Same instructions as Mission 1. This homework is based on Lectures 23-33. There are 5pts to earn for completely following formatting instructions. Feel free to use technology for any row-reductions, however, realize you may need to do some of these calculations in your next Boss Fight.

Problem 101: (1pt) Let $A = \begin{bmatrix} 50 & -60 \\ 75 & 110 \end{bmatrix}$. Find the complex eigenvalues and complex eigenvectors of A

Problem 102: (2pt) Given A is as above, find a dilation and rotation whose product forms A. If we define $X_{k+1} = AX_k$ then describe what happens as $k \to \infty$ if $X_0 = (1, 0)$.

Problem 103: (3pts) This matrix is complex diagonalizable, but not real diagonalizable. Let $A = \begin{bmatrix} -2 & 0 & 1 \\ -6 & -2 & 0 \\ 19 & 5 & -4 \end{bmatrix}$.

(a.) Find the real eigenvalue λ_1 and complex eigenvalue $\lambda_2 = \alpha + i\beta$ where $\beta > 0$,

(b.) Find the matrix P for which $P^{-1}AP = \lambda_1 \oplus R_2(\alpha + i\beta)$,

(c.) Find a complex matrix Q for which $Q^{-1}AQ = \lambda_1 \oplus (\alpha + i\beta) \oplus (\alpha - i\beta)$.

Problem 104: (2pts) Let $A = \begin{bmatrix} 5 & 6 & 2 \\ 0 & -1 & -8 \\ 1 & 0 & -2 \end{bmatrix}$. Explain why A is not diagonalizable over $\mathbb{R}$ or $\mathbb{C}$.

Problem 105: (2pt) Find the general real solution to $\frac{dx}{dt} = Ax$ for A given in

(a.) Problem 103

(b.) Problem 104

Problem 106: (1pt) Suppose A is a 3×3 real matrix for which there exists a nonzero vector v such that $Av = (2+3i)v$. If $tr(A) = 71$ then find all the real and complex eigenvalues of A.

Problem 107: (2pt) Find the extrema of $f(x,y) = xy$ on the unit-circle $x^2 + y^2 = 1$.

Problem 108: (2pts) Consider the curve $4x^2 - 12xy + 4y^2 = 1$. Classify the curve, is it an ellipse, hyperbola or parabola ? Use linear algebra to support your claim.

Problem 109: (2pt) Suppose $Q(x, y) = 5x^2 + 4xy + 5y^2$.

(a.) Find a coordinate system $\bar{x}, \bar{y}$ for which the expression $5x^2 + 4xy + 5y^2$ transforms into an expression with no cross-term. Graph $Q(x, y) = 1$.

(b.) Also, find the maximum and minimum values which Q attains on the unit-circle.

Problem 110: (3pt) Suppose $Q(x) = 5x_1^2 + 2x_2^2 + 4x_3^2 + 4x_1x_2$.

(a.) find symmetric matrix A for which $Q(x) = x^T Ax$,

(b.) find the eigenvalues and orthonormal eigenbasis for A,

(c.) find maximal and minimal values for Q on the unit-sphere.

Problem 111: (3pt) Find a SVD for $A = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$.

Problem 112: (3pt) Suppose $A \in \mathbb{R}^{n\times n}$ is square and invertible. Find the SVD for A^{-1} in terms of the SVD for A.

Problem 113: (3pt) Let $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}$. Calculate the Moore-Penrose inverse of A and calculate the least squares approximation of the inconsistent equation $Ax = (0, 1, 0)$

Problem 114: (1pt) Suppose $M \in \mathbb{R}^{m\times m}$ and $N \in \mathbb{R}^{n\times n}$ such that x_1, x_2 are solutions $Mx_1 = y_1$ and $Nx_2 = y_2$. Let A be a block-diagonal matrix of the form:

$$A = \left[\begin{array}{c|c} M & 0 \\ \hline 0 & N \end{array}\right]$$

If $det(M), det(N) \neq 0$ then solve $Az = w$ where $w = [2y_1, 3y_2]^T$. (show work below)

Problem 115: (1pt) We defined $O(n) = \{R \in \mathbb{R}^{n\times n} \mid R^T R = I\}$. Suppose $A, B \in O(2)$ and define

$$R = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right]$$

Show $R \in O(4)$. (show work below)

Problem 116: (1pt) Suppose R is an orthogonal $n \times n$ matrix. Let $R_j = Col_j(R)$ and let

$$A = c_1 R_1 R_1^T + c_2 R_2 R_2^T + \cdots + c_n R_n R_n^T.$$

Show A is symmetric with eigenvalue $c_1, c_2, \ldots, c_n$. (show work below)

Problem 117: (1pt) Calculus of complex-valued functions of a real variable is defined by component-wise rules;

$$\frac{df}{dt} = \frac{d}{dt}(u + iv) = \frac{du}{dt} + i\frac{dv}{dt}$$

where u, v are the real and imaginary component functions of $f : \mathbb{R} \to \mathbb{C}$. We define

$$e^{(a+ib)t} = e^{at}(\cos bt + i \sin bt).$$

Prove $\frac{d}{dt}(e^{(a+ib)t} = (a + ib)e^{(a+ib)t}$. (show work below)

Problem 118: (1pt) Let T_1 and T_2 be orthogonal transformations. Show that $T_1 \circ T_2$ is also an orthogonal transformation. (show work below)

Problem 119: (3pt) Let $A = \begin{bmatrix} -2 & 7 & 0 & -1 \\ 7 & -2 & -1 & 0 \\ 0 & -1 & -2 & 7 \\ -1 & 0 & 7 & -2 \end{bmatrix}$. Find a matrix P for which $P^T AP = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ if possible. Also, Find a matrix Q for which $Q^T AQ = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ if possible.

Problem 120: (3pt) Linear transformations on $\mathbb{R}^2$ sometimes correspond to multiplication by a complex number. In particular, if $T(x+iy) = (a+ib)(x+iy)$ for some $a+ib \in \mathbb{C} = \mathbb{R}^2$ where $e_1 = 1$ and $e_2 = i$ then we say T is **complex-linear**.

(a.) If $T(x+iy) = (a+ib)(x+iy)$ for all $x+iy \in \mathbb{R}^2$ then show $[T] = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

(b.) Let $M(a+ib) = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and show $M(z)M(w) = M(zw)$ for all $z, w \in \mathbb{C}$.

(c.) The multiplicative inverse of a complex number $z = x+iy$ is given by $z^{-1} = \dfrac{x-iy}{x^2+y^2}$. Show $M(z^{-1}) = (M(z))^{-1}$ for any $z \in \mathbb{C} - \{0\}$.