Show your work and box answers. This pdf should be printed and your solution should be handwritten on the printout. Once complete, please staple in upper left corner. Thanks.

Suggested Reading You may find the following helpful resources beyond lecture,

(a.) Chapter 6 and 8 of my lecture notes for Math 221

Problem 76: Consider $A : \mathbb{R} \to \mathbb{R}^{n \times n}$ where $A_{ij} : \mathbb{R} \to \mathbb{R}$ is a smooth function for each component function. We defined $\left(\frac{dA}{dt}\right)_{ij} = \frac{d}{dt}[A_{ij}]$. In other words, the derivative of a matrix is done component-wise. Calculate $\frac{d}{dt}(A^2)$ and $\frac{d}{dt}(A^3)$.

$$\frac{d}{dt}(A^2) = \frac{d}{dt}(AA) = \underline{\frac{dA}{dt}A + A\frac{dA}{dt}}.$$

$$\frac{d}{dt}(A^3) = \frac{d}{dt}(A^2A) = \frac{d}{dt}(A^2)A + A^2\frac{dA}{dt}$$

$$= \left(\frac{dA}{dt}A + A\frac{dA}{dt}\right)A + A^2\frac{dA}{dt}$$

$$= \underline{\frac{dA}{dt}A^2 + A\frac{dA}{dt}A + A^2\frac{dA}{dt}}.$$

Problem 77: Suppose $\frac{dx}{dt} = x + 2y$ and $\frac{dy}{dt} = 2x + y$. Find the general solution using the eigenvector method we derived in lecture.

$$\begin{aligned}\frac{dx}{dt} &= x + 2y \\ \frac{dy}{dt} &= 2x + y\end{aligned} \quad \Rightarrow \quad \begin{pmatrix} x \\ y \end{pmatrix}' = \underbrace{\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$\begin{aligned}\det(A - \lambda I) = \det\begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} &= (\lambda - 1)^2 - 2^2 \\ &= (\lambda - 1 + 2)(\lambda - 1 - 2) \\ &= \underline{(\lambda + 1)(\lambda - 3)}.\end{aligned}$$

- We find e-values $\lambda_1 = -1$ and $\lambda_2 = 3$

$\underline{\lambda_1 = -1}$ $(A + I)u_1 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{aligned} 2u + 2v &= 0 \\ v &= -u \end{aligned}$

choose $\underline{u_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}}.$

$\underline{\lambda_2 = 3}$ $(A - 3I)u_2 = \underbrace{\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}}_{u = v} \Rightarrow \underline{u_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}}.$

Thus,

$$\vec{r} = \begin{bmatrix} x \\ y \end{bmatrix} = \boxed{c_1 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}. \quad \text{(general sol}^{\underline{n}}\text{)}$$

Solution in terms of x, y,

$$\boxed{\begin{aligned} x &= c_1 e^{-t} + c_2 e^{3t} \\ y &= -c_1 e^{-t} + c_2 e^{3t} \end{aligned}}$$

Problem 78: Suppose $\left\{\begin{array}{l} \frac{dx}{dt} = 9x + 7y - 13z \\ \frac{dy}{dt} = 7x + 9y - 13z \\ \frac{dz}{dt} = -13x - 13y + 29z \end{array}\right\}$. Find the general solution using the eigenvector method we derived in lecture.

Fun fact: $\lambda^3 - 47\lambda^2 + 216\lambda - 252$ has $\lambda = 2$ as a zero.

$$\begin{array}{r} \lambda^2 - 45\lambda + 126 \\ \lambda - 2 \,\big)\, \overline{\lambda^3 - 47\lambda^2 + 216\lambda - 252} \\ -(\lambda^3 - 2\lambda^2) \\ \hline -45\lambda^2 + 216\lambda - 252 \\ -(-45\lambda^2 + 90\lambda) \\ \hline 126\lambda - 252 \\ \underline{126\lambda - 252} \end{array}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix}' = \begin{bmatrix} 9 & 7 & -13 \\ 7 & 9 & -13 \\ -13 & -13 & 29 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda - 9 & -7 & 13 \\ -7 & \lambda - 9 & 13 \\ 13 & 13 & \lambda - 29 \end{bmatrix}$$

