MATH 221	MISSION 7: QUADRADIC FORMS AND ORTHOGONALITY, LEAST SQUARES (30PTS)

Show your work and box answers. This pdf should be printed and your solution should be handwritten on the printout. Once complete, please staple in upper left corner. Thanks.

Suggested Reading You may find the following helpful resources beyond lecture,

(a.) Chapter 7 and §9.2 of my lecture notes for Math 221

(b.) Chapter 6 and §7.1, 7.2 of Lay's *Linear Algebra*

Problem 91: Consider the quadratic form given by

$$Q(x,y) = x^2 + 4xy + y^2 = (x,y)\begin{bmatrix}1&2\\2&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}$$

Find the diagonalized formula for Q in terms of eigencoordinates $\bar{x}, \bar{y}$.

$$[Q] = \begin{bmatrix}1&2\\2&1\end{bmatrix} \quad \det\begin{pmatrix}\lambda-1 & -2\\ -2 & \lambda-1\end{pmatrix} = (\lambda-1)^2 - 2^2 = (\lambda-1+2)(\lambda-1-2)$$
$$= (\lambda+1)(\lambda-3)$$

Thus $\lambda_1 = -1$ and $\lambda_2 = 3$ are eigenvalues of $[Q]$

and we know $\boxed{Q(x,y) = -\bar{x}^2 + 3\bar{y}^2}$ if $(\bar{x},\bar{y}) = [\beta]^{-1}(x,y)$

where $\beta = \{u_1, u_2\}$ is eigenbasis for $[Q]$

That is, $\boxed{Q(\bar{x}u_1 + \bar{y}u_2) = -\bar{x}^2 + 3\bar{y}^2}$ ← better (solved)

(commentary) (we know the formula holds as above from our theoretical calculations in Lecture. Now, we can make u_1, u_2 explicit and find formulas explicitly changing x, y to $\bar{x}, \bar{y}$)

$$[Q]\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1&2\\2&1\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}3\\3\end{bmatrix} = 3\begin{bmatrix}1\\1\end{bmatrix} \quad \therefore \quad u_2 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\1\end{bmatrix}$$

$$[Q]\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}1&2\\2&1\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}-1\\1\end{bmatrix} = -1\begin{bmatrix}1\\-1\end{bmatrix} \quad \therefore \quad u_1 = \frac{1}{\sqrt{2}}\begin{bmatrix}1\\-1\end{bmatrix}$$

Thus the change of coordinates is given by

$$\begin{bmatrix}\bar{x}\\\bar{y}\end{bmatrix} = [\beta]^{-1}\begin{bmatrix}x\\y\end{bmatrix} = [\beta]^T\begin{bmatrix}x\\y\end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix}1&-1\\1&1\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix} = \begin{pmatrix}\frac{1}{\sqrt{2}}(x-y)\\ \frac{1}{\sqrt{2}}(x+y)\end{pmatrix}$$

Problem 92: Consider the quadratic form given by

$$Q(x,y,z) = 9x^2 + 9y^2 + 29z^2 + 3.5xy - 6.5xz - 6.5yz.$$

Find the diagonalized formula for Q in terms of eigencoordinates $\bar{x}, \bar{y}, \bar{z}$.

$$[Q] = \begin{bmatrix} 9 & 1.75 & -3.25 \\ 1.75 & 9 & -3.25 \\ -3.25 & -3.25 & 29 \end{bmatrix}$$

This seems like a good time for technology!

$\lambda_1 = 7.25$, $u_1' = (-1,1,0) \to (-0.707, 0.707, 0) = u_1$

$\lambda_2 \cong 9.66$, $u_2' \cong (2.98, 2.98, 1) \to (0.69, 0.69, 0.23) = u_2$

$\lambda_3 \cong 30.1$, $u_3' \cong (-0.17, -0.17, 1) \to \underbrace{(-0.16, -0.16, 0.97) = u_3}_{\text{orthonormal}}$

$$Q(\bar{x}u_1 + \bar{y}u_2 + \bar{z}u_3) = \lambda_1\bar{x}^2 + \lambda_2\bar{y}^2 + \lambda_3\bar{z}^2$$
$$= \boxed{7.25\bar{x}^2 + 9.66\bar{y}^2 + 30.1\bar{z}^2}$$

