Show your work and box answers. This pdf should be printed and your solution should be handwritten on the printout. Once complete, please staple in upper left corner. Thanks.

Suggested Reading You may find the following helpful resources beyond lecture,

(a.) Chapter 5 of my lecture notes for Math 221

Let us be clear on some notation which is not in Lay, but is in my notes. If $\beta = \{v_1, v_2, \dots, v_n\}$ is a basis for $\mathbb{R}^n$ then for any $x \in \mathbb{R}^n$ we define $[x]_\beta = (y_1, \dots, y_n)$ if and only if $x = y_1v_1 + \cdots + y_nv_n$. Notice,

$$x = y_1v_1 + \cdots + y_nv_n = \underbrace{[v_1|v_2|\cdots|v_n]}_{[\beta]} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = [\beta][x]_\beta \quad \Rightarrow \quad \boxed{[x]_\beta = [\beta]^{-1}x}$$

Likewise, if we consider a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ then the matrix of T with respect to the β basis is defined by

$$[T]_{\beta,\beta} = [[T(v_1)]_\beta|[T(v_2)]_\beta|\cdots|[T(v_n)]_\beta].$$

In the special case of the standard basis we define

$$[T] = [T(e_1)|T(e_2)|\cdots|T(e_n)].$$

and note $T(x) = [T]x$ for all x. Let's derive how to calculate $[T]_{\beta,\beta}$,

$$\begin{aligned} [T]_{\beta,\beta} &= [[T(v_1)]_\beta|[T(v_2)]_\beta|\cdots|[T(v_n)]_\beta] \\ &= [[\beta]^{-1}T(v_1)|[\beta]^{-1}T(v_2)|\cdots|[\beta]^{-1}T(v_n)] \\ &= [\beta]^{-1}[T(v_1)|T(v_2)|\cdots|T(v_n)] \\ &= [\beta]^{-1}[[T]v_1|[T]v_2|\cdots|[T]v_n] \\ &= [\beta]^{-1}[T][v_1|v_2|\cdots|v_n] \\ &= [\beta]^{-1}[T][\beta] \quad \Rightarrow \quad \boxed{[T]_{\beta,\beta} = [\beta]^{-1}[T][\beta]}. \end{aligned}$$

You will need to use the boxed formulas to solve some of the problems in this mission.

Problem 61: Let $\beta = \{(1,2),(3,4)\}$ and suppose $x = (-1,0)$. Calculate $[x]_\beta$.

$$[\beta] = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} \Rightarrow [\beta]^{-1} = \frac{1}{4-6}\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -4 & 3 \\ 2 & -1 \end{bmatrix}$$

$$\therefore [x]_\beta = [\beta]^{-1}x = \frac{1}{2}\begin{bmatrix} -4 & 3 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 \\ -2 \end{bmatrix} = \boxed{\begin{bmatrix} 2 \\ -1 \end{bmatrix}}$$

Check answer: $2\begin{pmatrix} 1 \\ 2 \end{pmatrix} - 1\begin{pmatrix} 3 \\ 4 \end{pmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ ✓

Problem 62: Let $\beta = \left\{\frac{1}{3}(1,2,2), \frac{1}{\sqrt{2}}(0,1,-1), \frac{1}{\sqrt{18}}(-4,1,1)\right\}$ and define coordinates by $(y_1, y_2, y_3) = [\beta]^{-1}x$ for each $x \in \mathbb{R}^n$. Fun helpful fact, you can check $[\beta]^T[\beta] = I$, this makes β an orthonormal basis. Find formulas y_1, y_2, y_3 in terms of x_1, x_2, x_3 and show that:

$$x_1^2 + x_2^2 + x_3^2 = y_1^2 + y_2^2 + y_3^2.$$

Does the analog of the identity hold for $x = (-1, 0)$ and $y = [x]_\beta$ which you calculated in the previous problem ?

