves both $AX = I$ & $XA = I$</u>.

Problem 23: Given $x = (x_1, x_2, x_3)$ and $A = \begin{bmatrix} 1 & 3 & -5 \\ 1 & 4 & -8 \\ -3 & -7 & 9 \end{bmatrix}$, solve $Ax = 0$ and express the solution in parametric vector form.

Consider,

$$A = \begin{bmatrix} 1 & 3 & -5 \\ 1 & 4 & -8 \\ -3 & -7 & 9 \end{bmatrix} \xrightarrow[r_3+3r_1]{r_2-r_1} \begin{bmatrix} 1 & 3 & -5 \\ 0 & 1 & -3 \\ 0 & 2 & -6 \end{bmatrix} \xrightarrow{r_3-2r_2} \begin{bmatrix} 1 & 3 & -5 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_1-3r_2} \underbrace{\begin{bmatrix} 1 & 0 & 4 \\ 0 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{rref }(A)}$$

Thus, by column-by-column structure of row reduction,

$$\text{rref}\,[A|0] = \left[\begin{array}{ccc|c} 1 & 0 & 4 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Consequently $AX = 0 \Rightarrow X_1 + 4X_3 = 0$ and $X_2 - 3X_3 = 0$

Hence $X_1 = -4X_3$ and $X_2 = 3X_3$ for some $X_3 \in \mathbb{R}$.

Thus our solution for $AX = 0$ can be written as

$$X = \begin{bmatrix} X_1 \\ X_2 \\ X_3 \end{bmatrix} = \begin{bmatrix} -4X_3 \\ 3X_3 \\ X_3 \end{bmatrix} = \boxed{X_3 \begin{bmatrix} -4 \\ 3 \\ 1 \end{bmatrix}}$$

In other words, the solution set is $\underline{\text{span}\,\{(-4,3,1)\}}$.

Problem 24: Let $v_1 = (1,2,2)$ and $v_2 = (0,1,-1)$.

(a.) what condition must be given for a, b, c for $(a,b,c) \in \text{span}(v_1, v_2)$?

(b.) what condition is needed for $\{v_1, v_2, (a,b,c)\}$ to by a linearly independent (LI) set ?

(a.) $\underline{c_1 V_1 + c_2 V_2 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}}\ *$ $\iff$ $\begin{bmatrix} V_1 | V_2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$

Hence consider $\left[\begin{array}{cc|c} 1 & 0 & a \\ 2 & 1 & b \\ 2 & -1 & c \end{array}\right] \xrightarrow[r_3 - 2r_1]{r_2 - 2r_1} \left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b-2a \\ 0 & -1 & c-2a \end{array}\right]$

Continuing, $\xrightarrow{r_3 + r_2} \left[\begin{array}{cc|c} 1 & 0 & a \\ 0 & 1 & b-2a \\ 0 & 0 & -4a+b+c \end{array}\right]$

Thus $*$ is consistent only if $\underbrace{\boxed{-4a+b+c=0}}_{\text{Condition needed for } (a,b,c) \in \text{span}\{V_1, V_2\}}$

(b.) $\{V_1, V_2, (a,b,c)\}$ LI

iff $c_1 V_1 + c_2 V_2 + c_3 (a,b,c) = 0 \implies c_1 = c_2 = c_3 = 0$

Hence consider

$$\left[\begin{array}{c|c|c|c} V_1 & V_2 & \begin{matrix} a \\ b \\ c \end{matrix} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & a & 0 \\ 0 & 1 & b-2a & 0 \\ 0 & 0 & -4a+b+c & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

provided $-4a+b+c \neq 0$ (we can complete the backwords pass with $r_3/(-4a+b+c)$ then $\xrightarrow{r_2 - (b-2a)r_3}$ and $\xrightarrow{r_1 - ar_3}$)

$\boxed{\text{Thus the set } \{V_1, V_2, (a,b,c)\} \text{ is LI if } -4a+b+c \neq 0}$

Problem 25: Solve the chemical reaction $B_2S_3 + H_2O \to H_3BO_3 + H_2S$ via linear algebra.

