Same instructions as Mission 1. This homework is based on Lectures 16 - 22. There are 5pts to earn for completely following formatting instructions. Feel free to use technology for any row-reductions, however, realize you may need to do some of these calculations in your next Boss Fight.

Problem 61: Let $A = \begin{bmatrix} 1 & -3 & 0 \\ -2 & 4 & 1 \\ 5 & -2 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 5 & 4 & 7 \\ 0 & 0 & 2 & 1 \\ 0 & 5 & 1 & 8 \\ 0 & 0 & 0 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 7 \\ 1 & 8 \end{bmatrix}$.

(5pts) Calculate the following determinants:

(a.) $det(A^2)$

(b.) $det(B^T)$

(c.) $det(C)$

(d.) $det(A \oplus B \oplus D)$

(e.) $det(10C^{-1})$

Problem 62: (1pt) Let $A = \begin{bmatrix} 1 & 2 & 0 \\ k & 1 & k \\ 0 & 2 & 1 \end{bmatrix}$. Find k for which A^{-1} exists. (show work below)

Problem 63: (1pts) Suppose A is a 2×2 matrix with $det(45A) = 2025$ and B is a 3×3 matrix with $det(B) = 10$. If $M = \left[\begin{array}{c|c} A^{999} & 0 \\ \hline 0 & 3B \end{array}\right]$ then calculate $det(M)$. (show work below)

Problem 64: (1pts) Calculate: (show work below)

$$\det \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 5 & 5 & 0 & 0 & 0 & 0 \\ 5 & 5 & 3 & 0 & 0 & 0 \\ 5 & 5 & 5 & 5 & 0 & 0 \\ 5 & 5 & 5 & 5 & 0 & -3 \\ 5 & 5 & 5 & 5 & 3 & 0 \end{bmatrix}$$

Problem 65: (1pt) The matrix $A = \begin{bmatrix} 1 & x & x^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$ is an example of a **Vandermonde matrix**. Use properties about row and column operations to show that $det(A) = (x-b)(x-c)(c-b)$. (show work below)

Problem 66: (2pt) What conditions are needed for the following sets to be a basis ?

(a.) $S = \{(1, 0, -1), (1, 1, 1), (a, b, c)\}$

(b.) $S = \{(1, 0, -1), (x, y, z), (a, b, c)\}$

Problem 67: (1pt) Let $A = \begin{bmatrix} x & 1 & y \\ 1 & 0 & 1 \\ z & 1 & w \end{bmatrix}$ be invertible. Calculate A^{-1}.

Problem 68: (1pt) Let $W = \text{span}\{(1, 1, 1, 1, 1), (1, -1, 0, 0, 0), (1, 1, -2, 0, 0), (2, 0, 4, 0, 6)\}$. Find an orthonormal basis for W.

Problem 69: (2pts) Find a basis for $S^\perp$ given

(a.) $S = \{(1, -2, -3), (2, -4, -6)\}$

(b.) $S = \{(1, 1, 1, 1), (2, 4, 6, 8), (0, 0, 1, 1)\}$

(c.) $S = \{(1, 1), (2, 3)\}$

(d.) $S = \{e_1, e_n\} \subseteq \mathbb{R}^n$

Problem 70: (2pts) Let $S = \{(1, 1, 1, 0), (1, 2, 2, 1)\}$. Find an orthonormal basis for $S^\perp$. Also, find the point in $S^\perp$ closest to $(1, 2, 3, 4)$.

Problem 71: (4pts) Let $S = \{(1, 1, 0, 0), (1, -1, 0, 0), (2, 0, 1, 1)\}$.

(a.) Use the Gram Schmidt Algorithm to create an orthonormal set $S'' = \{v_1'', v_2'', v_3''\}$ for which $\text{span}(S) = \text{span}(S'')$

(b.) Find an orthonormal basis $\{v\}$ for $S^\perp$

(c.) Let $R = [v_1''|v_2''|v_3''|v]$ and calculate $R^T R$.

(d.) If $W = \text{span}(S)$ then find the closest point in W to the point $(2, 2, 3, 4)$

Problem 72: (2pts) Let $A = \begin{bmatrix} 1 & 3 & 1 \\ 0 & 2 & 4 \\ 1 & 3 & -1 \\ 1 & 3 & -1 \\ 1 & 3 & 1 \end{bmatrix}$.

(a.) Find the QR-decomposition for A,

(b.) Find the best approximation to the equation $Ax = (0, 3, 2, 0, 1)$.

Problem 73: (3pts) Suppose $W = \text{span}\{(1, 2, 2, 2), (0, 1, 1, 0), (1, 3, 3, 2)\}$.

(a.) Find an orthonormal basis for W

(b.) Find an orthonormal basis for $W^{\perp}$

(c.) Find the point on W which is closest to (a, b, c, d).

Problem 74: (1pts) Find the line which is closest to the points $(-2, -6), (-1, 1), (0, 9), (1, 13), (2, 18)$.

Problem 75: (2pts) Find equation the plane which is closest to the given points:

(a.) $(1, 2, 5), (1, -4, 2), (0, -11, 0), (11, 0, -11)$

(b.) $(1, 3, 5), (2, 2, 2), (1, 0, 1), (13, 11, 2)$

Problem 76: (1pts) Let $\gamma = \{\frac{1}{\sqrt{3}}(1,1,1), \frac{1}{\sqrt{2}}(-1,0,1), \frac{1}{\sqrt{6}}(-1,2,-1)\}$. Calculate $[(a,b,c)]_\gamma$

Problem 77: Let $\gamma = \{u_1, u_2, u_3\}$ where u_1, u_2, u_3 were given in the previous problem. Define $T(\bar{x}u_1 + \bar{y}u_2 + \bar{z}u_3) = 6\bar{x}u_1 + 4\bar{y}u_2 + 12\bar{z}u_3$.

(a.) (1pt) Calculate $[T]_{\gamma,\gamma}$

(b.) (2pt) Calculate $[T]$

solve the remaining three problem's separate paper, box answers where appropriate.

Problem 78: (3pts) Let the plane W be given by $x + y - 3z = 0$ where we define $n = \langle 1, 1, -3\rangle$ to point the **upward**-direction for W.

(a.) Find u, v which form an orthonormal basis $\beta = \{u, v\}$ for $\{\langle 1, 1, -3\rangle\}^\perp$.

(b.) Consider $p = (x, y, z)$, calculate $\det(u|v|p)$ and explain geometrically what it means for this determinant to be positive, negative or zero.

(c.) How far off the plane W is the point p ?

Problem 79: (3pts) Conside the plane W in $\mathbb{R}^3$ given by $x + 2y + 2z = 11$.

(a.) Find a pair of orthonormal tangent vectors to this plane, let's denote these by A and B,

(b.) Let $X(u, v) = (1, 2, 3) + uA + vB$. Show that $X(u, v) \in S$ for all $u, v \in \mathbb{R}$,

(c.) Parametrize a circle of radius R centered at $(1, 2, 3)$; that is, provide a mapping $\gamma : [0, 2\pi] \to \mathbb{R}^3$ for which $\|\gamma(t) - (1, 2, 3)\| = R$ and $\gamma(t) \in S$ for each t.

Problem 80: (1pt) Suppose $T : \mathbb{R}^n \to \mathbb{R}^n$ is an orthogonal linear transformation. Let $C_R(p)$ be a circle of radius R centered at p. Show that $T(C_R(p))$ is also a circle of radius R.