$$= (\lambda - 9)\left[(\lambda - 9)(\lambda - 29) - 169\right] + 7\left[-7(\lambda - 29) - 169\right] + 13\left[-91 - 13(\lambda - 9)\right]$$

$$= (\lambda - 9)\left[\lambda^2 - 38\lambda + 92\right] - 49\lambda + 238 - 169\lambda + 338$$

$$= \lambda^3 - 38\lambda^2 + 92\lambda - 9\lambda^2 + 342\lambda - 828 - 218\lambda + 576$$

$$= \lambda^3 - 47\lambda^2 + 216\lambda - 252$$

$$= (\lambda - 2)(\lambda^2 - 45\lambda + 126)$$

$$= (\lambda - 2)(\lambda - 3)(\lambda - 42)$$

$\underbrace{\lambda_1 = 2 \text{ gives } u_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}}_{\text{Can see } Au_1 = 2u_1}$, $\underbrace{\lambda_2 = 3 \text{ gives } u_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}}_{\text{Can see } Au_2 = 3u_2}$, $\underbrace{\lambda_3 = 42 \text{ gives } u_3 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}}_{\text{Can see } Au_3 = 42u_3}$

$$\boxed{\begin{pmatrix} x \\ y \\ z \end{pmatrix} = c_1 e^{2t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + c_3 e^{42t} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}}$$

Problem 79: Eigenvector problems:

(a.) Let $A = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$. Find a basis for the eigenspace with eigenvalue $\lambda = 4$

$$(A - 4I)u_1 = \begin{bmatrix} 6 & -9 \\ 4 & -6 \end{bmatrix} u_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad u_1 = \begin{bmatrix} u \\ v \end{bmatrix}$$

we can write less

$$\begin{bmatrix} 6 & -9 \\ 4 & -6 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 \\ 2 & -3 \end{bmatrix} \sim \begin{bmatrix} 2 & -3 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -3/2 \\ 0 & 0 \end{bmatrix} \quad \underline{u = \tfrac{3}{2}v}$$

$$u_1 = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} (3/2)v \\ v \end{bmatrix} = v\begin{bmatrix} 3/2 \\ 1 \end{bmatrix} \quad \text{choose } v = 2$$

$$\boxed{\mathcal{E}'_{\lambda=4} = \text{span}\left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\}} \quad \left(\left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\} \text{ BASIS FOR } \mathcal{E}'_{\lambda=4} \right)$$

(b.) Let $A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}$ find bases for the eigenspaces of A. Note $\lambda_1 = 1$ and $\lambda_2 = 5$.

$\underline{\lambda_1 = 1}$

$$(A - I)u_1 = \begin{bmatrix} 6 & 4 \\ -3 & -2 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow u_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix} \text{ will do nicely}$$

$\underline{\left\{ \begin{bmatrix} 2 \\ -3 \end{bmatrix} \right\} \text{ basis for } \mathcal{E}'_{\lambda_1 = 1}}$.

$\underline{\lambda_2 = 5}$

$$(A - 5I)u_2 = \begin{bmatrix} 2 & 4 \\ -3 & -6 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \Rightarrow u_2 = \begin{bmatrix} 4 \\ -2 \end{bmatrix} = 2\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

$\underline{\left\{ \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right\} \text{ gives basis for } \mathcal{E}'_{\lambda_2 = 5}}$.

Problem 80: Eigenvector problems:

(a.) Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -3 & 0 \\ 4 & -13 & 1 \end{bmatrix}$. Find a basis for the eigenspace with eigenvalue $\lambda = -2$

$$A + 2I = \begin{bmatrix} 3 & 0 & -1 \\ 1 & -1 & 0 \\ 4 & -13 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 3 & 0 & -1 \\ 4 & -13 & 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & -1 \\ 0 & -9 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 3 & -1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1/3 \\ 0 & 1 & -1/3 \\ 0 & 0 & 0 \end{bmatrix}$$

$u = \frac{+1}{3} w$

$v = \frac{1}{3} w$

w free

select $w = 3 \rightarrow \boxed{u_1 = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}}$ check $Au_1 = -2u_1$ ✓