Remark: I meant to double, not halve the values to make Q. The intended problem was $Q(x,y,z) = 9x^2 + 9y^2 + 29z^2 + 14xy - 26xz - 26yz$ which gives

$$[Q] = \begin{bmatrix} 9 & 7 & -13 \\ 7 & 9 & -13 \\ -13 & -13 & 29 \end{bmatrix} \Rightarrow \begin{array}{l} \lambda_1 = 2 \\ \lambda_2 = 3 \\ \lambda_3 = 42 \end{array}$$

Problem 93: Consider a subspace W of $\mathbb{R}^4$ which contains the vectors $(1, 1, 2, 3)$ and $(1, 0, 4, 5)$. Find a basis for $W^{\perp}$.

$$W^{\perp} = \text{Null}\begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 0 & 4 & 5 \end{bmatrix} = \left\{ x \in \mathbb{R}^4 \;\middle|\; \begin{matrix} x \cdot (1,1,2,3) = 0 \\ x \cdot (1,0,4,5) = 0 \end{matrix} \right\}$$

So we just need to find the nullspace basis by our usual calculation,

$$\begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 0 & 4 & 5 \end{bmatrix} \xrightarrow{r_2 - r_1} \begin{bmatrix} 1 & 1 & 2 & 3 \\ 0 & -1 & 2 & 2 \end{bmatrix} \xrightarrow{r_1 + r_2} \begin{bmatrix} 1 & 0 & 4 & 5 \\ 0 & -1 & 2 & 2 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 4 & 5 \\ 0 & 1 & -2 & -2 \end{bmatrix}$$

$x \in \text{Null}\begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 0 & 4 & 5 \end{bmatrix}$ has
$$\begin{aligned} x_1 &= -4x_3 - 5x_4 \\ x_2 &= 2x_3 + 2x_4 \end{aligned}$$

$$\begin{aligned} x &= (x_1, x_2, x_3, x_4) \\ &= (-4x_3 - 5x_4, 2x_3 + 2x_4, x_3, x_4) \\ &= x_3(-4, 2, 1, 0) + x_4(-5, 2, 0, 1) \end{aligned}$$

Thus $\boxed{\{(-4, 2, 1, 0), (-5, 2, 0, 1)\} \text{ is basis for } W^{\perp}}$

Problem 94: Find an orthonormal basis for

$$W = \text{span}\{(1,1,1,1), (0,1,-1,1), (2,0,2,0)\}$$

by using the Gram-Schmidt algorithm on the given generating vectors. Also, find an orthonormal basis for $W^{\perp}$.

$$u_1' = (1,1,1,1)$$

$$u_2' = (0,1,-1,1) - \frac{(1,1,1,1)\cdot(0,1,-1,1)}{4}(1,1,1,1) = \left(-1/4,\ 3/4,\ -5/4,\ 3/4\right)$$

$$u_3' = (2,0,2,0) - \frac{(1,1,1,1)\cdot(2,0,2,0)}{4}(1,1,1,1) - \frac{(-1,3,-5,3)\cdot(2,0,2,0)}{(-1,3,-5,3)\cdot(-1,3,-5,3)}(-1,3,-5,3)$$

$$u_3' = (2,0,2,0) - (1,1,1,1) + \frac{12}{44}(-1,3,-5,3)$$

$$u_3' = (1,-1,1,-1) + \left(\frac{-6}{22}, \frac{18}{22}, \frac{-30}{22}, \frac{18}{22}\right)$$

$$u_3' = \left(\frac{22-6}{22}, \frac{-22+18}{22}, \frac{22-30}{22}, \frac{-22+18}{22}\right)$$

$$u_3' = \frac{1}{22}(16,-4,-8,-4) = \frac{1}{11}(8,-2,-4,-2)$$

$$u_3 = \frac{2}{11}(4,-1,-2,-1)$$

Normalizing, the orthonormal basis for W is given by $\boxed{\beta_W = \left\{ \frac{1}{2}(1,1,1,1), \frac{1}{\sqrt{44}}(-1,3,-5,3), \frac{1}{\sqrt{22}}(4,-1,-2,-1)\right\}}$

$W^{\perp} = \text{Null}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 2 & 0 & 2 & 0 \end{bmatrix}$ hence calculate,

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 2 & 0 & 2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & -2 & 0 & -2 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & -2 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{array}{l} \to x_1 = 0 \\ x_2 = -x_4 \\ x_3 = 0 \end{array}$$

Thus $x \in W^{\perp}$ has $x = (0, -x_4, 0, x_4) = x_4(0,-1,0,1)$

hence $\boxed{\beta_{W^{\perp}} = \left\{\frac{1}{\sqrt{2}}(0,-1,0,1)\right\}}$

Problem 95: Let $f(x,y,z) = 9x^2 + 14\sin(xy) + y\sinh y + 29e^{z^2} - 26z(x+y)$.