If $y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = [x]_\beta = [\beta]^{-1}x = [\beta]^T x$ since β orthonormal

then note that $[\beta]^T[\beta] = I$ and $[\beta][\beta]^T = I$,

$$\begin{aligned} y_1^2 + y_2^2 + y_3^2 = y^T y &= (([\beta]^T x)^T [\beta]^T x \\ &= x^T [\beta][\beta]^T x \\ &= x^T x \\ &= x_1^2 + x_2^2 + x_3^2 \ // \end{aligned}$$

Or, more explicitly,

$$y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ -4/\sqrt{18} & 1/\sqrt{18} & 1/\sqrt{18} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \Rightarrow \begin{aligned} y_1 &= \frac{x_1 + 2x_2 + 2x_3}{3} \\ y_2 &= \frac{x_2 - x_3}{\sqrt{2}} \\ y_3 &= \frac{-4x_1 + x_2 + x_3}{\sqrt{18}} \end{aligned}$$

Hence,

$$\begin{aligned} y_1^2 + y_2^2 + y_3^2 = &\frac{1}{9}(x_1 + 2x_2 + 2x_3)(x_1 + 2x_2 + 2x_3) \\ &+ \frac{1}{2}(x_2 - x_3)(x_2 - x_3) \\ &+ \frac{1}{18}(-4x_1 + x_2 + x_3)(-4x_1 + x_2 + x_3) \end{aligned}$$

$$= \left(\frac{1}{9} + \frac{16}{18}\right)x_1^2 + \left(\frac{4}{9} + \frac{1}{2} + \frac{1}{18}\right)x_2^2 + \left(\frac{4}{9} + \frac{1}{2} + \frac{1}{18}\right)x_3^2$$
$$\left(\frac{4}{9} - \frac{8}{18}\right)x_1x_2 + \left(\frac{4}{9} - \frac{8}{18}\right)x_1x_3 + \left(\frac{8}{9} - 1 + \frac{2}{18}\right)x_2x_3$$

$$= x_1^2 + x_2^2 + x_3^2 \ //$$

Problem 63: Let $\beta = \{(1,0,1),(0,1,1),(1,1,0)\}$. Calculate $[(a,b,c)]_\beta$ and use your result to find c_1, c_2, c_3 for which $c_1(1,0,1)+c_2(0,1,1)+c_3(1,1,0)=(3,5,7)$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & 1 & b \\ 1 & 1 & 0 & c \end{array}\right] \xrightarrow{r_3 - r_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & 1 & b \\ 0 & 1 & -1 & c-a \end{array}\right] \xrightarrow{r_3 - r_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & 1 & b \\ 0 & 0 & -2 & c-a-b \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & 1 & b \\ 0 & 0 & 1 & (a+b-c)/2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & a - \frac{1}{2}(a+b-c) \\ 0 & 1 & 0 & b - \frac{1}{2}(a+b-c) \\ 0 & 0 & 1 & (a+b-c)/2 \end{array}\right]$$

Thus $c_1(1,0,1) + c_2(0,1,1) + c_3(1,1,0) = (a,b,c)$

for $c_1 = \frac{1}{2}(a-b+c)$, $c_2 = \frac{1}{2}(b-a+c)$, $c_3 = \frac{1}{2}(a+b-c)$

$$\boxed{[(a,b,c)]_\beta = \begin{bmatrix} \frac{1}{2}(a-b+c) \\ \frac{1}{2}(b-a+c) \\ \frac{1}{2}(a+b-c) \end{bmatrix}}$$

$$[(3,5,7)]_\beta = \left(\tfrac{1}{2}(3-5+7),\ \tfrac{1}{2}(5-3+7),\ \tfrac{1}{2}(3+5-7)\right)$$
$$= \boxed{(5/2,\ 9/2,\ 1/2) = (c_1, c_2, c_3)}$$

Check it, $\frac{5}{2}(1,0,1) + \frac{9}{2}(0,1,1) + \frac{1}{2}(1,1,0) = (3,5,7)$ ✓

Problem 64: Let $\beta = \{\underbrace{(0,1,0,0)}_{e_2}, \underbrace{(0,0,1,0)}_{e_3}, \underbrace{(0,0,0,1)}_{e_4}, \underbrace{(1,0,0,0)}_{e_1}\}$. Calculate $[(a,b,c,d)]_\beta$.

$$(a,b,c,d) = ae_1 + be_2 + ce_3 + de_4$$
$$= be_2 + ce_3 + de_4 + ae_1 \implies \boxed{[(a,b,c,d)]_\beta = \begin{bmatrix} b \\ c \\ d \\ a \end{bmatrix}}$$

Yes, we could calculate

$$[\beta]^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

and compute $[\beta]^{-1}(a,b,c,d)$, but ↑ is <u>much</u> easier.

