Show your work and box answers. This pdf should be printed and your solution should be handwritten on the printout. Once complete, please staple in upper left corner. Thanks.

Suggested Reading You may find the following helpful resources beyond lecture;

(a.) Chapter 3 of my lecture notes for Math 221

Problem 46: Calculate det(A) where $A = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 5 & 3 & 1 \end{bmatrix}$

$$\det A = 2\det\begin{bmatrix} 2 & 2 \\ 3 & 1 \end{bmatrix} - 2\det\begin{bmatrix} 0 & 2 \\ 5 & 1 \end{bmatrix} + 0\det\begin{bmatrix} 0 & 2 \\ 5 & 3 \end{bmatrix} = 2(2-6) - 2(-10)$$
$$= 2(-4) + 20$$
$$= \boxed{12}$$

Problem 47: Calculate det(B) where $B = \begin{bmatrix} 2 & 2 & 0 & 2 & 2 \\ 0 & 2 & 0 & -1 & 0 \\ 2 & 2 & 0 & 0 & 3 \\ 7 & 7 & 2 & 7 & 7 \\ 5 & 3 & 0 & 0 & 0 \end{bmatrix}$

$$\det B = -2\det\begin{bmatrix} 2 & 2 & 2 & 2 \\ 0 & 2 & -1 & 0 \\ 2 & 2 & 0 & 3 \\ 5 & 3 & 0 & 0 \end{bmatrix}$$
$$= -2\left[-5\det\begin{bmatrix} 2 & 2 & 2 \\ 2 & -1 & 0 \\ 2 & 0 & 3 \end{bmatrix} + 3\det\begin{bmatrix} 2 & 2 & 2 \\ 0 & -1 & 0 \\ 2 & 0 & 3 \end{bmatrix}\right]$$
$$= 10\big(2(-3) - 2(6) + 2(2)\big) - 6\big(2(-3) - 2(0) + 2(2)\big)$$
$$= 10(-14) + 12$$
$$= \boxed{-128}$$

Problem 48: Let A, B be as given in the previous problems. If $M = \left[\begin{array}{c|c} 2A & 0 \\ \hline 0 & 3B \end{array}\right]$ then calculate det(M) via application of properties of determinants given in the lecture notes and the results of the previous pair of problems.

$$\det(M) = \det(2A)\det(3B) = \left(2^3\det(A)\right)\left(3^5\det(B)\right)$$
$$= (2^3)(3^5)(12)(-128)$$
$$= \boxed{-2985984}$$

Problem 49: For which values of x is the matrix $M = \begin{bmatrix} x & 2 & 2 \\ 1 & 1 & 1 \\ 7 & 5 & 3 \end{bmatrix}$ invertible?

$$\det(M) = \det\begin{bmatrix} x & 2 & 2 \\ 1 & 1 & 1 \\ 7 & 5 & 3 \end{bmatrix} \xrightarrow{r_1 - 2r_2} = \det\begin{bmatrix} x-2 & 0 & 0 \\ 1 & 1 & 1 \\ 7 & 5 & 3 \end{bmatrix} = (x-2)(3-5) \neq 0$$

$$\Rightarrow \boxed{x \neq 2}$$

Problem 50: Solve $\alpha x + 3y = 7$ and $5x - \beta y = 6$ by Cramer's rule. Comment on needed conditions on α, β for the solution to exist.

$$\begin{bmatrix} \alpha & 3 \\ 5 & -\beta \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 7 \\ 6 \end{bmatrix}$$

$$x = \frac{\det\begin{bmatrix} 7 & 3 \\ 6 & -\beta \end{bmatrix}}{\det\begin{bmatrix} \alpha & 3 \\ 5 & -\beta \end{bmatrix}} = \frac{-7\beta - 18}{-\alpha\beta - 15} = \frac{18 + 7\beta}{\alpha\beta + 15}$$

$$y = \frac{\det\begin{bmatrix} \alpha & 7 \\ 5 & 6 \end{bmatrix}}{\det\begin{bmatrix} \alpha & 3 \\ 5 & -\beta \end{bmatrix}} = \frac{6\alpha - 35}{-\alpha\beta - 15} = \frac{35 - 6\alpha}{\alpha\beta + 15}$$

$$\boxed{\text{Thus the solution is } \left(\frac{18 + 7\beta}{\alpha\beta + 15}, \frac{35 - 6\alpha}{\alpha\beta + 15}\right) \text{ for } \alpha\beta \neq -15}$$

Problem 51: Let A be a matrix which is similar to $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. In other words, suppose there exists an invertible matrix P for which $\underbrace{B = P^{-1}AP}_{A = PBP^{-1}}$. Calculate $\det(A)$ and $\operatorname{trace}(A)$.