(a.) Calculate partial derivatives $f_x, f_y, f_z, f_{xx}, f_{xy}, f_{xz}, f_{yy}, f_{yz}, f_{zz}$

(b.) calculate the multivariate Taylor series based at $(0,0,0)$ up to second order. You should find that $(0,0,0)$ is a critical point hence $f(x,y,z) = f(0,0,0) + Q(z,y,z) + \cdots$ where the quadratic form Q has matrix with entries fixed by the values of the second derivatives of f at $(0,0,0)$:

$$[Q] = \begin{bmatrix} f_{xx}(0,0,0) & f_{xy}(0,0,0) & f_{xz}(0,0,0) \\ f_{xy}(0,0,0) & f_{yy}(0,0,0) & f_{yz}(0,0,0) \\ f_{xz}(0,0,0) & f_{yz}(0,0,0) & f_{zz}(0,0,0) \end{bmatrix}$$

(c.) classify the nature of the critical point $(0,0,0)$ by diagonalizing Q. Is the function minimized, maximized or is it at a saddle point at the origin ?

Remark: you stumble across a homeless mathematician with chalk in hand scribbling on the ground, you notice the phrase 78 WAS A GOOD YEAR, *strange, what does this mean.*

(a.) $f = 9x^2 + 14\sin(xy) + y\sinh y + 29e^{z^2} - 26z(x+y)$

$f_x = 18x + 14y\cos(xy) - 26z$

$f_y = 14x\cos(xy) + \sinh y + y\cosh y - 26z$

$f_z = 58ze^{z^2} - 26(x+y)$

$f_{xx} = 18 - 14xy\sin(xy)$

$f_{xy} = 14\cos(xy) - 14xy\sin(xy)$

$f_{xz} = -26$

$f_{yy} = -14x^2\sin(xy) + \cosh y + \cosh y + y\sinh y$

$f_{yz} = -26$

$f_{zz} = 58e^{z^2} + 58z(2ze^{z^2})$

Remark: $\nabla f(0,0,0) = \langle 0,0,0\rangle$ gives $(0,0,0)$ as critical point.

(b.) $[Q] = \begin{bmatrix} 18 & 14 & -26 \\ 14 & 2 & -26 \\ -26 & -26 & 58 \end{bmatrix}$ from evaluating formulas.

Remark: sorry, I meant for $f_{yy}(0,0,0) = 18$, then P78 solves it

(c.) technology shows $[Q]$ has $\lambda_1 \cong -8.99$, $\lambda_2 \cong 5.27$, $\lambda_3 \cong 81.7$ thus $f(0,0,0)$ is a saddle point, neither min nor max.

P96 Show set is orthogonal and express x as linear comb.

§6.2 #8

$u_1 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $u_2 = \begin{bmatrix} -2 \\ 6 \end{bmatrix}$ has $u_1 \cdot u_2 = -6+6=0$.

Given $x = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$ notice $x \cdot u_1 = -18+3 = -15$

$x \cdot u_2 = 12+18 = 30$

Then $x = \underbrace{\left(\frac{x \cdot u_1}{u_1 \cdot u_1}\right) u_1 + \left(\frac{x \cdot u_2}{u_2 \cdot u_2}\right) u_2}_{} = \left(\frac{-15}{10}\right) u_1 + \left(\frac{30}{40}\right) u_2$

using Th^{m} 5 from pg. 385 of Lay, or my notes.

$$\boxed{x = (-1.5)\begin{bmatrix} 3 \\ 1 \end{bmatrix} + (0.75)\begin{bmatrix} -2 \\ 6 \end{bmatrix} = \begin{bmatrix} -4.5-1.5 \\ -1.5+4.5 \end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix} -6 \\ 3 \end{bmatrix}}.$$