Problem 65: Find a matrix A for which $T(x) = Ax$ for each linear transformation below. Also, calculate $rref(A)$ and determine if the given map is onto, one-to-one or both. Find the rank and nullity for each map.

(a.) $T(x,y) = (x-y, 3x+2y)$

(b.) $T(x,y) = (x+y, 2x+2y, y)$

(c.) $T(x,y,z) = (x+2y+z, 2x+4y-2z, x+2y+z)$

(a.) $$T(x,y) = \begin{pmatrix} x-y \\ 3x+2y \end{pmatrix} = \underbrace{\begin{bmatrix} 1 & -1 \\ 3 & 2 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & -1 \\ 0 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = rref(A) \Rightarrow \begin{array}{l} \text{rank}(A) = 2 \\ \text{nullity}(A) = 0 \end{array}$$

Hence $\boxed{\text{T is both 1-to-1 and onto}}$.

Also, rank$(T) = 2$ and nullity$(T) = 2$.

(b.) $$T(x,y) = \begin{bmatrix} x+y \\ 2x+2y \\ y \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 2 \\ 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow \begin{array}{l} \text{nullity}(A) = \text{nullity}(T) = 0. \\ \text{rank}(A) = \text{rank}(T) = 2. \end{array}$$

Since $T: \mathbb{R}^2 \to \mathbb{R}^3$ and rank$(T) \neq 3$ we find T is $\boxed{\text{not onto.}}$ However, nullity$(T) = 0$ gives $\boxed{\text{T one-to-one}}$

(c.) $$T(x,y,z) = \begin{bmatrix} x+2y+z \\ 2x+4y-2z \\ x+2y+z \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & -2 \\ 1 & 2 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$A \sim \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & -4 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = rref(A)$$

$\Rightarrow$ rank$(T) = 2$ and nullity$(T) = 1$ $\therefore$ $\boxed{\text{T not onto and T not 1-to-1}}$

Problem 66: Let $T(x,y) = (x+2y, -x+3y, 3x)$ and $S(u,v) = (u, 2u+v)$.

(a.) find standard matrices of T and S,

(b.) calculate $T \circ S$ by working through $(T \circ S)(u,v) = T(S(u,v))$ and find the standard matrix for $T \circ S$.

(c.) show $[T \circ S] = [T][S]$

Remark: this is why we defined matrix multiplication as we did, at least this is a common motivation in many texts. There are others, some of which date to antiquity

(a.) $$T(x,y) = \begin{bmatrix} x+2y \\ -x+3y \\ 3x \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 3 & 0 \end{bmatrix}}_{[T]} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$S(u,v) = \begin{bmatrix} u \\ 2u+v \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}}_{[S]} \begin{bmatrix} u \\ v \end{bmatrix}$$

(b.) $$\begin{aligned} (T \circ S)(u,v) &= T(S(u,v)) \\ &= T(u, 2u+v) \qquad x = u, \quad y = 2u+v \\ &= (u+2(2u+v), -u+3(2u+v), 3u) \\ &= (5u+2v, 5u+3v, 3u) \\ &= \underbrace{\begin{bmatrix} 5 & 2 \\ 5 & 3 \\ 3 & 0 \end{bmatrix}}_{[T \circ S]} \begin{bmatrix} u \\ v \end{bmatrix} \end{aligned}$$

(c.) $$[T][S] = \begin{bmatrix} 1 & 2 \\ -1 & 3 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 2 \\ 5 & 3 \\ 3 & 0 \end{bmatrix} \overset{\checkmark}{=} [T \circ S]$$

Problem 67: Suppose $T(x) = Ax+b$ where $A \in \mathbb{R}^{n\times n}$ and $A^TA = I$ and $b \in \mathbb{R}^n$. The distance between points in $\mathbb{R}^n$ is given by $d(P,Q) = \sqrt{(Q-P)^T(Q-P)}$. Show that[1]

$$d(P,Q) = d(T(P),T(Q)).$$

$$\begin{aligned}
d(P,Q) &= \sqrt{(Q-P)^T(Q-P)} \\
&= \sqrt{(Q-P)^T A^T A\,(Q-P)} \\
&= \sqrt{(A(Q-P))^T A(Q-P)} \\
&= \sqrt{(AQ+b-(AP+b))^T(AQ+b-(AP+b))} \\
&= \sqrt{(T(Q)-T(P))^T(T(Q)-T(P))} \\
&= d(T(P),T(Q))
\end{aligned}$$

Remark: this solution is easier to understand if you read it bottom to top ☺.