$$\begin{aligned} \det(A) &= \det(PBP^{-1}) \\ &= \det(P)\det(B)\det(P^{-1}) \\ &= \det(PP^{-1})\det(B) \\ &= \det(I)\det(B) \\ &= \det(B) \\ &= \boxed{15} \end{aligned}$$

$M = PB$ & $N = P^{-1}$

$\operatorname{trace}(MN) = \operatorname{trace}(NM)$

Likewise, $\operatorname{trace}(A) = \operatorname{trace}(PBP^{-1}) = \operatorname{trace}(P^{-1}(PB))$

$$\begin{aligned} &= \operatorname{trace}(B) \\ &= 1 + 3 + 5 \\ &= \boxed{9} \end{aligned}$$

Problem 52: Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$. Calculate $\det(A)$ using properties of the determinant based on row-reductions. Almost certainly using Laplace's expansion by minors here is a really bad idea.

swapping rows $3 \leftrightarrow 4$ and $5 \leftrightarrow 6$

$$\det A = (-1)^2 \det \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7 \end{bmatrix} = 1 \cdot 3 \cdot 1 \cdot 2 \cdot 1 \cdot 7 = \boxed{42}$$

Problem 53: Find the volume of the paralell piped with edges $(1,2,3), (2,3,3), (-1,-2,0)$.

$$\pm \text{Vol} = \det \begin{bmatrix} 1 & 2 & -1 \\ 2 & 3 & -2 \\ 3 & 3 & 0 \end{bmatrix}$$

$$= 1(0+6) - 2(0+6) - 1(6-9)$$

$$= 6 - 12 + 3$$

$$= -3 \quad \therefore \boxed{\text{Vol} = 3}$$

(volume is positive or zero)

Problem 54: Lay, §3.2#40 (problem to help learn properties of determinant)

Let $A, B \in \mathbb{R}^{4\times 4}$ with $\det(A) = -1$ and $\det(B) = 2$

(a.) $\det(AB) = \det(A)\det(B) = (-1)(2) = \boxed{-2}$

(b.) $\det(B^5) = (\det(B))^5 = 2^5 = \boxed{32}$

(c.) $\det(2A) = 2^4 \det(A) = 16(-1) = \boxed{-16}$

(d.) $\det(A^T A) = \det(A^T)\det(A) = (\det A)^2 = (-1)^2 = \boxed{1}$

(e.) $\det(B^{-1}AB) = \det(B^{-1})\det(A)\det(B) = \frac{1}{2}(-1)(2) = \boxed{-1}$

Problem 55: Lay, Chapter 3, page 212 #7 on (equation of line)

$$\det \begin{bmatrix} 1 & x & y \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{bmatrix} = 0$$

In my notes I used row operation-based calculation to derive this easily. I'll try brute force now.

$$0 = 1(x_1y_2 - x_2y_1) - x(y_2 - y_1) + y(x_2 - x_1)$$

Now solve for y,

$$y(x_2 - x_1) = x_2y_1 - x_1y_2 + x(y_2 - y_1)$$

$$y(x_2 - x_1) = y_2(x_2 - x_1) - y_2x_2 + x_2y_1 + x(y_2 - y_1)$$

$$y = y_2 + \frac{x_2y_1 - x_2y_2}{x_2 - x_1} + x\left(\frac{y_2 - y_1}{x_2 - x_1}\right)$$

$$y = y_2 + \frac{-x_2(y_2 - y_1)}{x_2 - x_1} + x\frac{(y_2 - y_1)}{x_2 - x_1}$$

$$\boxed{y = y_2 + \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_2)}$$

Problem 56: Lay, §3.3#6 (Cramer's rule)

$$2x_1 + x_2 + x_3 = 4$$
$$-x_1 \quad + 2x_3 = 2$$
$$3x_1 + x_2 + 3x_3 = -2$$

$$\overbrace{\begin{bmatrix} 2 & 1 & 1 \\ -1 & 0 & 2 \\ 3 & 1 & 3 \end{bmatrix}}^{A} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \overbrace{\begin{bmatrix} 4 \\ 2 \\ -2 \end{bmatrix}}^{b}$$

$$\det A = 2(-2) - 1(-3-6) + 1(-1) = 4$$

$$x_1 = \frac{1}{4}\det\begin{bmatrix} 4 & 1 & 1 \\ 2 & 0 & 2 \\ -2 & 1 & 3 \end{bmatrix} = \frac{1}{4}\left[4(-2) - 1(6+4) + 1(2)\right] = \frac{-16}{4} = -4$$

$$x_2 = \frac{1}{4}\det\begin{bmatrix} 2 & 4 & 1 \\ -1 & 2 & 2 \\ 3 & -2 & 3 \end{bmatrix} = \frac{1}{4}\left[2(6+4) - 4(-3-6) + 1(2-6)\right] = \frac{52}{4} = 13$$

$$x_3 = \frac{1}{4}\det\begin{bmatrix} 2 & 1 & 4 \\ -1 & 0 & 2 \\ 3 & 1 & -2 \end{bmatrix} = \frac{1}{4}\left[2(-2) - 1(2-6) + 4(-1)\right] = \frac{-4}{4} = -1$$

Solution $\boxed{(-4, 13, -1)}$

Problem 57: Is $(a, b, c) \in \text{span}\{(1,2,3), (0,1,1)\}$? Use determinants and the theory of linear algebra we have discussed to answer this question.