§6.2#10 $u_1 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix}$, $u_2 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$, $u_3 = \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}$ and $x = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$

notice $u_1 \cdot u_2 = 0 = u_1 \cdot u_3 = u_2 \cdot u_3$ and $u_1 \cdot u_1 = 18$ and $u_2 \cdot u_2 = 9$ and $u_3 \cdot u_3 = 18$ hence,

$$x = \left(\frac{x \cdot u_1}{18}\right) u_1 + \frac{(x \cdot u_2)}{9} u_2 + \frac{(x \cdot u_3)}{18} u_3$$

$$x = \left(\frac{24}{18}\right) u_1 + \left(\frac{3}{9}\right) u_2 + \left(\frac{6}{18}\right) u_3$$

$$\boxed{x = \tfrac{4}{3}(3,-3,0) + \tfrac{1}{3}(2,2,-1) + \tfrac{1}{3}(1,1,4)}.$$

P97

§6.2 #14 $y = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ and $u = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$

write y as sum of vector in span$\{u\}$ and a vector $\perp$ to u

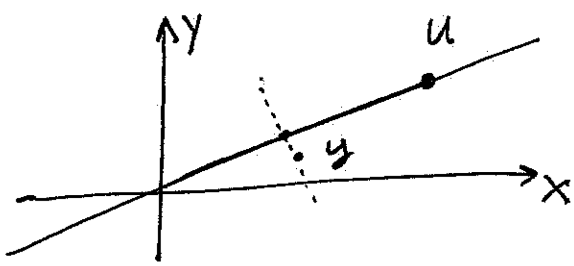
$$\mathrm{Proj}_u(y) = \left(\frac{y \cdot u}{u \cdot u}\right) u = \left(\frac{14+6}{49+1}\right) u = \frac{20}{50} u = \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix}$$

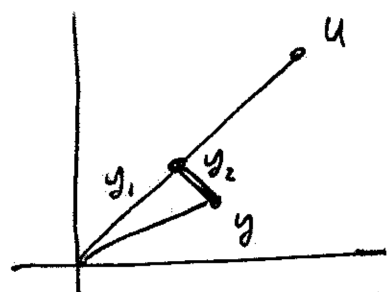
$$\mathrm{Orth}_u(y) = y - \mathrm{Proj}_u(y) = \begin{bmatrix} 2 \\ 6 \end{bmatrix} - \begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix} = \begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix}$$

$$y = \underbrace{\begin{bmatrix} 14/5 \\ 2/5 \end{bmatrix}}_{\parallel u} + \underbrace{\begin{bmatrix} -4/5 \\ 28/5 \end{bmatrix}}_{\perp u}$$

§6.2 #15 $y = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$ and $u = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$. Compute

distance from y to line through u and the origin





$$y_1 = \mathrm{Proj}_u(y) = \left(\frac{y \cdot u}{u \cdot u}\right) u = \frac{30}{100} u = \frac{3}{10}\begin{bmatrix} 8 \\ 6 \end{bmatrix} = \begin{bmatrix} 24/10 \\ 18/10 \end{bmatrix}$$

$$y_2 = y - \mathrm{Proj}_u(y) = \begin{bmatrix} 3 \\ 1 \end{bmatrix} - \begin{bmatrix} 24/10 \\ 18/10 \end{bmatrix} = \begin{bmatrix} 6/10 \\ -8/10 \end{bmatrix}$$

$$\text{distance} = \|y_2\| = \sqrt{\left(\frac{6}{10}\right)^2 + \left(-\frac{8}{10}\right)^2} = \sqrt{\frac{100}{100}} = \boxed{1}$$

$\boxed{P98}$ (§6.3#4) verify $\{u_1, u_2\}$ is an orthogonal set and then find orthogonal projection of y onto $\underbrace{\text{Span}\{u_1, u_2\}}_{W}$.

$$y = \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix}, \quad u_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \quad u_2 = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}$$

Notice $u_1 \cdot u_2 = 3 + 1 - 4 = 0$ then

$$\begin{aligned}
\text{Proj}_W(y) &= \left(\frac{y \cdot u_1}{u_1 \cdot u_1}\right)u_1 + \left(\frac{y \cdot u_2}{u_2 \cdot u_2}\right)u_2 \\
&= \left(\frac{-3-2+12}{9+1+4}\right)u_1 + \left(\frac{-1-2-12}{1+1+4}\right)u_2 \\
&= \frac{7}{14}u_1 - \frac{15}{6}u_2 \\
&= \frac{1}{2}(3,-1,2) - \frac{5}{2}(1,-1,-2) \\
&= \left(3/2 - 5/2,\ -1/2 + 5/2,\ 2/2 + 10/2\right) \\
&= \boxed{(-1,\ 2,\ 6)}
\end{aligned}$$

Check, $\text{Orth}_W(y) = y - \text{Proj}_W(y) = \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix} - \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

well, this is rather silly.

