Show your work and box answers. This pdf should be printed and your solution should be handwritten on the printout. Once complete, please staple in upper left corner. Thanks.

Suggested Reading You may find the following helpful resources beyond lecture,

(a.) Chapter 8 of my lecture notes for Math 221

(b.) §7.4 of Lay's *Linear Algebra*

Problem 106: Let $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Calculate e^{tJ} and express your answer in terms of $\cosh t$ and $\sinh t$ as well as I and J.

$$J^2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I \Rightarrow \underline{J^{2k} = I \text{ and } J^{2k+1} = J} \ *$$

$$e^{tJ} = \sum_{k=0}^{\infty} \frac{(tJ)^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(tJ)^{2k+1}}{(2k+1)!} \quad *$$

$$= \left(\sum_{k=0}^{\infty} \frac{t^{2k}}{(2k)!}\right) I + \left(\sum_{k=0}^{\infty} \frac{t^{2k+1}}{(2k+1)!}\right) J$$

$$= \boxed{(\cosh t)I + (\sinh t)J}$$

$$= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$$

Didn't ask for it, but to solve $\frac{dx}{dt} = Jx$ we may use

$$\underline{x = c_1 \begin{bmatrix} \cosh t \\ \sinh t \end{bmatrix} + c_2 \begin{bmatrix} \sinh t \\ \cosh t \end{bmatrix}}.$$

Feel free to work these on your own paper, but please lable them as indicated below. Thanks!

Problem 116: Lay §7.4#3 (SVD)

Problem 117: Lay §7.4#7 (SVD)

Problem 118: Lay §7.4#11 (SVD)

Problem 119: Lay §7.4#17 (SVD)

Problem 120: Lay §7.4#23 (SVD)

Problem 107: Suppose $M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. Calculate e^M directly from the power series definition of the matrix exponential. *Hint: convergence is not an issue here.*

$$M = N_4^T \Rightarrow M^T = N_4 \Rightarrow (M^T)^4 = N_4^4 = 0$$

$$\Rightarrow (M^4)^T = 0$$

$$\Rightarrow M^4 = 0.$$

Of cource we can just calculate directly,

$$M^2 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$M^3 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

Thus,

$$e^M = I + M + \frac{1}{2}M^2 + \frac{1}{6}M^3$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1/6 & 0 & 0 & 0 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1/2 & 1 & 1 & 0 \\ 1/6 & 1/2 & 1 & 1 \end{bmatrix}}$$

I meant to ask for e^{tM}. Note $e^{tM} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ t & 1 & 0 & 0 \\ t^2/2 & t & 1 & 0 \\ t^3/6 & t^2/2 & t & 1 \end{bmatrix}$

Problem 108: Let $A = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$. Calculate e^{tA} and solve $\frac{dx}{dt} = Ax$.

$$A^2 = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}\begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} -9 & 0 \\ 0 & -9 \end{bmatrix} = -3^2 \cdot I$$

$$A^3 = A(-3^2 \cdot I) = \begin{bmatrix} 0 & -3^3 \\ 3^3 & 0 \end{bmatrix} = -3^3 \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$A^4 = (-3^2)(-3^2)\, I \cdot I = 3^4 I$$

Let $J = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ then $A = 3J$ and $A^n = 3^n J^n$

and $J^2 = -I$ and $J^{2k} = (-I)^{k} = (-1)^k I$

whereas $J^{2k+1} = (-1)^{k+1} J$ thus

$$A^{2k} = 3^{2k}(-1)^k I$$

$$A^{2k+1} = 3^{2k+1}(-1)^k J$$

Therefore,

$$e^{tA} = \sum_{k=0}^{\infty} \frac{(3t)^{2k}(-1)^k}{(2k)!} I + \sum_{k=0}^{\infty} \frac{(3t)^{2k+1}(-1)^k}{(2k+1)!} J$$

$$= \cos(3t)\, I + \sin(3t)\, J$$

$$= \boxed{\begin{bmatrix} \cos(3t) & \sin(3t) \\ -\sin(3t) & \cos(3t) \end{bmatrix}}$$