[1] *Remark: a transformation like T is known as a* **rigid motion** *if $det(A) = 1$. These are the transformations which preserve the shape of rigid objects. The determinant has to be one in order that the transformation not turn things inside out. It is a far more difficult task, but it can be shown that maps such as T are the* **only** *functions on $\mathbb{R}^n$ which preserve the Euclidean distance between points.*

Problem 68: Lay §1.9#2, 4 (on constructing linear transformations)

#2) $T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2$

$$[T] = \underbrace{[T(e_1) | T(e_2) | T(e_3)]}_{\text{given}} = \boxed{\begin{bmatrix} 1 & 4 & -5 \\ 3 & -7 & 4 \end{bmatrix}}$$

#4) $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ rotates points about origin through $-\pi/4$

we're given $T(e_1) = (1/\sqrt{2}, -1/\sqrt{2})$

$$\underbrace{\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}}_{\substack{\text{form of rotation} \\ \text{on } \mathbb{R}^2 \\ \text{about origin}}} = \begin{bmatrix} 1/\sqrt{2} & -\sin\theta \\ -1/\sqrt{2} & \cos\theta \end{bmatrix} \Rightarrow \begin{matrix} \cos\theta = 1/\sqrt{2} \\ \sin\theta = -1/\sqrt{2} \end{matrix}$$

$$\therefore \boxed{[T] = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix}}$$

Problem 69: Lay §1.9#6, 9 (on constructing linear transformations)

#6) $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ horizontal shear transformation that leaves e_1 unchanged and maps $e_2 \longmapsto e_2 + 3e_1$

$$[T] = [T(e_1) | T(e_2)] = [e_1 | e_2 + 3e_1] = \boxed{\begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}}$$

#9) $T: \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by horizontal shear that transforms $e_2 \longmapsto e_2 - 2e_1$ and $e_1 \longmapsto e_1$ then reflects through line $x_2 = -x_1$

$$T = T_2 \circ T_1 \quad \left\{ \begin{aligned} T_1(x_1, x_2) &= \begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 - 2x_2 \\ x_2 \end{bmatrix} \\ T_2(x_1, x_2) &= \underbrace{\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}}_{\text{from page 85 in Lay.}}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ -x_1 \end{bmatrix} \end{aligned} \right.$$

$$[T] = [T_2][T_1]$$

$$= \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 & -2 \\ 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 0 & -1 \\ -1 & 2 \end{bmatrix}}$$

Problem 70: Consider $T(x) = Ax$ where $A = \begin{bmatrix} 9 & 7 & -13 \\ 7 & 9 & -13 \\ -13 & -13 & 29 \end{bmatrix}$.

(a) show T is onto and one-to-one map on $\mathbb{R}^3$

(b) find the standard matrix for T^{-1}.

(a.)

$\det(A) = 9(9(29) - 169) - 7(7(29) - 169) - 13(7(-13) + 9(13)) = 252$

Thus $\det(A) \neq 0 \Rightarrow \text{rref}(A) = I \Rightarrow \text{nullity}(A) = 0, \text{rank}(A) = 3$

Hence $\text{nullity}(T) = 0$ and $\text{rank}(T) = 3$

Thus $\underline{T \text{ is one-to-one and } T \text{ onto}}$.

(b.)

$$\text{rref}(A \mid I) = \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 23/63 & -17/126 & 13/126 \\ 0 & 1 & 0 & -17/126 & 23/63 & 13/126 \\ 0 & 0 & 1 & 13/126 & 13/126 & 8/63 \end{array}\right]$$

where the right block is A^{-1}.