(b.) Let $A = \begin{bmatrix} 3 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$. Find a basis for the eigenspace with eigenvalue $\lambda = 4$

$$A - 4I = \begin{bmatrix} -1 & 0 & 2 & 0 \\ 1 & -1 & 1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 2 & 0 \\ 0 & -1 & 3 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -1 & 0 & 2 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\left(\det \begin{bmatrix} -1 & 0 & 2 \\ 1 & -1 & 1 \\ 0 & 1 & -3 \end{bmatrix} = -1(3-1) + 2(1) = 0 \text{ rats.} \right)$$

$u_1 = +2u_3$

$u_2 = 3u_3$

u_3 & u_4 free

$$(A - 4I)u = 0 \Rightarrow u = \begin{bmatrix} +2u_3 \\ 3u_3 \\ u_3 \\ u_4 \end{bmatrix} = u_3 \begin{bmatrix} +2 \\ 3 \\ 1 \\ 0 \end{bmatrix} + u_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\boxed{W_{\lambda=4} = \text{span}\{(+2, 3, 1, 0), (0, 0, 0, 1)\}}$$

Problem 81: Let $A = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix}$.

(a.) Find a basis for eigenspace with $\lambda = 0$. Hint: guess.

Observe $\underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}_{u_1}$ and $\underbrace{\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}}_{u_2}$ have $Au_1 = 0$ and $Au_2 = 0$

By part (b.) $\dim(\mathcal{E}_{\lambda=0}) \leq 2$ and since $\{u_1, u_2\}$ LI we find $\boxed{\{(1,-1,0), (1,0,-1)\} \text{ is basis for } \mathcal{E}_{\lambda=0}}$

(b.) Calculate $\det(xI - A)$ and determine the other eigenvalue of A.

$$\det\left[xI - A\right] = \det\begin{bmatrix} x-5 & -5 & -5 \\ -5 & x-5 & -5 \\ -5 & -5 & x-5 \end{bmatrix}$$

$$= \det\begin{bmatrix} x & 0 & -x \\ -5 & x-5 & -5 \\ -5 & -5 & x-5 \end{bmatrix}$$

$$= \det\begin{bmatrix} x & 0 & -x \\ -5 & x-5 & -5 \\ 0 & -x & x \end{bmatrix}$$

$$= x(x(x-5) - 5x) - x(5x)$$

$$= x(x^2 - 10x) - 5x^2$$

$$= x^3 - 15x^2$$

$$= x^2(x-15) \Rightarrow \lambda = 0 \text{ or } \lambda = 15$$

(c.) Check that trace$(A) = \lambda_1 + \lambda_2 + \lambda_3$ (here $\lambda_1 = \lambda_2 = 0$)

$$\text{trace}(A) = tr\begin{pmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{pmatrix}$$

$$= 5 + 5 + 5$$

$$= 15 = \lambda_1 + \lambda_2 + \lambda_3 = 0 + 0 + 15. \checkmark$$

Problem 82: Let λ be the eigenvalue of an invertible matrix A. Prove A^{-1} has eigenvalue $1/\lambda$.

Let λ be e-value of invertible matrix A.

Then $\lambda \neq 0$ since $Ax = 0$ for $x \neq 0 \Rightarrow A^{-1}$ d.n.e.

(or $\det A = \lambda_1 \lambda_2 \cdots \lambda_n$ so $\lambda_j = 0 \Rightarrow \det A = 0 \Rightarrow A^{-1}$ d.n.e.)

So we may assume $\lambda \neq 0$ and $\exists x \neq 0$ s.t.

$$Ax = \lambda x \Rightarrow A^{-1}Ax = A^{-1}(\lambda x)$$
$$\Rightarrow x = \lambda A^{-1} x$$
$$\Rightarrow \frac{1}{\lambda} x = A^{-1} x$$

$\therefore$ A^{-1} has e-value $\frac{1}{\lambda}$ for e-vector x.

Problem 83: Suppose $Au = \lambda u$ and $Av = \mu v$ for $u, v \neq 0$ and define

$$x_k = c_1 \lambda^k u + c_2 \mu^k v$$

for $k = 0, 1, 2, \ldots$ where $c_1, c_2 \in \mathbb{R}$. Show that $Ax_k = x_{k+1}$ for $k = 0, 1, 2, \ldots$.