P99 §6.3#12 find closest pt to y in $W = \text{span}\{v_1, v_2\}$

$$y = \begin{bmatrix} 3 \\ -1 \\ 1 \\ 13 \end{bmatrix}, \quad v_1 = \begin{bmatrix} 1 \\ -2 \\ -1 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ 3 \end{bmatrix} \qquad \underbrace{v_1 \cdot v_2 = 0}_{\text{noticed.}}$$

$$\text{Proj}_W(y) = \left(\frac{y \cdot v_1}{v_1 \cdot v_1}\right) v_1 + \left(\frac{y \cdot v_2}{v_2 \cdot v_2}\right) v_2$$

$$= \left(\frac{3+2-1+26}{1+4+1+4}\right) v_1 + \left(\frac{-12-1+39}{16+1+9}\right) v_2$$

$$= \left(\frac{30}{10}\right) v_1 + \left(\frac{26}{26}\right) v_2$$

$$= 3v_1 + v_2$$

$$= 3(1, -2, -1, 2) + (-4, 1, 0, 3)$$

$$= \boxed{(-1, -5, -3, 9)}$$

§6.3#16 Let y, v_1, v_2 as above, find distance from y to $W = \text{Span}\{v_1, v_2\}$

$$\text{Orth}_W(y) = y - \text{Proj}_W(y)$$

$$= (3, -1, 1, 13) - (-1, -5, -3, 9)$$

$$= (4, 4, 4, 4)$$

distance from y to W $= \|\text{Orth}_W(y)\| = \sqrt{16+16+16+16}$

$$= \sqrt{64}$$

$$= \boxed{8}$$

$\boxed{P100}$ (§6.4 #11) find orthogonal basis for Col(A)
(§6.4 #15) find QR – decomp. for A

$$A = \begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix} = [A_1 | A_2 | A_3]$$

$$u_1' = (1,-1,-1,1,1)$$

$$u_2' = (2,1,4,-4,2) - \frac{(2,1,4,-4,2)\cdot(1,-1,-1,1,1)}{(1,-1,-1,1,1)\cdot(1,-1,-1,1,1)}(1,-1,-1,1,1)$$

$$u_2' = (2,1,4,-4,2) - \frac{2-1-4-4+2}{5}(1,-1,-1,1,1)$$

$$u_2' = (2,1,4,-4,2) + (1,-1,-1,1,1)$$

$$\underline{u_2' = (3,0,3,-3,3)} = 3(1,0,1,-1,1)$$

$$u_3' = (5,-4,-3,7,1) - \frac{20}{5}(1,-1,-1,1,1) + \frac{4}{4}(1,0,1,-1,1)$$

$$u_3' = (5,-4,-3,7,1) - (4,-4,-4,4,4) + (1,0,1,-1,1)$$

$$\underline{u_3' = (2,0,2,2,-2)}.$$

Thus
$$\boxed{\begin{aligned} u_1 &= \tfrac{1}{\sqrt{5}}(1,-1,-1,1,1) \\ u_2 &= \tfrac{1}{2}(1,0,1,-1,1) \\ u_3 &= \tfrac{1}{2}(1,0,1,1,-1) \end{aligned}}$$
← orthonormal basis for Col(A).