I'll use $\begin{bmatrix} \text{Boron} \\ \text{Sulfur} \\ \text{Hydrogen} \\ \text{oxygen} \end{bmatrix}$ we seek x, y, z, w such that $x\begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} = z\begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \end{bmatrix} + w\begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$

$$\begin{array}{cccc} x & y & z & w \end{array}$$

$$\left[\begin{array}{cccc|c} 2 & 0 & -1 & 0 & 0 \\ 3 & 0 & 0 & -1 & 0 \\ 0 & 2 & -3 & -2 & 0 \\ 0 & 1 & -3 & 0 & 0 \end{array}\right] \xrightarrow[\substack{2r_2 \\ 2r_4}]{3r_1} \left[\begin{array}{cccc|c} 6 & 0 & -3 & 0 & 0 \\ 6 & 0 & 0 & -2 & 0 \\ 0 & 2 & -3 & -2 & 0 \\ 0 & 2 & -6 & 0 & 0 \end{array}\right] \xrightarrow[r_4 - r_3]{r_2 - r_1} \left[\begin{array}{cccc|c} 6 & 0 & -3 & 0 & 0 \\ 0 & 0 & 3 & -2 & 0 \\ 0 & 2 & -3 & -2 & 0 \\ 0 & 0 & -3 & 2 & 0 \end{array}\right]$$

$$\xrightarrow[\substack{r_2 + r_4 \\ r_3 - r_4}]{r_1 - r_4} \left[\begin{array}{cccc|c} 6 & 0 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & -4 & 0 \\ 0 & 0 & -3 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 0 & -1/3 & 0 \\ 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 1 & -2/3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(Select $w = 3$ to obtain integer solution)

Thus $x = w/3$, $y = 2w$ and $z = 2w/3 \longrightarrow x = 1,\ y = 6,\ z = 2$

$$\boxed{B_2S_3 + 6H_2O \longrightarrow 2H_3BO_3 + 3H_2S}$$

Problem 26: Suppose $Av = b$ and $Aw = b$ for a given matrix A and column vector b. If $v \neq w$ then show that the columns of A are linearly dependent.

Observe $x = v - w \neq 0$ and $Ax = A(v-w) = Av - Aw = b - b = 0$.

Thus $\exists\, x_1, x_2, \dots, x_n \in \mathbb{R}$, not all zero for which

$$Ax = x_1 \mathrm{col}_1(A) + x_2 \mathrm{col}_2(A) + \cdots + x_n \mathrm{col}_n(A) = 0.$$

Hence $\{\mathrm{col}_1(A), \mathrm{col}_2(A), \dots, \mathrm{col}_n(A)\}$ is not LI which is to say the set of columns of A is <u>linearly dependent</u>.

Problem 27: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ a & b & c \\ 3 & 2 & 1 \end{bmatrix}$ and $R_1 = \begin{bmatrix} a & b & c \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$ and $R_2 = \begin{bmatrix} 1 & 2 & 3 \\ a & b & c \\ 2 & 0 & -2 \end{bmatrix}$. Notice both R_1 and R_2 are obtained by performing an elementary row operation on A. Find 3×3 matrices E_1 and E_2 for which $R_1 = E_1 A$ and $R_2 = E_2 A$.

$$A \xrightarrow{r_1 \leftrightarrow r_2} R_1 \Rightarrow \boxed{E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}} \quad \text{has } E_1 A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ a & b & c \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} a & b & c \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \checkmark$$

$$A \xrightarrow{r_3 - r_1} R_2 \Rightarrow \boxed{E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}}$$

I'll check it, $E_2 A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ a & b & c \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ a & b & c \\ 2 & 0 & -2 \end{bmatrix} \stackrel{\checkmark}{=} R_2$

Problem 28: If $A \to R$ under a row-operation then there exists an elementary matrix E for which $R = EA$. Find elementary matrices $E_1, E_2, \dots, E_k$ for which $\text{rref}(A) = E_k E_{k-1} \dots E_2 E_1 A$ for:

(a) $k = 3$ will suffice for $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \end{bmatrix} \xrightarrow{r_2 - r_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix} \xrightarrow{r_2/2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{r_1 - r_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

see solution for (a.)