Thus $\frac{dx}{dt} = Ax$ has solution,

$$\boxed{x = c_1 \begin{bmatrix} \cos(3t) \\ -\sin(3t) \end{bmatrix} + c_2 \begin{bmatrix} \sin(3t) \\ \cos(3t) \end{bmatrix}}$$

Problem 109: Let $A = \lambda I + N$ where $\lambda \in \mathbb{R}$ and $N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and I is the usual 3×3 identity matrix. Notice I and N commute. Calculate e^{tA}.

$$e^{tA} = e^{\lambda t I + tN}$$

$$= e^{\lambda t I} e^{tN}$$

$\left[e^A e^B = e^{A+B} \text{ if } AB = BA \right.$
Notice $(\lambda t I)(tN) = (tN)(\lambda t I)$
since I commutes with all matrices.

$$= e^{\lambda t} I \left(I + tN + \frac{1}{2} t^2 N^2 + \frac{1}{3!} t^3 N^3 + \cdots \right)$$

$$= e^{\lambda t} \left(I + tN + \frac{t^2}{2} N^2 \right)$$

Here $N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and $N^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $N^k = 0 \ \forall k \geq 3$.

$$e^{tA} = e^{\lambda t} \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} e^{\lambda t} & te^{\lambda t} & \frac{1}{2} t^2 e^{\lambda t} \\ 0 & e^{\lambda t} & te^{\lambda t} \\ 0 & 0 & e^{\lambda t} \end{bmatrix}}$$

Problem 110: Let $\beta = \{v_1, v_2, v_3, v_4, v_5\}$ be a basis such that

$$T(v_1) = 7v_1, \quad T(v_2) = 7v_2 + v_1, \quad T(v_3) = 7v_3 + v_2$$

and

$$T(v_4) = 11v_4, \quad T(v_5) = 11v_5 + v_4.$$

Calculate $[T]_{\beta,\beta}$ and explain why T is not diagonalizable. Classify each vector in β as an eigenvector or generalized eigenvector of a particular order.

$$[T]_{\beta,\beta} = \big[[T(V_1)]_\beta \,|\, [T(V_2)]_\beta \,|\, [T(V_3)]_\beta \,|\, [T(V_4)]_\beta \,|\, [T(V_5)]_\beta \big]$$

$$= \big[[7V_1]_\beta \,|\, [7V_2+V_1]_\beta \,|\, [7V_3+V_2]_\beta \,|\, [11V_4]_\beta \,|\, [11V_5+V_4]_\beta \big]$$

$$= \boxed{\left[\begin{array}{ccc|cc} 7 & 1 & 0 & 0 & 0 \\ 0 & 7 & 1 & 0 & 0 \\ 0 & 0 & 7 & 0 & 0 \\ \hline 0 & 0 & 0 & 11 & 1 \\ 0 & 0 & 0 & 0 & 11 \end{array}\right]}$$

$$= J_3(7) \oplus J_2(11)$$

- V_1 and V_4 are eigenvectors with $\lambda = 7, 11$ respective.
- $\underbrace{V_2 \text{ and } V_5}_{\text{order two generalized e-vect.}}$ are generalized e-vect. with $\lambda = 7, 11$ respectively
- V_3 is generalized e-vect. of order 3 with $\lambda = 7$

Problem 111: If $A = [T]_{\beta,\beta}$ as given in the previous problem then solve $\frac{dx}{dt} = Ax$ where $x = (x_1, x_2, x_3, x_4, x_5)$ using the matrix exponential technique as shown in lecture.

$$* \begin{cases} e^{tA}u_1 = e^{7t}u_1 \\ e^{tA}u_2 = e^{7t}(u_2 + tu_1) \\ e^{tA}u_3 = e^{7t}(u_3 + tu_2 + \frac{t^2}{2}u_1) \end{cases}$$