Observe,

$$x_{k+1} = c_1 \lambda^{k+1} u + c_2 \mu^{k+1} v$$

Thus,

$$\begin{aligned} Ax_k &= c_1 \lambda^k Au + c_2 \mu^k Av \\ &= c_1 \lambda^k \lambda u + c_2 \mu^k \mu v \\ &= c_1 \lambda^{k+1} u + c_2 \mu^{k+1} v \\ &= x_{k+1}. \end{aligned}$$

Problem 84: For each matrix below find the factored form of the characteristic equation and state the eigenvalues for A

(a.) $A = \begin{bmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ 6 & 7 & -2 \end{bmatrix}$.

$$\det(x - A) = \det\begin{bmatrix} x-5 & 2 & -3 \\ 0 & x-1 & 0 \\ -6 & -7 & x+2 \end{bmatrix}$$

$$= (x-5)\det\begin{bmatrix} x-1 & 0 \\ -7 & x+2 \end{bmatrix} - 6\det\begin{bmatrix} 2 & -3 \\ x-1 & 0 \end{bmatrix}$$

$$= (x-5)(x-1)(x+2) - 6(-3)(-1)(x-1)$$

$$= (x-1)\left[(x+2)(x-5) - 18\right]$$

$$= (x-1)\left[x^2 - 3x - 28\right] = \underline{(x-1)(x-7)(x+4)}.$$

$$= \underline{x^3 - 4x^2 - 25x + 28}.$$

$$\left[\begin{array}{l} \lambda_1 = 1,\ \lambda_2 = 7,\ \lambda_3 = -4 \\ \lambda_1\lambda_2\lambda_3 = -28 \\ \lambda_1 + \lambda_2 + \lambda_3 = 4 = \text{trace}(A) \end{array}\right]$$

(b.) $A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 8 & -4 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 1 & -5 & 2 & 1 \end{bmatrix}$ $\Rightarrow$ $\lambda_1 = 5,\ \lambda_2 = -4,\ \lambda_3 = 1$ (repeated)

$$\boxed{\det(\lambda I - A) = (\lambda - 5)(\lambda + 4)(\lambda - 1)^2}$$

(c.) $A = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 & 0 \\ 3 & 8 & 0 & 0 & 0 \\ 0 & -7 & 2 & 1 & 0 \\ -4 & 1 & 9 & -2 & 3 \end{bmatrix}$. $\Rightarrow$ $\left.\begin{array}{l} \lambda_1 = 3 \\ \lambda_2 = 1 \\ \lambda_3 = 0 \\ \lambda_4 = 1 \\ \lambda_5 = 3 \end{array}\right\}$ upper - Δ matrix can read e-values off the diag.

$$\boxed{\det(\lambda I - A) = -(\lambda - 3)^2(\lambda - 1)^2(\lambda)}$$

Problem 85: Consider the stochastic matrix $A = \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.3 & 0.8 & 0.3 \\ 0.2 & 0 & 0.4 \end{bmatrix}$. Let $v_1 = (0.3, 0.6, 0.1)$ and $v_2 = (1, -3, 2)$ and $v_3 = (-1, 0, 1)$ and $w = (1, 1, 1)$.

(a.) Show v_1, v_2, v_3 are eigenvectors of A.

(a.)
$$Av_1 = \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.3 & 0.8 & 0.3 \\ 0.2 & 0 & 0.4 \end{bmatrix}\begin{bmatrix} 0.3 \\ 0.6 \\ 0.1 \end{bmatrix} = \begin{bmatrix} 0.3 \\ 0.6 \\ 0.1 \end{bmatrix} \quad (\lambda_1 = 1)$$

$$Av_2 = \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.3 & 0.8 & 0.3 \\ 0.2 & 0 & 0.4 \end{bmatrix}\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 0.5 \\ -1.5 \\ 1 \end{bmatrix} = 0.5\begin{bmatrix} 1 \\ -3 \\ 2 \end{bmatrix} \quad (\lambda_2 = 0.5)$$

$$Av_3 = \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.3 & 0.8 & 0.3 \\ 0.2 & 0 & 0.4 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.2 \\ 0 \\ 0.2 \end{bmatrix} = 0.2\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad (\lambda_3 = 0.2)$$

(b.) Suppose $x_o = (\alpha, \beta, \gamma)$ where $\alpha + \beta + \gamma = 1$ and $\alpha, \beta, \gamma \geq 0$. If $x_o = c_1 v_1 + c_2 v_2 + c_3 v_3$ then derive why $c_1 = 1$. *Hint: multiply by* w^T

Calculate $W^T X_o$,

$$\underbrace{W^T X_o}_{} = c_1 (W^T V_1) + c_2 (W^T V_2) + c_3 (W^T V_3)$$

$$\alpha + \beta + \gamma = c_1 (0.3 + 0.6 + 0.1) + c_2 (0) + c_3 (0)$$

$$\therefore \boxed{1 = c_1}$$

(c.) Define $x_k = A^k x_o$. Show $x_k \to v_1$ as $k \to \infty$.