I forgot to prove this in-class, so, here it is now,

$T \circ T^{-1} = Id = T^{-1} \circ T$ where $Id(x) = x \quad \forall x \in \mathbb{R}^3$

then $[T \circ T^{-1}] = [Id] \Rightarrow [T][T^{-1}] = I$

Hence $\boxed{[T^{-1}] = [T]^{-1}}$ now, apply this

noting $[T] = A$ hence

$$\boxed{[T^{-1}] = \frac{1}{126}\begin{bmatrix} 46 & -17 & 13 \\ -17 & 46 & 13 \\ 13 & 13 & 16 \end{bmatrix}}$$

$\frac{1}{\sqrt{3}}(1,1,1)$ makes β orthonormal

Problem 71: Let $\beta = \left\{\frac{1}{\sqrt{3}}(1,1,1), \frac{1}{\sqrt{2}}(1,-1,0), \frac{1}{\sqrt{6}}(1,1,-2)\right\}$. For T given in the previous problem, calculate both $[T]_{\beta,\beta}$ and $[T^{-1}]_{\beta,\beta}$.

$$[T]_{\beta\beta} = [\beta]^{-1}[T][\beta] = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{bmatrix} \begin{bmatrix} 9 & 7 & -13 \\ 7 & 9 & -13 \\ -13 & -13 & 29 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix}$$

$$= \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{3} & \sqrt{2} & 42/\sqrt{6} \\ \sqrt{3} & -\sqrt{2} & 42/\sqrt{6} \\ \sqrt{3} & 0 & -84/\sqrt{6} \end{bmatrix}$$

$$= \underline{\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 42 \end{bmatrix}}.$$

$$\begin{aligned}
[T^{-1}]_{\beta\beta} &= [\beta]^{-1}[T^{-1}][\beta] \\
&= [\beta]^{-1}[T]^{-1}[\beta] \\
&= \left([\beta]^{-1}[T][\beta]\right)^{-1} \\
&= \left([T]_{\beta\beta}\right)^{-1}
\end{aligned}$$

But, clearly,

$$\left([T]_{\beta\beta}\right)^{-1} = \underbrace{\begin{bmatrix} 1/3 & & \\ & 1/2 & \\ & & 1/42 \end{bmatrix}}$$

$$\therefore \quad [T^{-1}]_{\beta\beta}$$

Problem 72: Calculate $det([T])$ and $det([T]_{\beta,\beta})$ as well as trace$([T])$ and trace$([T]_{\beta,\beta})$. Do you see any patterns ?

$\det[T] = \det(A) = \underline{252}$. from P70a

$\det[T]_{\beta\beta} = \det\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 42 \end{bmatrix} = 3\cdot 2\cdot 42 = \underline{252}$.

$\operatorname{trace}[T] = \operatorname{trace}(A) = \operatorname{trace}\begin{bmatrix} 9 & 7 & -13 \\ 7 & 9 & -13 \\ -13 & -13 & 29 \end{bmatrix} = 18 + 29 = \underline{47}$.

$\operatorname{trace}[T]_{\beta\beta} = 3 + 2 + 42 = \underline{47}$.

Pattern is clear, $\det[T]_{\beta\beta} = \det[T]$ & $\operatorname{trace}[T]_{\beta\beta} = \operatorname{trace}(T)$

(last problem on Test 1 explored this.)

Remark: gave a hint at start of class 3-6-24 for P74, that way was less clever and much more computational.

Problem 73: Find all transformations on $\mathbb{R}^3$ for which $T(1,2,3) = (1,0,0)$ and $T(1,0,1) = (0,1,0)$. The answer should be a set of linear transformations indexed by some parameter.

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = [T] \quad \text{has} \quad A\begin{bmatrix}1\\2\\3\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} \quad \text{and} \quad A\begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix}$$

Calculate,

$$\text{rref}\begin{array}{c} \begin{array}{ccccccccc|c} a & b & c & d & e & f & g & h & i & \end{array} \\ \left[\begin{array}{ccccccccc|c} 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 0 \\ \hline 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \end{array}\right] \end{array} = \left[\begin{array}{ccc|ccc|ccc|c} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & -1/2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]$$

We should use c, f, i as free parameters since

$$a = -c, \quad b = 1/2 - c, \quad d = 1 - f, \quad e = \tfrac{-1}{2} - f, \quad g = -i, \quad h = -i$$

$$\boxed{\left\{ T : \mathbb{R}^3 \to \mathbb{R}^3 \;\middle|\; T(x) = Ax \text{ with } A = \left[\begin{array}{c|c|c} -c & \frac{1}{2} - c & c \\ \hline 1-f & -\frac{1}{2} - f & f \\ \hline -i & -i & i \end{array}\right], \; c, f, i \in \mathbb{R} \right\}}$$