Applying $(A-7I)u_1 = 0$

$(A-7I)u_2 = u_1 \qquad (A-7I)^2 u_2 = 0$

$(A-7I)u_3 = u_2 \qquad (A-7I)^2 u_3 = u_1 \qquad (A-7I)^3 u_3 = 0$

To the magic formula with $\lambda = 7$

$$e^{tA} = e^{7t}\left(I + t(A-7I) + \frac{t^2}{2}(A-7I)^2 + \frac{t^3}{3!}(A-7I)^3 + \dots\right)$$

when we calculate $e^{tA}u$ for $u = u_1, u_2, u_3$ we obtain $*$. Likewise for $(A-11I)u_4 = 0$ and $(A-11I)u_5 = u_4$ we find

$$e^{tA}u_4 = e^{11t}u_4$$
$$e^{tA}u_5 = e^{11t}(u_5 + tu_4)$$

Therefore, the general solution is,

$$\boxed{x = c_1 e^{7t}u_1 + c_2 e^{7t}(u_2 + tu_1) + c_3 e^{7t}(u_3 + tu_2 + \frac{t^2}{2}u_1) + c_4 e^{11t}u_4 + c_5 e^{11t}(u_5 + tu_4)}$$

(where u_1, u_2, u_3, u_4, u_5 are given in P110)

Problem 112: One place we can anticipate the need for something more than eigenvectors is in the case of the differential equation $y'' = 0$ where $y' = \frac{dy}{dt}$. The solution is obtained by twice integrating to find $y = c_1 + c_2 t$. But, what does this have to do with systems of first order differential equations ? Well, let us make a **reduction of order** by introducing

$$x_1 = y \qquad \& \qquad x_2 = y'$$

then $x_1' = y' = x_2$ whereas $x_2' = y'' = 0$ hence we face:

$$\frac{dx_1}{dt} = x_2$$
$$\frac{dx_2}{dt} = 0$$

That is, we face $\frac{dx}{dt} = Ax$ where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Show A is not diagonalizable by showing there are not enough linearly independent eigenvectors to form an eigenbasis for A.

Remark: notice the general solution $y = c_1 + c_2 t$ gives us $y' = c_2$ and hence $x_1 = c_1 + c_2 t$ and $x_2 = c_2$ thus the general solution has the following form in terms of our reduced variables:

$$x = \begin{bmatrix} c_1 + c_2 t \\ c_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} t \\ 1 \end{bmatrix}.$$

We can understand the solution with c_1 as its coefficient as an eigensolution stemming from $\lambda = 0$ which makes $e^{\lambda t} = e^0 = 1$, however the term with coefficient c_2 is not something which was we could cipher with mere eigenvectors. It requires a deeper magic.

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda & -1 \\ 0 & \lambda \end{pmatrix} = \lambda^2 = 0 \quad \therefore \lambda = 0 \text{ is e-value.}$$

$$(A - 0 \cdot I)\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} v = 0 \\ u - \text{free} \end{array}$$

$$W_{\lambda=0} = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \neq \mathbb{R}^2$$

Thus $\nexists$ eigenbasis for A.

I didn't ask for this, but notice $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = J_2(0)$ thus $\{e_1, e_2\}$ is generalized eigenbasis, indeed a 2-chain for $\lambda = 0$ where $Ae_2 = e_1$ and

$$e^{tA}e_1 = e^0 \cdot e_1 = e_1 \quad \& \quad e^{tA}e_2 = e^0(e_2 + te_1)$$

Thus $x = c_1 e_1 + c_2(e_2 + te_1) = \begin{bmatrix} c_1 + c_2 t \\ c_2 \end{bmatrix}$ solves $\frac{dx}{dt} = Ax$.

this is in agreement with $x_1 = y = c_1 + c_2 t$ & $x_2 = y' = c_2$.

