MATH 221 MISSION 3: INVERSE MATRICES AND THE CCP (30PTS)

Show your work and box answers. This pdf should be printed and your solution should be handwritten on the printout. Once complete, please staple in upper left corner. Thanks.

Suggested Reading You may find the following helpful resources beyond lecture.

(a.) Chapter 2 and §9.1 of my lecture notes for Math 221

Problem 31: Consider $v = (1,2,3,4)$ and $w = (0,1,1,0)$. Determine if $b_1 = (2,3,5,8) \in \text{span}(v,w)$. Is $b_2 = (1,0,0,0) \in \text{span}(v,w)$? Calculate $\text{rref}[v|w|b_1|b_2]$ and use the CCP (column correspondence property) to answer the questions.

$$[v|w|b_1|b_2] = \left[\begin{array}{cc|c|c} 1 & 0 & 2 & 1 \\ 2 & 1 & 3 & 0 \\ 3 & 1 & 5 & 0 \\ 4 & 0 & 8 & 0 \end{array}\right] \begin{array}{c} \xrightarrow{r_2-2r_1} \\ \xrightarrow{r_3-3r_1} \\ \xrightarrow{r_4-4r_1} \end{array} \left[\begin{array}{cc|c|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 1 & -1 & -3 \\ 0 & 0 & 0 & -4 \end{array}\right]$$

$$\xrightarrow{r_3-r_2} \left[\begin{array}{cc|c|c} 1 & 0 & 2 & 1 \\ 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & -4 \end{array}\right] \sim \left[\begin{array}{cc|c|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] = \text{rref}[v|w|b_1|b_2] = R$$

From rref $[v|w|b_1|b_2]$ we can easily see that

$$col_3(R) = 2\,col_1(A) - col_2(R)$$

thus by CCP we find

$$b_1 = 2v - w \quad \therefore \quad \boxed{b_1 \in span\{v, w\}}$$

Likewise, since $col_4(R) \notin span\{col_1(R), col_2(R)\}$ the CCP gives $\boxed{b_2 \notin span\{v, w\}}$.

Problem 32: Suppose that $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 & 0 & 4 \\ 0 & 1 & -2 & 0 & -3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Given that $\text{col}_1(A) = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix}$ and $\text{col}_3(A) = \begin{bmatrix} -1 \\ -1 \\ -1 \\ 3 \end{bmatrix}$ and $\text{col}_4(A) = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$, use the CCP to find A.

$$\text{Col}_3(A) = \text{Col}_1(A) - 2\,\text{col}_2(A) \quad \text{by CCP}$$

Thus $\text{Col}_2(A) = \frac{1}{2}\left(\text{Col}_1(A) - \text{col}_3(A)\right) = \frac{1}{2}\left(\begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix} - \begin{bmatrix} -1 \\ -1 \\ -1 \\ 3 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix}$

By CCP,

$$\text{Col}_5(A) = 4\,\text{col}_1(A) - 3\,\text{col}_2(A) + 2\,\text{col}_4(A)$$

$$= 4\begin{bmatrix} 3 \\ 3 \\ 3 \\ 3 \end{bmatrix} - 3\begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 12 - 6 + 2 \\ 12 - 6 + 0 \\ 12 - 6 + 2 \\ 12 - 0 + 0 \end{bmatrix}$$

$$= \begin{bmatrix} 8 \\ 6 \\ 8 \\ 12 \end{bmatrix}$$

Put it all together,

$$\boxed{A = \begin{bmatrix} 3 & 2 & -1 & 1 & 8 \\ 3 & 2 & -1 & 0 & 6 \\ 3 & 2 & -1 & 1 & 8 \\ 3 & 0 & 3 & 0 & 12 \end{bmatrix}}$$

Problem 33: Let $A = \begin{bmatrix} 1 & 9 & 4 & 14 & 8 \\ 2 & 2 & 2 & 6 & 0 \\ 3 & 1 & 0 & 4 & -2 \end{bmatrix}$.