Alternatively, $\begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \end{bmatrix} \xrightarrow{r_2 - r_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix} \xrightarrow{r_1 - \frac{1}{2}r_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix} \xrightarrow{r_2/2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ this gives

(b) $k = 2$ will suffice for $A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 0 & 0 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \xrightarrow{r_1 - 2r_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$.

$$\underbrace{\begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_2}\underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E_3}\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 0 & 0 \end{bmatrix}}_{A} = \begin{bmatrix} 1 & -2 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} = \text{rref}(A).$$

PROBLEM 28

(a.) $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \end{bmatrix} \xrightarrow{r_2 - r_1} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix} \xrightarrow{r_2/2} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{r_1 - r_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}$

$$\underbrace{\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}}_{E_3}\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}}_{E_2}\underbrace{\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}}_{E_1}\underbrace{\begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \end{bmatrix}}_{A} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \text{rref}(A).$$

Alternatively,

$$\underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}}_{E_3}\underbrace{\begin{bmatrix} 1 & -1/2 \\ 0 & 1 \end{bmatrix}}_{E_2}\underbrace{\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}}_{E_1}\begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & -1/2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 \\ 0 & 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix} = \text{rref}(A).$$

There are many correct answers in that row-reduction can be done in many different ways.

Problem 29: We defined the matrix product of $A \in \mathbb{R}^{m\times p}$ with $B \in \mathbb{R}^{p\times n}$ by $(AB)_{ij} = \sum_{k=1}^{p} A_{ik}B_{kj}$ for all $1 \leq i \leq m$ and $1 \leq j \leq n$. In the case $m = n$ we note the matrix $AB \in \mathbb{R}^{n\times n}$. Let us define the trace of a square matrix M to be the sum of its diagonals. To be precise,

$$\text{trace}(M) = \sum_{i=1}^{n} M_{ii}.$$

(a) Let I be the $n \times n$ identity matrix. Calculate trace(I).

(b) Suppose $A \in \mathbb{R}^{n\times p}$ and $B \in \mathbb{R}^{p\times n}$. Show trace($AB$) = trace($BA$).

(a.) $$\text{trace}(I_n) = \sum_{i=1}^{n} (I_n)_{ii} = \sum_{i=1}^{n} \delta_{ii} = \sum_{i=1}^{n} 1 = \underbrace{1+1+\cdots+1}_{n\text{-terms}} = \boxed{n}$$

(b.) $$\text{trace}(AB) = \sum_{i=1}^{n} (AB)_{ii}$$
$$= \sum_{i=1}^{n} \sum_{j=1}^{p} A_{ij} B_{ji}$$
$$= \sum_{j=1}^{p} \sum_{i=1}^{n} B_{ji} A_{ij}$$
commute #'s and swap sums.
$$= \sum_{j=1}^{p} (BA)_{jj} = \text{trace}(BA).$$

Problem 30: Let $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$. Use Matlab, to calculate $(AA^T)^{100}$ and $(A^TA)^{100}$.

$$(A^* A^\wedge T)^\wedge 100 = 10^{55} \begin{bmatrix} 5.0906 & 0 & 3.1461 \\ 0 & 0 & 0 \\ 3.1461 & 0 & 1.9444 \end{bmatrix}$$

$$(A^\wedge T * A)^\wedge 100 = 10^{55} \begin{bmatrix} 3.6836 & 1.4070 & 2.2766 & 2.2766 \\ 1.4070 & 0.5374 & 0.8696 & 0.8696 \\ 2.2766 & 0.8696 & 1.4070 & 1.4070 \\ 2.2766 & 0.8696 & 1.4070 & 1.4070 \end{bmatrix}$$

Assuming that
$$1.0e+55 = 10^{55}$$