Problem 113: Let us work through an analysis similar to the previous problem. Except this time let's look at the family of differential equations of the form $y'' - 2ay' + a^2y = 0$ where $a \in \mathbb{R}$.

(a) show $y_1 = e^{at}$ and $y_2 = te^{at}$ serve as solutions to the DEqn.

(b) let $x_1 = y$ and $x_2 = y'$ and rewrite the given second order differential equation as $\frac{dx}{dt} = Ax$ where $x = (x_1, x_2)$

(c) find an eigenvalue and eigenvector of A

(d) given $y = c_1e^{at} + c_2te^{at}$ is the general solution to $y'' - 2ay' + a^2y = 0$ find the corresponding solution to $\frac{dx}{dt} = Ax$. Which part of the vector solution is an eigensolution and which part is not ?

(a.) $y_1 = e^{at} \Rightarrow y_1' = ae^{at} \Rightarrow y_1'' = a^2e^{at}$ thus $y_1'' - 2ay_1' + a^2y_1 =$

$= a^2e^{at} - 2a(ae^{at}) + a^2e^{at} = (a^2 - 2a^2 + a^2)e^{at} = 0.$

Likewise, $y_2 = te^{at} \Rightarrow y_2' = (1+at)e^{at} \Rightarrow y_2'' = (a + (1+at)a)e^{at}$

thus $y_2'' - 2ay_2' + a^2y_2 = (a^2t + 2a)e^{at} - 2a(1+at)e^{at} + a^2te^{at} = 0.$

(b.) $* \begin{cases} x_1' = y' = x_2 \\ x_2' = y'' = 2ay' - a^2y = -a^2x_1 + 2ax_2 \end{cases}$ $\quad \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}' = \begin{bmatrix} 0 & 1 \\ -a^2 & 2a \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

Identify $*$ is $\frac{dx}{dt} = Ax$ with $\underline{A = \begin{bmatrix} 0 & 1 \\ -a^2 & 2a \end{bmatrix}}$.

(c.) $\det[\lambda I - A] = \det\begin{bmatrix} \lambda & -1 \\ a^2 & \lambda - 2a \end{bmatrix} = \lambda(\lambda - 2a) + a^2$
$= \lambda^2 - 2\lambda a + a^2$
$= (\lambda - a)^2 \quad \therefore \underline{\lambda = a}$ eigenvalue.

$(A - aI)\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -a & 1 \\ -a^2 & a \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$-au + v = 0 \Rightarrow \underline{v = au}. \Rightarrow \underline{u_1 = \begin{bmatrix} 1 \\ a \end{bmatrix}}$ eigenvector.

(d.) $x_1 = c_1e^{at} + c_2te^{at} = y$
$x_2 = y' = c_1ae^{at} + c_2(1+at)e^{at}$

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = c_1\begin{bmatrix} e^{at} \\ ae^{at} \end{bmatrix} + c_2\begin{bmatrix} te^{at} \\ (1+at)e^{at} \end{bmatrix} = \boxed{\underbrace{c_1e^{at}\begin{bmatrix} 1 \\ a \end{bmatrix}}_{\text{eigensolution}} + \underbrace{c_2e^{at}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ a \end{bmatrix}\right)}_{\text{not an eigensolution}}}$

Remark: $(A - aI)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ a \end{bmatrix} \therefore \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ gen. e-vector of order two with $\lambda = a$.