(a.) Calculate rref(A) and use the CCP to write each non-pivot column of A as a linear combination of the pivot columns

(b.) Find a basis for Null(A) $= \{x \in \mathbb{R}^5 \mid Ax = 0\}$ and express an arbitary element of the nullspace of A as a linear combination of the basis

(a.)

$$A = \begin{bmatrix} 1 & 9 & 4 & 14 & 8 \\ 2 & 2 & 2 & 6 & 0 \\ 3 & 1 & 0 & 4 & -2 \end{bmatrix} \xrightarrow[r_3 - 3r_1]{r_2 - 2r_1} \begin{bmatrix} 1 & 9 & 4 & 14 & 8 \\ 0 & -16 & -6 & -22 & -16 \\ 0 & -26 & -12 & -38 & -26 \end{bmatrix}$$

$$\xrightarrow{r_3 - \frac{26}{16} r_2} \begin{bmatrix} 1 & 9 & 4 & 14 & 8 \\ 0 & -16 & -6 & -22 & -16 \\ 0 & 0 & -9/4 & -9/4 & 0 \end{bmatrix}$$

$$\xrightarrow[-4r_3/9]{r_2/-16} \begin{bmatrix} 1 & 9 & 4 & 14 & 8 \\ 0 & 1 & 3/8 & 11/8 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\xrightarrow[r_1 - 4r_3]{r_2 - \frac{3}{8} r_3} \begin{bmatrix} 1 & 9 & 0 & 10 & 8 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix} \xrightarrow{r_1 - 9r_2} \begin{bmatrix} 1 & 0 & 0 & 1 & -1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}$$

By CCP and examination of rref (A) ↗ we see that

$$\boxed{\begin{aligned} col_4(A) &= col_1(A) + col_2(A) + col_3(A) \\ col_5(A) &= -col_1(A) + col_2(A) \end{aligned}}$$

as well as

easily verified by explicit calculation from A.

(b.) $Ax = 0$ implies $x_1 + x_4 - x_5 = 0$ etc. hence,

$$\begin{aligned} x &= (-x_4 + x_5, -x_4 - x_5, -x_4, x_4, x_5) \\ &= x_4(-1,-1,-1,1,0) + x_5(1,-1,0,0,1) \end{aligned}$$

← How any $x \in Null(A)$ is linear combo of basis

$\boxed{\{(-1,-1,-1,1,0), (1,-1,0,0,1)\}}$ is basis for $Null(A)$

Btw, $nullity(A) = 2$, $rank(A) = 3 = dim(Col(A))$.

Problem 34: Let $A = \begin{bmatrix} 1 & 0 & 2 \\ -2 & 0 & 1 \\ 0 & 3 & 0 \end{bmatrix}$. Calculate A^{-1} and solve $Ax = (a, b, c)$ where a, b, c are constants.

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_2 + 2r_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 0 & 5 & 2 & 1 & 0 \\ 0 & 3 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{r_1 - \frac{2}{5}r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & -2/5 & 0 \\ 0 & 0 & 5 & 2 & 1 & 0 \\ 0 & 3 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & -2/5 & 0 \\ 0 & 0 & 1 & 2/5 & 1/5 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/3 \end{array}\right]$$

$$\xrightarrow{r_2 \leftrightarrow r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/5 & -2/5 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1/3 \\ 0 & 0 & 1 & 2/5 & 1/5 & 0 \end{array}\right] \quad \boxed{A^{-1} = \begin{bmatrix} 1/5 & -2/5 & 0 \\ 0 & 0 & 1/3 \\ 2/5 & 1/5 & 0 \end{bmatrix}}$$

A^{-1}

$$Ax = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \Rightarrow A^{-1}Ax = x = A^{-1}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \frac{1}{15}\begin{bmatrix} 3 & -6 & 0 \\ 0 & 0 & 5 \\ 6 & 3 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

$$x = \begin{bmatrix} \frac{1}{15}(3a - 6b) \\ c/3 \\ \frac{1}{15}(6a + 3b) \end{bmatrix}$$

a.k.a

$$\boxed{x = \left(\frac{a-2b}{5}, \frac{c}{3}, \frac{2a+b}{5}\right)}$$

Problem 35: Let $A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 0 \\ 0 & 2 & 1 \end{bmatrix}$. Calculate A^{-1}.

