Math 221	Mission 5: Eigenvectors & Jordan Form (40+5)pts)

Same instructions as Mission 1. This homework is based on Lectures 23 - 27. There are 5pts to earn for completely following formatting instructions. Feel free to use technology for any row-reductions, however, realize you may need to do some of these calculations in your next Boss Fight.

Problem 81: (1pt) Let $A = \begin{bmatrix} t & t^2 \\ 1 & t^3 \end{bmatrix}$. Calculate A^2 and verify that $\frac{d}{dt}(A^2) = \frac{dA}{dt}A + A\frac{dA}{dt}$. (show work below)

Problem 82: (2pt) Let $A = \begin{bmatrix} 0 & -6 \\ 1 & 5 \end{bmatrix}$. Find the eigenvalues and eigenvectors for A.

Problem 83: (1pt) For A in the previous problem, calculate A^n for arbitrary $n \in \mathbb{Z}$.

Problem 84: (3pts) Let $A = \begin{bmatrix} 7 & 2 & 3 \\ 3 & 10 & 3 \\ 2 & 0 & 6 \end{bmatrix}$. Find the eigenvalues and write each eigenspace as span of an appropriate vector or set of LI vectors for:

(a.) A,

(b.) A^2,

(c.) A^{-1}.

Problem 85: (2pts) Let $A = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}$. Find the eigenvalues and write each eigenspace as span of an appropriate vector or set of LI vectors.

Problem 86: (2pts) Let $A = \begin{bmatrix} 4 & 0 & 1 \\ -2 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$. Diagonalize A by finding P for which $P^{-1}AP$ is a diagonal matrix.

Problem 87: (3pt) For each of the matrices below. Find the eigenvalues by inspection and find one eigenvector by common sense, no need for calculation here if you understand matrix multiplication.

(a.) $A = \begin{bmatrix} -1 & 6 \\ 0 & 5 \end{bmatrix}$

(b.) $A = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 7 & 0 \\ 4 & 8 & 1 \end{bmatrix}$

(c.) $A = \begin{bmatrix} 1/2 & 2 & 0 & 0 \\ 0 & 1/3 & 2 & 0 \\ 0 & 0 & 1/4 & 2 \\ 0 & 0 & 0 & 1/5 \end{bmatrix}$

Problem 88: (2pt) Find the general real solution to $\frac{dx}{dt} = Ax$ for A given in

(a.) Problem 82

(b.) Problem 85

Problem 89: (6pts) Let $A = A_1 \oplus A_2$ where $A_1 = \begin{bmatrix} 3 & 0 \\ 8 & -1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} -1 & -4 \\ 8 & 11 \end{bmatrix}$

(a.) Find eigenvalues and eigenvectors for A_1

(b.) Find eigenvalues and eigenvectors for A_2

(c.) Find an eigenbasis for A, if β is that basis then calculate $[\beta]^{-1}A[\beta]$ without actually calculating $[\beta]^{-1}$ (use theory)

Problem 90: (4pts) Let $A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$

(a.) Find the eigenvalue of A (it will be repeated)

(b.) Find a basis $\beta = \{v_1, v_2\}$ consisting of 2-chain for A

(c.) Show $[\beta]^{-1}A[\beta] = J_2(\lambda)$

(d.) Solve $\frac{dx}{dt} = Ax$ with the help of the matrix exponential.

Problem 91: (3pt) Let $A = [7I_{59}] \oplus J_2(7) \oplus J_6(7) \oplus J_3(2) \oplus J_3(2) \oplus [2I_6]$.

(a.) Find the eigenvalues of A,

(b.) Find all the Jordan chains for A of length two or more,

(c.) Find a basis for each eigenspace,

(d.) Find a basis for each generalized eigenspace,

(e.) Find the characteristic polynomial for A,

(f.) Find the minimal polynomial for A.

Problem 92: (1pt) Show $e^{A \oplus B} = e^A \oplus e^B$. Then calculate $e^{t[J_2(6) \oplus J_3(7)]}$

Problem 93: (1pt) Let $A = J_3(8)^T \oplus J_2(0)$. Find a Jordan basis for A using the standard basis for $\mathbb{R}^5$.

Problem 94: (1pt) Let $J = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}$. Calculate e^{tJ} and express your answer using elementary functions and I and J.

Problem 95: (1pt) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$. Let $P(t) = det(A - tI)$. Find c_0, c_1, c_2 for which $P(t) = c_0 + c_1 t + c_2 t^2$ and show $P(A) = c_0 I + c_1 A + c_2 A^2 = 0$. (this exemplifies the Cayley Hamilton Theorem) (show work below)

answers and work for remaining problems to be shown on separate paper

Problem 96: (1pt) Consider $A = J_2(3) \oplus J_2(3) \oplus J_2(-3)$. Find the formula for A^{-1} as a polynomial in A. Check your answer by matrix multiplication.

Problem 97: (1pt) Let $A, B \in \mathbb{R}^{n \times n}$ and suppose $[A, [A, B]] = 0$ and $[B, [B, A]] = 0$ where $[A, B] = AB - BA$. Show that $e^A e^B = e^{A+B+\frac{1}{2}[A,B]}$.

Problem 98: (1pt) Let $Av = \lambda v$. Show $e^{tA} v = e^{\lambda t} v$.

Problem 99: (3pt) To show that $\det(e^A) = e^{\text{trace}(A)}$ for any $A \in \mathbb{C}^{n \times n}$ we procede in steps:

- **(a.)** show $P^{-1} e^A P = e^{P^{-1}AP}$,
- **(b.)** calculate the formula for e^J when $J = J_{\lambda_1}(m_1) \oplus \cdots \oplus J_{\lambda_1}(m_s)$ where $\lambda_1, \dots, \lambda_s$ are the eigenvalues of the s-chains which form a Jordan basis for A,
- **(c.)** Recall, there exists a Jordan form J for which $P^{-1}AP = J$ and show $\det(e^A) = e^{\text{trace}(A)}$.

Problem 100: (1pt) Consider $A = J_2(6) \oplus J_2(7)$ and $B = J_2(7) \oplus J_2(6)$. Show $A \sim B$ (this is read "A is similar to B") by finding a matrix P for which $P^{-1}AP = B$.