(c.) Define
$$X_k = A^k X_o = c_1 (A^k V_1) + c_2 (A^k V_2) + c_3 (A^k V_3)$$
$$= c_1 \lambda_1^k V_1 + c_2 \lambda_2^k V_2 + c_3 \lambda_3^k V_3$$

Noting $\lambda_2^k, \lambda_3^k \longrightarrow 0$ as $k \to \infty$ yet $\lambda_1 = 1$ we find $X_k \longrightarrow c_1 V_1 = V_1$.

Problem 86: Observe $A = \begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix} = PDP^{-1}$. Calculate A^k explicitly as a 2×2 matrix. Simplify where possible.

$$A^k = (PDP^{-1})(PDP^{-1})\cdots(PDP^{-1})(PDP^{-1})$$

$$= P\underbrace{DD\cdots D}_{k}P^{-1}$$

$$= PD^kP^{-1}$$

$$= \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2^k & 0 \\ 0 & 1^k \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 3(2)^k & 4 \\ 2^k & 1 \end{bmatrix}\begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} -3(2)^k + 4 & 12(2)^k - 12 \\ -2^k + 1 & 4(2)^k - 3 \end{bmatrix}}$$

Problem 87: Let $A = \begin{bmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{bmatrix}$. Find P for which $P^{-1}AP$ is a diagonal matrix. Verify your claim by multiplying out $P^{-1}AP$. You can use technology to find P and calculate P^{-1}.

$$A = \begin{bmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{bmatrix}$$

$\lambda_1 = 1$, $u_1 = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$

$\lambda_2 = 3$, $u_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ & $u_3 = \begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix}$

used website to find these e-vectors.

Let $P = \begin{bmatrix} -2 & -1 & -4 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ then

$$P^{-1}AP = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

Problem 88: Solve $\frac{dx}{dt} = 5x - 2y$ and $\frac{dy}{dt} = x + 3y$ via the eigenvector technique.

$$A = \begin{bmatrix} 5 & -2 \\ 1 & 3 \end{bmatrix}$$ find complex e-value / vector

$$\det(A - \lambda I) = \det\begin{bmatrix} 5-\lambda & -2 \\ 1 & 3-\lambda \end{bmatrix}$$
$$= (\lambda-5)(\lambda-3) + 2$$
$$= \lambda^2 - 8\lambda + 17$$
$$= (\lambda-4)^2 + 1 \Rightarrow \underline{\lambda = 4 \pm i}.$$

$$(A - (4+i))u_1 = \begin{bmatrix} 1-i & -2 \\ 1 & -1-i \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}. \qquad (-1-i)(1-i) = -(1-i^2) = -2.$$

$u = (1+i)v$ choose $\underline{v = 1} \hookrightarrow \underline{u = 1+i}$.

$$\underline{u_1 = \begin{bmatrix} 1+i \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + i\begin{bmatrix} 1 \\ 0 \end{bmatrix} \text{ for } \lambda = 4+i}.$$

Likewise, $\underline{u_2 = \begin{bmatrix} 1-i \\ 1 \end{bmatrix} \text{ for } \lambda = 4-i}$.

Solution to $\begin{pmatrix} x \\ y \end{pmatrix}' = A\begin{pmatrix} x \\ y \end{pmatrix}$ given by

$$\boxed{\begin{pmatrix} x \\ y \end{pmatrix} = c_1 e^{4t}\left(\cos t\begin{bmatrix} 1 \\ 1 \end{bmatrix} - \sin t\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + c_2 e^{4t}\left(\sin t\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \cos t\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)}$$

Alternatively,

$$\boxed{\begin{aligned} x &= c_1 e^{4t}(\cos t - \sin t) + c_2 e^{4t}(\sin t + \cos t) \\ y &= c_1 e^{4t}(\cos t) + c_2 e^{4t}\sin t \end{aligned}}$$

Problem 89: We saw $x_k = A^k x_o$ where $x_o = c_1u_1 + c_2u_2$ and $Au_1 = \lambda_1 u_1$ and $Au_2 = \lambda_2 u_2$ yields $x_k = c_1(\lambda_1)^k + c_2(\lambda_2)^k$. Classify the behavior of x_k as $k \to \infty$.