If $c_1(1,2,3) + c_2(0,1,1) = (a,b,c)$ then

$\{(1,2,3), (0,1,1), (a,b,c)\}$ is <u>linearly</u> <u>dependent</u>

thus $\det\begin{bmatrix} 1 & 0 & a \\ 2 & 1 & b \\ 3 & 1 & c \end{bmatrix} = 0$

$$\Rightarrow 1(c-b) - 0(2c-3b) + a(2-3) = 0$$

$$\Rightarrow c - b + 2a - 3a = 0$$

$$\Rightarrow -a - b + c = 0$$

$\boxed{\text{Thus, } (a,b,c) \in \text{span}\{(1,2,3),(0,1,1)\} \text{ only if } c = a+b.}$

Problem 58: Suppose you have a square matrix A for which the matrix equation $A^T J A = J$ holds for some invertible matrix J. Find the possible values for $det(A)$.

$$A^T J A = J \Rightarrow \det(A^T J A) = \det(J)$$
$$\Rightarrow \det(A^T)\,\cancel{\det(J)}\,\det(A) = \cancel{\det(J)} \quad (\det J \neq 0 \text{ since } J^{-1} \text{ exists})$$
$$\Rightarrow (\det A)^2 = 1 \quad (\text{since } \det A^T = \det A)$$
$$\Rightarrow \boxed{\det A = \pm 1}$$

Problem 59: The cross product: For all $a, b \in \mathbb{R}^3$ we define

$$T(a,b) = \sum_{j=1}^{3} (\det[a|b|e_j])\, e_j.$$

Show $a \cdot T(a,b) = 0$ and $b \cdot T(a,b) = 0$ and for any $c \in \mathbb{R}^3$ we have $\underline{T(a,b) \cdot c = \det[a|b|c]}$.

$$T(a,b) \cdot c = \left(\sum_{j=1}^{3} \det(a|b|e_j)\, e_j \right) \cdot c$$
$$= \sum_{j=1}^{3} \det(a|b|e_j)\, e_j \cdot c \qquad \underline{c \cdot e_j = c_j}$$
$$= \sum_{j=1}^{3} \det(a|b|c_j e_j)$$
$$= \det\left(a|b|\sum_{j=1}^{3} c_j e_j\right)$$
$$= \underline{\det(a|b|c)}.$$

Then $T(a,b) \cdot a = \det(a|b|a) = 0 = a \cdot T(a,b)$.
Likewise $b \cdot T(a,b) = T(a,b) \cdot b = \det(a|b|b) = 0$.
Where I've used $\det(A) = 0$ if $\exists$ a linear dependence among the columns. Or, if you prefer, that any repeated column $\Rightarrow$ zero det.

Problem 60: A natural candidate for the cross product in $\mathbb{R}^4$ is given by extending the formula in the previous problem: for all $a, b, c \in \mathbb{R}^4$ we define

$$T(a,b,c) = \sum_{j=1}^{4} (\det[a|b|c|e_j])\, e_j$$

Show: $a \cdot T(a,b,c) = 0$ and $b \cdot T(a,b,c) = 0$ and $c \cdot T(a,b,c) = 0$.
I should mention, the equations above tell us a, b, c are perpendicular to $T(a,b,c)$ and we can prove that implies $\{a, b, c, T(a,b,c)\}$ is linearly independent provided $T(a,b,c) \neq 0$. In other words, if you want a fourth vector which is outside the span of $a, b, c \in \mathbb{R}^4$ then $T(a,b,c)$ is a nice choice. It is the normal to the hypervolume spanned by a, b, c.

$$T(a,b,c) \cdot d = \sum_{j=1}^{4} \det(a|b|c|e_j)(d \cdot e_j) \qquad d \cdot e_j = d_j$$

$$= \sum_{j=1}^{4} \det(a|b|c|d_j e_j) \qquad d = \sum_{j=1}^{4} d_j e_j$$

$$= \det\left(a|b|c|\sum_{j=1}^{4} d_j e_j\right)$$

$$= \det(a|b|c|d).$$

Now apply the identity above and use repeated column $\Rightarrow$ zero det. property,

$$a \cdot T(a,b,c) = T(a,b,c) \cdot a = \det(a|b|c|a) = 0$$
$$b \cdot T(a,b,c) = T(a,b,c) \cdot b = \det(a|b|c|b) = 0$$
$$c \cdot T(a,b,c) = T(a,b,c) \cdot c = \det(a|b|c|c) = 0.$$