Remark: see ↓ for an alternate method of calculation ↓

$$\boxed{T(x,y,z) = (2x + 2y - z,\ 4x - 2y + 2z,\ 7x + y)} \leftarrow \text{from } *$$

Problem 74: Find the formula for the linear transformation on $\mathbb{R}^3$ for which $T(1,1,1) = (3,4,8)$ and $T(0,1,1) = (1,0,1)$ and $T(0,0,1) = (-1,2,0)$.

$$\underbrace{[T]\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}3\\4\\8\end{bmatrix}}_{\text{I}}, \quad \underbrace{[T]\begin{bmatrix}0\\1\\1\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix}}_{\text{II}}, \quad \underbrace{[T]\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}-1\\2\\0\end{bmatrix}}_{\text{III}}$$

Ha, I see an easier way than my hint today!

$$\text{col}_3[T] = \begin{bmatrix}-1\\2\\0\end{bmatrix} \text{ by III}.$$

$$\text{II}: \text{col}_2([T]) + \text{col}_3([T]) = \begin{bmatrix}1\\0\\1\end{bmatrix} \Rightarrow \text{col}_2[T] = \begin{bmatrix}1\\0\\1\end{bmatrix} - \begin{bmatrix}-1\\2\\0\end{bmatrix} = \begin{bmatrix}2\\-2\\1\end{bmatrix},$$

$$\text{III}: \text{col}_1[T] + \text{col}_2[T] + \text{col}_3[T] = \begin{bmatrix}3\\4\\8\end{bmatrix} \Rightarrow \text{col}_1[T] = \begin{bmatrix}3\\4\\8\end{bmatrix} - \begin{bmatrix}2\\-2\\1\end{bmatrix} - \begin{bmatrix}-1\\2\\0\end{bmatrix}$$

$$\text{Hence } \text{col}_1[T] = \begin{bmatrix}3\\4\\8\end{bmatrix} - \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}2\\4\\7\end{bmatrix} \quad \therefore \quad \boxed{[T] = \begin{bmatrix} 2 & 2 & -1 \\ 4 & -2 & 2 \\ 7 & 1 & 0 \end{bmatrix}} - (*)$$

P73 a weirder solution

Notice $\{(1,0,0), (1,0,1), (1,2,3)\} = \beta$ serves as basis for $\mathbb{R}^3$. Observe

$$[\beta]^{-1} = \begin{bmatrix} 1 & 1 & -1 \\ 0 & -3/2 & 1 \\ 0 & 1/2 & 0 \end{bmatrix}$$

You can check $[\beta][\beta]^{-1} = I$. Let $\beta = \{v_1, v_2, v_3\}$

Ok, $[T]_{\beta\beta} = \big[\, [T(v_1)]_\beta \mid [T(v_2)]_\beta \mid [T(v_3)]_\beta \,\big] = [\beta]^{-1}[T][\beta]$