Problem 114: Consider $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$. Show $(A-3I)e_1 = 0$ and $(A-3I)e_2 = e_1$. Find the general solution of $\frac{dx}{dt} = Ax$ using the magic formula with $\lambda = 3$. How does your result compare the previous problem ?

$$e^{tA} = e^{3t}\left(I + t(A-3I) + \tfrac{t^2}{2}(A-3I)^2 + \cdots\right)$$

$$e^{tA}e_1 = e^{3t}\left(Ie_1 + t(A-3I)e_1 + \cdots\right) = e^{3t}e_1$$

$$e^{tA}e_2 = e^{3t}\left(Ie_2 + t(A-3I)e_2 + \cdots\right)$$
$$= e^{3t}\left(e_2 + te_1 + \tfrac{t^2}{2}(A-3I)e_1 + \cdots\right)$$
$$= e^{3t}(e_2 + te_1)$$

$$\boxed{x = c_1 e^{3t}e_1 + c_2 e^{3t}(e_2 + te_1)}$$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} c_1 e^{3t} + c_2 t e^{3t} \\ c_2 e^{3t} \end{bmatrix}$$

$$= c_1 e^{3t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{3t}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right).$$

Solution compared to $\boxed{P113}$, both have coefficient matrix which is not diagonalizable and both solutions are easily seen as stemming from the magic formula. Can't compare directly since pattern bit different $\begin{bmatrix} 0 & 1 \\ -a^2 & 2a \end{bmatrix}$ vs. $\begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$

Problem 115: If we faced a problem with a spring under a force tuned to the natural frequency of the spring then we would find the system has a pure resonance. Reduction of order for such a problem leads to $\frac{dx}{dt} = Ax$ where A has a complex eigenvector $v_1 = a_1 + ib_1$ and a generalized complex eigenvector $v_2 = a_2 + ib_2$ where $a_1, b_1, a_2, b_2 \in \mathbb{R}^4$ and there exists $\omega > 0$ for which

$$Av_1 = i\omega v_1 \qquad \& \qquad Av_2 = i\omega v_2 + v_1$$

Let $\beta = \{a_1, b_1, a_2, b_2\}$ serve as a basis for $\mathbb{R}^4$ and define $T(x) = Ax$.

(a.) show v_1, v_2 is a 2-chain of complex eigenvectors for A with $\lambda = i\omega$.

(b.) Calculate $[T]_{\beta,\beta}$.

(c.) find the real solution of $\frac{dx}{dt} = Ax$ in terms of the given vectors and ω.

(a.) $\left.\begin{array}{l}(A - i\omega)V_1 = 0 \\ (A - i\omega)V_2 = V_1\end{array}\right\}$ $\{V_1, V_2\}$ is 2-chain of complex e-vectors of A with $\lambda = i\omega$

(b.)
$$[T]_{\beta,\beta} = \big[\,[T(a_1)]_\beta \,|\, [T(b_1)]_\beta \,|\, [T(a_2)]_\beta \,|\, [T(b_2)]_\beta\,\big]$$
$$= \big[\,[-\omega b_1]_\beta \,|\, [\omega a_1]_\beta \,|\, [-\omega b_2 + a_1]_\beta \,|\, [\omega a_2 + b_1]_\beta\,\big]$$
$$= \boxed{\begin{bmatrix} 0 & \omega & 1 & 0 \\ -\omega & 0 & 0 & 1 \\ 0 & 0 & 0 & \omega \\ 0 & 0 & -\omega & 0 \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 0 & \omega \\ -\omega & 0 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = RJ_4(i\omega)$$

Since $Av_1 = i\omega v_1 \Rightarrow A(a_1 + ib_1) = i\omega(a_1 + ib_1)$
$$Aa_1 + iAb_1 = -\omega b_1 + i\omega a_1 \quad \left[\begin{array}{l} Aa_1 = -\omega b_1 \\ Ab_1 = \omega a_1 \end{array}\right]$$

$$AV_2 = i\omega V_2 + V_1 \Rightarrow A(a_2 + ib_2) = i\omega(a_2 + ib_2) + a_1 + ib_1$$
$$\left[\begin{array}{l} Aa_2 = a_1 - \omega b_2 \\ Ab_2 = b_1 + \omega a_2 \end{array}\right]$$