$$[A|I] = \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 2 & 1 & 0 & 0 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{r_2 - 2r_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -4 & -2 & 1 & 0 \\ 0 & 2 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{r_3 - 2r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & -4 & -2 & 1 & 0 \\ 0 & 0 & 9 & 4 & -2 & 1 \end{array}\right] \xrightarrow[r_2 + \frac{4}{9}r_3]{r_1 - \frac{2}{9}r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/9 & 4/9 & -2/9 \\ 0 & 1 & 0 & -2/9 & 1/9 & 4/9 \\ 0 & 0 & 9 & 4 & -2 & 1 \end{array}\right]$$

$$\xrightarrow{r_3/9} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/9 & 4/9 & -2/9 \\ 0 & 1 & 0 & -2/9 & 1/9 & 4/9 \\ 0 & 0 & 1 & 4/9 & -2/9 & 1/9 \end{array}\right] \quad \therefore \quad \boxed{A^{-1} = \frac{1}{9}\begin{bmatrix} 1 & 4 & -2 \\ -2 & 1 & 4 \\ 4 & -2 & 1 \end{bmatrix}}$$

Problem 36: Let $A = \begin{bmatrix} 1 & 2 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$. Calculate A^{-1} and solve $Ax = (1, 2, 3, 4, 5)$.

$$[A|I] = \left[\begin{array}{ccccc|ccccc} 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right] \sim \left[\begin{array}{ccccc|ccccc} 1 & 0 & 0 & 0 & 0 & 1 & -2 & 4 & -8 & 16 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & -2 & 4 & -8 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$$Ax = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix} \Rightarrow X = A^{-1}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -2 & 4 & -8 & 16 \\ 0 & 1 & -2 & 4 & -8 \\ 0 & 0 & 1 & -2 & 4 \\ 0 & 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{bmatrix}$$

$$= (1-4+12-32+80,\ 2-6+16-40,\ 3-8+20,\ 4-10,\ 5)$$

$$= \boxed{(57, -28, 15, -6, 5)}$$

Problem 37: Given that $A^{-1} = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 7 \end{bmatrix}$, calculate $(A^TB)^{-1}$.

$$(A^TB)^{-1} = B^{-1}(A^T)^{-1}$$

$$= B^{-1}(A^{-1})^T$$

$$= \begin{bmatrix} -1 & 1 \\ 1 & 7 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} -2 & 2 \\ 18 & 22 \end{bmatrix}}$$

Problem 38: Consider a 5×5 matrix A along with vectors $v_1, v_2, v_3, v_4, v_5 \in \mathbb{R}^5$ for which:

$$Av_1 = e_1 + e_2,\ \ Av_2 = e_1 - e_2,\ \ Av_3 = e_5,\ \ Av_4 = \cos\theta e_3 + \sin\theta e_4,,\ \ Av_5 = -\sin\theta e_3 + \cos\theta e_4$$

where $e_1 = (1,0,0,0,0)$ and $e_5 = (0,0,0,0,1)$ etc. Find the formula for A^{-1} in terms of the given vectors v_1, v_2, v_3, v_4, v_5.

$AA^{-1} = I$ so if $A^{-1} = [W_1 / W_2 / W_3 / W_4 / W_5]$

we need $A[W_1 | W_2 | W_3 | W_4 | W_5] = [AW_1 | AW_2 | AW_3 | AW_4 | AW_5] = I$

which means we need

$$AW_1 = e_1,\quad AW_2 = e_2,\quad AW_3 = e_3,\quad AW_4 = e_4,\quad AW_5 = e_5$$

Consider the given algebraic data and +/− equations,

$$AV_1 + AV_2 = (e_1 + e_2) + (e_1 - e_2) = 2e_1$$
$$AV_1 - AV_2 = (e_1 + e_2) - (e_1 - e_2) = 2e_2$$