(a.) $A = \begin{bmatrix} 1.7 & -0.3 \\ -1.2 & 0.8 \end{bmatrix}$.

$$A = \begin{bmatrix} 1.7 & -0.3 \\ -1.2 & 0.8 \end{bmatrix}$$

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda - 1.7 & 0.3 \\ 1.2 & \lambda - 0.8 \end{pmatrix} = (\lambda - 1.7)(\lambda - 0.8) - 0.36$$
$$= \lambda^2 - 2.5\lambda + 1$$
$$= (\lambda - 2)(\lambda - 0.5)$$

$\underline{\lambda_1 = 2, \ \lambda_2 = 0.5}$

$$X_k = \underbrace{c_1 (2)^k u_1}_{\text{blows up}} + \underbrace{c_2 \left(\tfrac{1}{2}\right)^k u_2}_{\text{goes to zero}}$$

$\boxed{\text{origin saddle pt.}}$

(b.) $A = \begin{bmatrix} 0.4 & 0.5 \\ -0.4 & 1.3 \end{bmatrix}$.

$$A = \begin{bmatrix} 0.4 & 0.5 \\ -0.4 & 1.3 \end{bmatrix}$$

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda - 0.4 & -0.5 \\ 0.4 & \lambda - 1.3 \end{pmatrix}$$
$$= (\lambda - 0.4)(\lambda - 1.3) + 0.2$$
$$= \lambda^2 - 1.7\lambda + 0.72$$
$$= \underline{(\lambda - 0.9)(\lambda - 0.8)}$$

$$X_k = \underbrace{c_1 (0.9)^k u_1 + c_2 (0.8)^k u_2}_{\text{both go to zero as } k \to \infty}$$

find $\boxed{\text{origin is stable attractor}}$

Problem 90: Let $A = \begin{bmatrix} 30 & 64 & 23 \\ -11 & -23 & -9 \\ 6 & 15 & 4 \end{bmatrix}$. Find the general solution of $\frac{d\vec{r}}{dt} = A\vec{r}$. Please use technology to find the necessary eigenvectors and/or complex eigenvectors and then use the theory for differential equations which was discussed in lecture.

$$A = \begin{bmatrix} 30 & 64 & 23 \\ -11 & -23 & -9 \\ 6 & 15 & 4 \end{bmatrix}$$

$$\lambda_1 = \quad \text{with} \quad u_1 = \begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix}$$

$$\lambda_2 = 5 + 2i \quad \text{with} \quad u_2 = \begin{bmatrix} 0.92 \\ -0.37 + 0.0088i \\ 0.038 - 0.056i \end{bmatrix}$$

$$\text{or} \quad u_2 = \begin{bmatrix} 23 - 34i \\ -9 + 14i \\ 3 \end{bmatrix} = \underbrace{\begin{bmatrix} 23 \\ -9 \\ 3 \end{bmatrix}}_{a} + i \underbrace{\begin{bmatrix} -34 \\ 14 \\ 0 \end{bmatrix}}_{b}$$

$$x = c_1 e^{t} u_1 + c_2 e^{5t}\left((\cos 2t)a - (\sin 2t)b\right) + c_3 e^{5t}\left((\sin 2t)a + (\cos 2t)b\right)$$

$$\boxed{= c_1 e^{t}\begin{bmatrix} -3 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{5t}\left((\cos 2t)\begin{bmatrix} 23 \\ -9 \\ 3 \end{bmatrix} - (\sin 2t)\begin{bmatrix} -34 \\ 14 \\ 0 \end{bmatrix}\right) + c_3 e^{5t}\left(\sin(2t)\begin{bmatrix} 23 \\ -9 \\ 3 \end{bmatrix} + \cos(2t)\begin{bmatrix} -34 \\ 14 \\ 0 \end{bmatrix}\right)}$$