(c.)
$$z_1 = e^{tA}V_1 = e^{i\omega t}(I + \cancel{t(A - i\omega I)}^{\,0})V_1 = (\cos\omega t + i\sin\omega t)(a_1 + ib_1)$$
$$z_2 = e^{tA}V_2 = e^{i\omega t}(V_2 + t(A - i\omega I)V_2 + \cdots) = (\cos\omega t + i\sin\omega t)[a_2 + ib_2 + t(a_1 + ib_1)]$$

Now z_1 and z_2 are complex-valued solutions of $\frac{dx}{dt} = Ax$, the real solution is given by $\underbrace{c_1 Re(z_1) + c_2 Im(z_1) + c_3 Re(z_2) + c_4 Im(z_2)}$,

I'll write it out ↲

P115 continued

$$\boxed{\begin{aligned} X = {} & c_1\left((\cos\omega t)a_1 - (\sin\omega t)b_1\right) \\ & + c_2\left((\cos\omega t)b_1 + (\sin\omega t)a_1\right) \\ & + c_3\left((\cos\omega t)(a_2 + t a_1) - (\sin\omega t)(b_2 + b_1 t)\right) \\ & + c_4\left((\cos\omega t)(b_2 + t b_1) + (\sin\omega t)(a_2 + t a_1)\right) \end{aligned}}$$

P116 §7.4 #3

Find the singular values of $\begin{bmatrix} \sqrt{6} & 1 \\ 0 & \sqrt{6} \end{bmatrix} = A$

Defn/ The singular values of A are the <u>square roots</u> of the eigenvalues of A^TA. We denote these by $\sigma_1, \sigma_2, \dots, \sigma_n$ arranged by $\sigma_1 \geq \sigma_2 \geq \dots \geq \sigma_n$

$$A^TA = \begin{bmatrix} \sqrt{6} & 0 \\ 1 & \sqrt{6} \end{bmatrix}\begin{bmatrix} \sqrt{6} & 1 \\ 0 & \sqrt{6} \end{bmatrix} = \begin{bmatrix} 6 & \sqrt{6} \\ \sqrt{6} & 7 \end{bmatrix}$$

$$\det(\lambda I - A^TA) = \det\begin{bmatrix} \lambda-6 & -\sqrt{6} \\ -\sqrt{6} & \lambda-7 \end{bmatrix} = (\lambda-6)(\lambda-7) - 6$$
$$= \lambda^2 - 13\lambda + 36$$
$$= (\lambda-4)(\lambda-9)$$

Cool. $\lambda_1 = 9, \ \lambda_2 = 4$

$\sigma_1 = \sqrt{\lambda_1} = \sqrt{9} = 3$ and $\sigma_2 = \sqrt{\lambda_2} = \sqrt{4} = 2$

We find singular values $\boxed{\sigma_1 = 3 \text{ and } \sigma_2 = 2}$

P117 §7.4 #7 Find an SVD for $A = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$

$$A^TA = \begin{bmatrix} 2 & 2 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix}$$

$$\det(\lambda I - A^TA) = \det\begin{bmatrix} \lambda-8 & -2 \\ -2 & \lambda-5 \end{bmatrix}$$
$$= (\lambda-8)(\lambda-5) - 4$$
$$= \lambda^2 - 13\lambda + 40 - 4$$
$$= (\lambda-9)(\lambda-4) \quad \text{(again!)}$$

$\Rightarrow \lambda_1 = 9, \ \lambda_2 = 4$

$\therefore \ \underline{\sigma_1 = 3 \text{ and } \sigma_2 = 2}.$

continued

P117 continued

Find orthonormal basis for $B = A^T A = \begin{bmatrix} 8 & 2 \\ 2 & 5 \end{bmatrix}$

$$\boxed{\lambda_1 = 9} \quad (B - 9I)\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \underline{V_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}}.$$

$$\boxed{\lambda_2 = 4} \quad (B - 4I)\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \underline{V_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix}}.$$