Hence $\underline{A\left(\frac{1}{2}(V_1 + V_2)\right) = e_1}$ (I) and $\underline{A\left(\frac{1}{2}(V_1 - V_2)\right) = e_2}$ (II)

Likewise multiplying by $\sin\theta$ and $\cos\theta$,

$$\begin{aligned} & A(\sin\theta V_4) = \sin\theta\cos\theta e_3 + \sin^2\theta e_4 \\ + \ & A(\cos\theta V_5) = -\cos\theta\sin\theta e_3 + \cos^2\theta e_4 \\ \hline & A(\sin\theta V_4 + \cos\theta V_5) = (\sin^2\theta + \cos^2\theta) e_4 = e_4 \end{aligned}$$ (III)

Likewise,

$$\begin{aligned} & A(\cos\theta V_4) = \cos^2\theta e_3 + \cos\theta\sin\theta e_4 \\ + \ & A(-\sin\theta V_5) = \sin^2\theta e_3 - \sin\theta\cos\theta e_4 \\ \hline & A(\cos\theta V_4 - \sin\theta V_5) = e_3 \end{aligned}$$ (IV)

Sorry. ☹

Notice $AV_3 = e_5 \Rightarrow$ choose $\underline{W_5 = V_3}$ (V)

Put it all together,

$$\boxed{A^{-1} = \left[\frac{1}{2}(V_1 + V_2) \,\middle|\, \frac{1}{2}(V_1 - V_2) \,\middle|\, \sin\theta V_4 + \cos\theta V_5 \,\middle|\, \cos\theta V_4 - \sin\theta V_5 \,\middle|\, V_3 \right]}$$

Problem 39: Solve $13x - 2y = a$ and $x - 7y = b$ for arbitrary a, b using matrix techniques. I recommend multiplication by inverse of the coefficient matrix.

$$\underbrace{\begin{bmatrix} 13 & -2 \\ 1 & -7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{V} = \underbrace{\begin{bmatrix} a \\ b \end{bmatrix}}_{C} \qquad A^{-1} = \frac{1}{-91+2}\begin{bmatrix} -7 & 2 \\ -1 & 13 \end{bmatrix} = \frac{1}{89}\begin{bmatrix} 7 & -2 \\ 1 & -13 \end{bmatrix}$$

$$V = A^{-1}C = \frac{1}{89}\begin{bmatrix} 7 & -2 \\ 1 & -13 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \frac{1}{89}\begin{bmatrix} 7a - 2b \\ a - 13b \end{bmatrix}$$

$$\boxed{\begin{array}{l} x = \frac{1}{89}(7a - 2b) \\ \hline y = \frac{1}{89}(a - 13b) \end{array}}$$

Problem 40: Let $A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 3 \\ 2 & 0 & k \end{bmatrix}$. For what values of k does A^{-1} exist ?

$$A = \begin{bmatrix} 1 & 3 & 4 \\ 2 & 1 & 3 \\ 2 & 0 & k \end{bmatrix} \xrightarrow[r_3 - 2r_1]{r_2 - 2r_1} \begin{bmatrix} 1 & 3 & 4 \\ 0 & -5 & -5 \\ 0 & -6 & k-8 \end{bmatrix} \xrightarrow{-r_2/5} \begin{bmatrix} 1 & 3 & 4 \\ 0 & 1 & 1 \\ 0 & -6 & k-8 \end{bmatrix}$$

$$\xrightarrow[r_3 + 6r_2]{r_1 - 3r_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & k-2 \end{bmatrix}$$

if $k = 2$ then $\text{rref}(A) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \neq I$

hence A^{-1} does not exist.