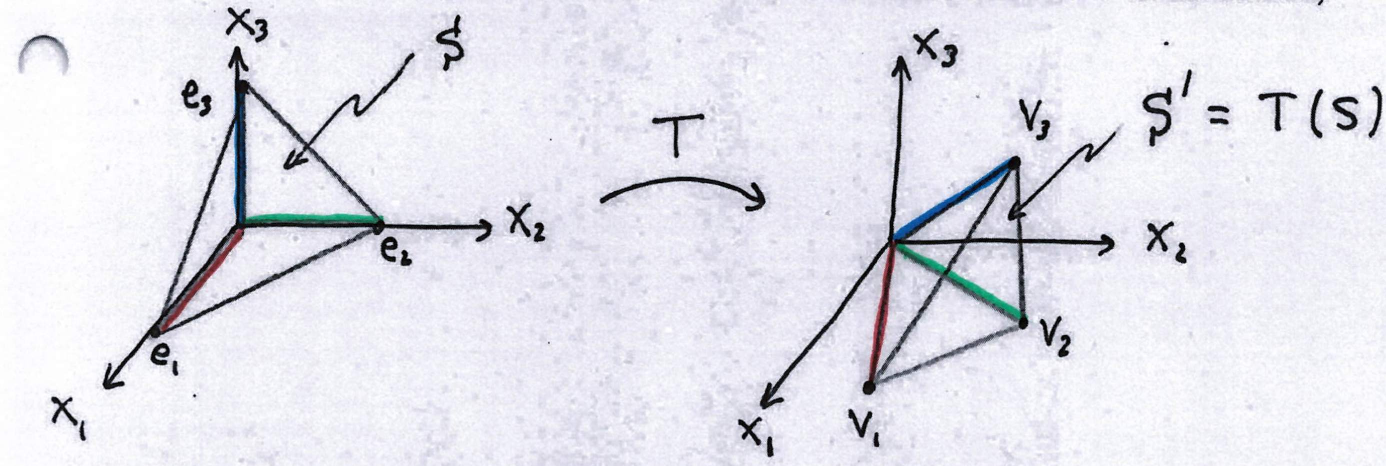
Thus,
$$\begin{aligned}
[T] &= [\beta]\big[\, [T(v_1)]_\beta \mid [T(v_2)]_\beta \mid [T(v_3)]_\beta \,\big][\beta]^{-1} \\
&= \big[\, [\beta][T(v_1)]_\beta \mid [\beta][T(v_2)]_\beta \mid [\beta][T(v_3)]_\beta \,\big][\beta]^{-1} \\
&= \big[\, T(v_1) \mid T(v_2) \mid T(v_3) \,\big][\beta]^{-1} \\
&= \left[\begin{array}{c|c|c} a & 0 & 1 \\ b & 1 & 0 \\ c & 0 & 0 \end{array}\right]\begin{bmatrix} 1 & 1 & -1 \\ 0 & -3/2 & 1 \\ 0 & 1/2 & 0 \end{bmatrix} \\
&= \begin{bmatrix} a & a+1/2 & -a \\ b & b-3/2 & -b+1 \\ c & c & -c \end{bmatrix}
\end{aligned}$$

$$\boxed{\begin{aligned} y &= [\beta]^{-1}x \\ x &= [\beta]y \\ &= [\beta][x]_\beta \end{aligned}}$$

$$\boxed{\left\{ T : \mathbb{R}^3 \to \mathbb{R}^3 \;\middle|\; T(x) = Ax \text{ with } A = \begin{bmatrix} a & a+1/2 & -a \\ b & b-3/2 & 1-b \\ c & c & -c \end{bmatrix}, a,b,c \in \mathbb{R} \right\}}$$

To obtain previous answer map
$$\begin{aligned} a &\longmapsto -c \\ b &\longmapsto 1-f \\ c &\longmapsto -i \end{aligned}$$
$\left(b - \frac{3}{2} \longmapsto 1 - f - \frac{3}{2} = -f - \frac{1}{2} \text{ etc...}\right)$

Problem 75: Lay §3.3#32. (using the concept of linear transformation to derive volume formula)



$$\left.\begin{aligned} T(e_1) &= V_1 \\ T(e_2) &= V_2 \\ T(e_3) &= V_3 \end{aligned}\right\} \quad \boxed{T(x,y,z) = \left[V_1 \mid V_2 \mid V_3\right]\begin{bmatrix} x \\ y \\ z \end{bmatrix}}$$

this transformation maps S to S'

(b.) $\text{Vol}(S) = \frac{1}{3}(\text{area of base}) \cdot \{\text{height}\}$

$= \frac{1}{3}\cdot\frac{1}{2}(1)(1)(1)$ (base is triangle)

$= \frac{1}{6}$

Linear transformation maps lines to lines

$$\text{Vol}(S') = \frac{1}{3}\left(\frac{1}{2}(V_1 \times V_2)\cdot V_3\right)$$

$$= \boxed{\frac{1}{6}\det\left[V_1 \mid V_2 \mid V_3\right]}$$

Or, since $T(S) = S'$ we know

$$\text{Vol}(S') = \det(T)\,\text{Vol}(S)$$

(since vol. of S fixed by lengths of sides)

$$= \underline{\det\left[V_1 \mid V_2 \mid V_3\right] \cdot \frac{1}{6}}.$$