Show your work and box answers. This pdf should be printed and your solution should be handwritten on the printout. Once complete, please staple in upper left corner. Thanks.

Suggested Reading You may find the following helpful resources beyond lecture,

(a.) Chapter 8 of my lecture notes for Math 221

(b.) §7.4 of Lay's *Linear Algebra*

Problem 106: Let $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Calculate e^{tJ} and express your answer in terms of $\cosh t$ and $\sinh t$ as well as I and J.

Problem 107: Suppose $M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. Calculate e^M directly from the power series definition of the matrix exponential. *Hint: convergence is not an issue here.*

Problem 108: Let $A = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$. Calculate e^{tA} and solve $\frac{dx}{dt} = Ax$.

Problem 109: Let $A = \lambda I + N$ where $\lambda \in \mathbb{R}$ and $N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and I is the usual 3×3 identity matrix. Notice I and N commute. Calculate e^{tA}.

Problem 110: Let $\beta = \{v_1, v_2, v_3, v_4, v_5\}$ be a basis such that

$$T(v_1) = 7v_1, \quad T(v_2) = 7v_2 + v_1, \quad T(v_3) = 7v_3 + v_2$$

and

$$T(v_4) = 11v_4, \quad T(v_5) = 11v_5 + v_4.$$

Calculate $[T]_{\beta,\beta}$ and explain why T is not diagonalizable. Classify each vector in β as an eigenvector or generalized eigenvector of a particular order.

Problem 111: If $A = [T]_{\beta,\beta}$ as given in the previous problem then solve $\frac{dx}{dt} = Ax$ where $x = (x_1, x_2, x_3, x_4, x_5)$ using the matrix exponential technique as shown in lecture.

Problem 112: One place we can anticipate the need for something more than eigenvectors is in the case of the differential equation $y'' = 0$ where $y' = \frac{dy}{dt}$. The solution is obtained by twice integrating to find $y = c_1 + c_2 t$. But, what does this have to do with systems of first order differential equations ? Well, let us make a **reduction of order** by introducing

$$x_1 = y \qquad \& \qquad x_2 = y'$$

then $x_1' = y' = x_2$ whereas $x_2' = y'' = 0$ hence we face:

$$\begin{aligned} \tfrac{dx_1}{dt} &= x_2 \\ \tfrac{dx_2}{dt} &= 0 \end{aligned}$$

That is, we face $\frac{dx}{dt} = Ax$ where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Show A is not diagonalizable by showing there are not enough linearly independent eigenvectors to form an eigenbasis for A.

Remark: notice the general solution $y = c_1 + c_2 t$ gives us $y' = c_2$ and hence $x_1 = c_1 + c_2 t$ and $x_2 = c_2$ thus the general solution has the following form in terms of our reduced variables:

$$x = \begin{bmatrix} c_1 + c_2 t \\ c_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} t \\ 1 \end{bmatrix}.$$

We can understand the solution with c_1 as its coefficient as an eigensolution stemming from $\lambda = 0$ which makes $e^{\lambda t} = e^0 = 1$, however the term with coefficient c_2 is not something which was we could cipher with mere eigenvectors. It requires a deeper magic.

Problem 113: Let us work through an analysis similar to the previous problem. Except this time let's look at the family of differential equations of the form $y'' - 2ay' + a^2y = 0$ where $a \in \mathbb{R}$.

(a) show $y_1 = e^{at}$ and $y_2 = te^{at}$ serve as solutions to the DEqn.

(b) let $x_1 = y$ and $x_2 = y'$ and rewrite the given second order differential equation as $\frac{dx}{dt} = Ax$ where $x = (x_1, x_2)$

(c) find an eigenvalue and eigenvector of A

(d) given $y = c_1e^{at} + c_2te^{at}$ is the general solution to $y'' - 2ay' + a^2y = 0$ find the corresponding solution to $\frac{dx}{dt} = Ax$. Which part of the vector solution is an eigensolution and which part is not ?

Problem 114: Consider $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$. Show $(A - 3I)e_1 = 0$ and $(A - 3I)e_2 = e_1$. Find the general solution of $\frac{dx}{dt} = Ax$ using the magic formula with $\lambda = 3$. How does your result compare the previous problem ?

Problem 115: If we faced a problem with a spring under a force tuned to the natural frequency of the spring then we would find the system has a pure resonance. Reduction of order for such a problem leads to $\frac{dx}{dt} = Ax$ where A has a complex eigenvector $v_1 = a_1 + ib_1$ and a generalized complex eigenvector $v_2 = a_2 + ib_2$ where $a_1, b_1, a_2, b_2 \in \mathbb{R}^4$ and there exists $\omega > 0$ for which

$$Av_1 = i\omega v_1 \qquad \& \qquad Av_2 = i\omega v_2 + v_1$$

Let $\beta = \{a_1, b_1, a_2, b_2\}$ serve as a basis for $\mathbb{R}^4$ and define $T(x) = Ax$.

(a.) show v_1, v_2 is a 2-chain of complex eigenvectors for A with $\lambda = i\omega$.

(b.) Calculate $[T]_{\beta,\beta}$.

(c.) find the real solution of $\frac{dx}{dt} = Ax$ in terms of the given vectors and ω.

Feel free to work these on your own paper, but please lable them as indicated below. Thanks!

Problem 116: Lay §7.4#3 (SVD)

Problem 117: Lay §7.4#7 (SVD)

Problem 118: Lay §7.4#11 (SVD)

Problem 119: Lay §7.4#17 (SVD)

Problem 120: Lay §7.4#23 (SVD)