Therefore, $\text{rref}(A) = I$ for $k \neq 2$, thus

$\underline{A^{-1} \text{ exists provided } k \neq 2.}$

Problem 41: (Lay §2.4#9) Symbolic block matrix algebra problem.

$$\begin{bmatrix} I & 0 & 0 \\ X & I & 0 \\ Y & 0 & I \end{bmatrix}\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \\ A_{31} & A_{32} \end{bmatrix} = \begin{bmatrix} B_{11} & B_{12} \\ 0 & B_{22} \\ 0 & B_{32} \end{bmatrix}$$

Solve for X, Y and B_{22} given that A_{11}^{-1} exists.

$$\boxed{\begin{array}{l|l} A_{11} = B_{11} & A_{12} = B_{12} \\ XA_{11} + A_{21} = 0 & XA_{12} + A_{22} = B_{22} \\ YA_{11} + A_{31} = 0 & YA_{12} + A_{32} = B_{32} \end{array}} \; - ☆$$

$$XA_{11} = -A_{21}$$

$$\Rightarrow \boxed{X = -A_{21}A_{11}^{-1}} - (*)$$

$$YA_{11} = -A_{31}$$

$$\Rightarrow \boxed{Y = -A_{31}A_{11}^{-1}}$$

Substitute $*$ into ☆ to find B_{22} sol^{n}

$$\boxed{B_{22} = XA_{12} + A_{22} = -A_{21}A_{11}^{-1}A_{12} + A_{22}}$$

Problem 42: Let $M = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \in \mathbb{R}^{4\times 4}$ where $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ and $B = \begin{bmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{bmatrix}$. Calculate A^{-1} and B^{-1} via the 2×2 inverse formula then check, via block-multiplication of the partitioned matrix M that $M^{-1} = \begin{bmatrix} A^{-1} & 0 \\ 0 & B^{-1} \end{bmatrix}$.

***

$$A^{-1} = \frac{1}{\cos^2\theta + \sin^2\theta}\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$$

$$B^{-1} = \frac{1}{\cosh^2\phi - \sinh^2\phi}\begin{bmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{bmatrix} = \begin{bmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{bmatrix}$$

$$MM^{-1} = \left[\begin{array}{cc|cc} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ \hline 0 & 0 & \cosh\phi & \sinh\phi \\ 0 & 0 & \sinh\phi & \cosh\phi \end{array}\right]\left[\begin{array}{cc|cc} \cos\theta & \sin\theta & 0 & 0 \\ -\sin\theta & \cos\theta & 0 & 0 \\ \hline 0 & 0 & \cosh\phi & -\sinh\phi \\ 0 & 0 & -\sinh\phi & \cosh\phi \end{array}\right]$$

$$= \left[\begin{array}{c|c|c|c} \cos^2\theta + \sin^2\theta & \cos\theta\sin\theta - \sin\theta\cos\theta & 0 & 0 \\ \hline \sin\theta\cos\theta - \cos\theta\sin\theta & \sin^2\theta + \cos^2\theta & 0 & 0 \\ \hline 0 & 0 & \cosh^2\phi - \sinh^2\phi & -\cosh\phi\sinh\phi + \sinh\phi\cosh\phi \\ \hline 0 & 0 & \sinh\phi\cosh\phi - \cosh\phi\sinh\phi & -\sinh^2\phi + \cosh^2\phi \end{array}\right]$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Look at *** I failed to follow instructions! It's way easier

P42 continued

$$M = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right]$$

Claim: $M^{-1} = \left[\begin{array}{c|c} A^{-1} & 0 \\ \hline 0 & B^{-1} \end{array}\right]$

$$MM^{-1} = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right]\left[\begin{array}{c|c} A^{-1} & 0 \\ \hline 0 & B^{-1} \end{array}\right]$$

$$= \left[\begin{array}{c|c} AA^{-1} + 0 & A0 + 0\cdot B^{-1} \\ \hline 0\cdot A^{-1} + B\cdot 0 & 0\cdot 0 + BB^{-1} \end{array}\right]$$

$$= \left[\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right]$$

$$= I.$$

Problem 43: If $v = (a,b,c)$ and $w = (x,y,z)$ are two vectors in $\mathbb{R}^3$ then the **dot-product** of v and w is given by $v \cdot w = v^T w = ax + by + cz$ and the **length** of the vector v is given by $\|v\| = \sqrt{v \cdot v} = \sqrt{a^2+b^2+c^2}$. Suppose we are given vectors of length one which are pairwise-perpendicular; that is $u_1 \cdot u_2 = 0$ and $u_1 \cdot u_3 = 0$ and $u_2 \cdot u_3 = 0$. Let $M = [u_1|u_2|u_3]$. Calculate M^TM and simplify in view of the given information. Calculate M^{-1} in terms of u_1, u_2, u_3.