(book gives u_1', u_2', u_3' for orthogonal basis)

P100 continued

$$A = QR \quad \text{where} \quad Q = [u_1 | u_2 | u_3]$$

$$R = Q^T A$$

$$= \begin{bmatrix} 1/\sqrt{5} & -1/\sqrt{5} & -1/\sqrt{5} & 1/\sqrt{5} & 1/\sqrt{5} \\ 1/2 & 0 & 1/2 & -1/2 & 1/2 \\ 1/2 & 0 & 1/2 & 1/2 & -1/2 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 \\ -1 & 1 & -4 \\ -1 & 4 & -3 \\ 1 & -4 & 7 \\ 1 & 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \sqrt{5} & -\sqrt{5} & 4\sqrt{5} \\ 0 & 6 & -2 \\ 0 & 0 & 4 \end{bmatrix}$$

$\boxed{P101}$ (§6.5 #4)

$$A = \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3 & 11 \end{bmatrix}, \quad A^Tb = \begin{bmatrix} 1 & 1 & 1 \\ 3 & -1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ 14 \end{bmatrix}$$

$$A^TA\hat{x} = A^Tb$$

$$\hat{x} = (A^TA)^{-1}A^Tb = \frac{1}{33-9}\begin{bmatrix} 11 & -3 \\ -3 & 3 \end{bmatrix}\begin{bmatrix} 6 \\ 14 \end{bmatrix} = \frac{1}{24}\begin{bmatrix} 24 \\ 24 \end{bmatrix} = \boxed{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$$

(§6.5#8) find least squares error for $\hat{x} = (1,1)$

$$\| A\hat{x} - b \| = \| (4,0,2) - (5,1,0) \| = \| (-1,-1,2) \|$$

$$= \sqrt{1+1+4}$$

$$= \boxed{\sqrt{6}}$$

$\boxed{P102}$ (§6.5 #11) (a.) find $\text{Proj}_{\text{col}(A)}(b)$
(b.) find least squares sol^n of $Ax = b$

(a.) $$A = \begin{bmatrix} 4 & 0 & 1 \\ 1 & -5 & 1 \\ 6 & 1 & 0 \\ 1 & -1 & -5 \end{bmatrix} = [A_1|A_2|A_3] \quad \& \quad b = \begin{bmatrix} 9 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$u_1' = \underline{(4,1,6,1)}.$$

$$u_2' = A_2 - \left(\frac{A_2 \cdot u_1'}{u_1' \cdot u_1'}\right)u_1' = (0,-5,1,-1) - 0(4,1,6,1) = \underline{(0,-5,1,-1)}.$$

I see, no need for G.S. here $A_1 \cdot A_2 = A_1 \cdot A_3 = A_2 \cdot A_3 = 0$

Then $u_1 = \dfrac{(4,1,6,1)}{\sqrt{54}}$, $u_2 = \dfrac{(0,-5,1,-1)}{\sqrt{27}}$, $u_3 = \dfrac{(1,1,0,-5)}{\sqrt{27}}$

$$\begin{aligned} \text{Proj}_{\text{col}(A)}(b) &= (b \cdot u_1)u_1 + (b \cdot u_2)u_2 + (b \cdot u_3)u_3 \\ &= \left(\frac{36}{54}\right)(4,1,6,1) + 0 \cdot u_2 + \frac{9}{27}(1,1,0,-5) \\ &= \boxed{(3,1,4,-1)}. \end{aligned}$$

(b.) $A^T A \hat{x} = A^T b$

$$A^T A = \begin{bmatrix} 4 & 1 & 6 & 1 \\ 0 & -5 & 1 & -1 \\ 1 & 1 & 0 & -5 \end{bmatrix} \begin{bmatrix} 4 & 0 & 1 \\ 1 & -5 & 1 \\ 6 & 1 & 0 \\ 1 & -1 & -5 \end{bmatrix} = \begin{bmatrix} 54 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 27 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 4 & 1 & 6 & 1 \\ 0 & -5 & 1 & -1 \\ 1 & 1 & 0 & -5 \end{bmatrix} \begin{bmatrix} 9 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 36 \\ 0 \\ 9 \end{bmatrix}$$

$$\hat{x} = (A^T A)^{-1} A^T b = \begin{bmatrix} 1/54 & & \\ & 1/27 & \\ & & 1/27 \end{bmatrix} \begin{bmatrix} 36 \\ 0 \\ 9 \end{bmatrix} = \boxed{\begin{bmatrix} 2/3 \\ 0 \\ 1/3 \end{bmatrix}}$$

P103 (§6.5 #15)

$$A = \begin{bmatrix} 2 & 3 \\ 2 & 4 \\ 1 & 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 2/3 & -1/3 \\ 2/3 & 2/3 \\ 1/3 & -2/3 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} 3 & 5 \\ 0 & 1 \end{bmatrix}}_{R} \quad , \quad b = \begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix}$$

Solve $A^T A \hat{x} = A^T b$ using the QR-decomp above.