$$M^T = \left[\begin{array}{c} u_1^T \\ \hline u_2^T \\ \hline u_3^T \end{array}\right]$$

$$(M^TM)_{ij} = \text{row}_i(M^T) \cdot \text{col}_j(M)$$

$$= u_i^T u_j$$

$$= \delta_{ij} \quad \therefore \quad M^TM = I$$

$$\Rightarrow \boxed{M^{-1} = M^T = \left[\begin{array}{c} u_1^T \\ \hline u_2^T \\ \hline u_3^T \end{array}\right]}$$

Problem 44: (Lay §2.5#3) Solution aided by a given LU-decomposition.

$$A = \begin{bmatrix} 2 & -1 & 2 \\ -6 & 0 & -2 \\ 8 & -1 & 5 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 4 & -1 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & -1 & 2 \\ 0 & -3 & 4 \\ 0 & 0 & 1 \end{bmatrix}}_{U} \quad \text{solve} \quad Ax = \begin{bmatrix} 1 \\ 0 \\ 4 \end{bmatrix}$$

To solve $LUx = b$ we let $y = Ux$ then solve $Ly = b$ to begin

$$[L|b] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ -3 & 1 & 0 & 0 \\ 4 & -1 & 1 & 4 \end{array}\right] \xrightarrow[r_3 - 4r_1]{r_2 + 3r_1} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & -1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Thus $\boxed{y = (1, 3, 3)}$ then consider

$$[U|y] = \left[\begin{array}{ccc|c} 2 & -1 & 2 & 1 \\ 0 & -3 & 4 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right] \xrightarrow[r_2 - 4r_3]{r_1 - 2r_3} \left[\begin{array}{ccc|c} 2 & -1 & 0 & -5 \\ 0 & -3 & 0 & -9 \\ 0 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & -1 & 0 & -5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\xrightarrow{r_1 + r_2} \left[\begin{array}{ccc|c} 2 & 0 & 0 & -2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right] \quad \therefore \quad \boxed{x = \begin{bmatrix} -1 \\ 3 \\ 3 \end{bmatrix}}$$

Problem 45: Let $A = \begin{bmatrix} 2 & -4 & 4 & -2 \\ 6 & -9 & 7 & -3 \\ -1 & -4 & 8 & 0 \end{bmatrix}$. Find the LU-decomposition for A. I believe this matrix does not require the introduction of a permutation matrix like I faced in the example solved https://math.stackexchange.com/a/186997/36530. There is a natural algorithm which helps us construct the LU-decomposition from following the forward-pass of the row-reduction on A. (this is Lay §2.5#15 but I don't think the slick method outlined in my linked answer is to be found in Lay)

$$A = \begin{bmatrix} 2 & -4 & 4 & -2 \\ 6 & -9 & 7 & -3 \\ -1 & -4 & 8 & 0 \end{bmatrix} \xrightarrow[r_3+\frac{1}{2}r_1]{r_2-3r_1} \begin{bmatrix} 2 & -4 & 4 & -2 \\ (3) & 3 & -5 & 3 \\ (-\frac{1}{2}) & -6 & 10 & -1 \end{bmatrix}$$

$$\xrightarrow{r_3+2r_2} \begin{bmatrix} 2 & -4 & 4 & -2 \\ (3) & 3 & -5 & 3 \\ (-\frac{1}{2}) & (-2) & 0 & 5 \end{bmatrix}$$

$$A = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -\frac{1}{2} & -2 & 1 \end{bmatrix}}_{L} \underbrace{\begin{bmatrix} 2 & -4 & 4 & -2 \\ 0 & 3 & -5 & 3 \\ 0 & 0 & 0 & 5 \end{bmatrix}}_{U}$$

(this agrees with answer in back of text and it checks by multiplication)