Since $Q^T Q = I$ notice as $A^T = (QR)^T = R^T Q^T$

$$\Rightarrow R^T Q^T Q R \hat{x} = R^T Q^T b$$

$$\Rightarrow R^T R \hat{x} = R^T Q^T b$$

$$\Rightarrow \underline{R\hat{x} = Q^T b}$$ nice equations.

$$\begin{bmatrix} 3 & 5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 2/3 & 2/3 & 1/3 \\ -1/3 & 2/3 & -2/3 \end{bmatrix}\begin{bmatrix} 7 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 21/3 \\ -3/3 \end{bmatrix} = \begin{bmatrix} 7 \\ -1 \end{bmatrix}$$

$$3u + 5v = 7$$

$$v = -1 \quad \Rightarrow u = \frac{7-5v}{3} = \frac{7+5}{3} = \frac{12}{3}$$

$$\therefore \boxed{\hat{x} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}}$$

$\boxed{P104}$ (§6.6 #1)

Find β_0, β_1 making $y = \beta_0 + \beta_1 x$ the least-squares line for data pts. $(0,1), (1,1), (2,2), (3,2)$. Plug in the data into $y = \beta_0 + \beta_1 X$,

$$\begin{aligned} 1 &= \beta_0 \\ 1 &= \beta_0 + \beta_1 \\ 2 &= \beta_0 + 2\beta_1 \\ 2 &= \beta_0 + 3\beta_1 \end{aligned} \longrightarrow \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix}}_{A} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix}}_{b}$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 4 & 6 \\ 6 & 14 \end{bmatrix}^{-1} \begin{bmatrix} 6 \\ 11 \end{bmatrix} = \frac{1}{56-36} \begin{bmatrix} 14 & -6 \\ -6 & 4 \end{bmatrix} \begin{bmatrix} 6 \\ 11 \end{bmatrix}$$

$$\begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 18 \\ 8 \end{bmatrix} = \begin{bmatrix} 0.9 \\ 0.4 \end{bmatrix} \Rightarrow \boxed{y = 0.9 + 0.4x}$$

$\boxed{P105}$ (§6.6 #7)

Data $(1, 1.8)$, $(2, 2.7)$, $(3, 3.4)$, $(4, 3.8)$, $(5, 3.9)$
is to fit the model $y = \beta_1 x + \beta_2 x^2$

(a.) write "design matrix" etc..
(b.) find least squares sol$^{\underline{n}}$ (use technology)

$$\begin{aligned} \beta_1 + \beta_2 &= 1.8 \\ 2\beta_1 + 4\beta_2 &= 2.7 \\ 3\beta_1 + 9\beta_2 &= 3.4 \\ 4\beta_1 + 16\beta_2 &= 3.8 \\ 5\beta_1 + 25\beta_2 &= 3.9 \end{aligned}$$

$$\underbrace{\begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \\ 5 & 25 \end{bmatrix}}_{A} \begin{bmatrix} \beta_1 \\ \beta_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 1.8 \\ 2.7 \\ 3.4 \\ 3.8 \\ 3.9 \end{bmatrix}}_{b}$$

$$A^T A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 2 & 4 \\ 3 & 9 \\ 4 & 16 \\ 5 & 25 \end{bmatrix} = \begin{bmatrix} 55 & 225 \\ 225 & 979 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 4 & 9 & 16 & 25 \end{bmatrix} \begin{bmatrix} 1.8 \\ 2.7 \\ 3.4 \\ 3.8 \\ 3.9 \end{bmatrix} = \begin{bmatrix} 52.1 \\ 201.5 \end{bmatrix}$$

$$\begin{aligned} A^T A \hat{\beta} &= A^T b \\ \hat{\beta} &= (A^T A)^{-1} (A^T b) \\ &= \frac{1}{3220} \begin{bmatrix} 979 & -225 \\ -225 & 55 \end{bmatrix} \begin{bmatrix} 52.1 \\ 201.5 \end{bmatrix} = \begin{bmatrix} 1.760 \\ -0.199 \end{bmatrix} \end{aligned}$$

$$\boxed{y = 1.76x - 0.199x^2}$$