Next calculate AV_1 and AV_2,

$$AV_1 = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}\left(\frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}\right) = \frac{1}{\sqrt{5}}\begin{bmatrix} 3 \\ 6 \end{bmatrix} = \frac{3}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$AV_2 = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}\left(\frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 2 \end{bmatrix}\right) = \frac{1}{\sqrt{5}}\begin{bmatrix} -4 \\ 2 \end{bmatrix} = \frac{2}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Rescale by dividing by singular values,

$$u_1 = \frac{AV_1}{\sigma_1} = \frac{AV_1}{3} = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$u_2 = \frac{AV_2}{\sigma_2} = \frac{AV_2}{2} = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

$$V = \begin{bmatrix} 2/\sqrt{5} & -1/\sqrt{5} \\ 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix}$$

Therefore, by SVD construction, $AV = U\Sigma$ and

$$\boxed{A = \underbrace{\begin{pmatrix} 1/\sqrt{5} & -2/\sqrt{5} \\ 2/\sqrt{5} & 1/\sqrt{5} \end{pmatrix}}_{U}\underbrace{\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}}_{\Sigma}\underbrace{\begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ -1/\sqrt{5} & 2/\sqrt{5} \end{pmatrix}}_{V^T}}$$

$$U = [u_1 \mid u_2]$$

$$V = [V_1 \mid V_2]$$

P118 § 7.4 # 11

$$A = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} \qquad \underline{\text{Hint}}: \; U = \frac{1}{3}\begin{bmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{bmatrix} \text{ is a possible choice for } U.$$

Find an SVD for A.

$$A^TA = \begin{bmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \end{bmatrix}\begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix}$$

$$\det(\lambda I - A^TA) = \det\begin{bmatrix} \lambda - 81 & 27 \\ 27 & \lambda - 9 \end{bmatrix}$$

$$= (\lambda - 81)(\lambda - 9) - (27)^2$$

$$= \lambda^2 - 90\lambda + 3^6 - 3^6$$

$$= \lambda(\lambda - 90) \Rightarrow \underbrace{\lambda_1 = 90, \; \lambda_2 = 0}_{\text{singular value of } \sigma_1 = \sqrt{90}}$$

$$(A^TA - 90I)\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} -9 & -27 \\ -27 & -81 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{array}{l} 9u + 27v = 0 \\ u = -3v \end{array}$$

choose $v = 1$ and normalize

Thus find eigenvector $\underline{v_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} -3 \\ 1 \end{bmatrix}}$.

$$Av_1 = \frac{1}{\sqrt{10}}\begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}\begin{bmatrix} -3 \\ 1 \end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix} 10 \\ -20 \\ -20 \end{bmatrix} = \frac{10}{\sqrt{10}}\begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix} \Rightarrow u_1 = \frac{Av_1}{\sqrt{90}}$$

$$\Rightarrow \underline{u_1 = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}}.$$

Now, we need to extend u_1 by u_2, u_3 s.t. u_1, u_2, u_3 orthonormal basis for $\mathbb{R}^3$. Looking at the hint, we can set

$$u_2 = \begin{bmatrix} 2/3 \\ -1/3 \\ 2/3 \end{bmatrix} \text{ and } u_3 = \begin{bmatrix} 2/3 \\ 2/3 \\ -1/3 \end{bmatrix}$$

(I'll leave u_1 as I found it, $\exists$ many choices for U.)

P118 continued

To complete V we need to find V_2 with $A^TAV_2 = 0$

$$A^TA\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 81 & 27 \\ -27 & 9 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{aligned} 81u + 27v &= 0 \\ v &= -3u \end{aligned}$$

setting $u=1$ find $v=-3$, normalized, $V_2 = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ -3 \end{bmatrix}$

Hence $\quad U = \begin{bmatrix} 1/3 & 2/3 & 2/3 \\ -2/3 & -1/3 & 2/3 \\ -2/3 & 2/3 & -1/3 \end{bmatrix}$ & $V = \begin{bmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & -3/\sqrt{10} \end{bmatrix}$

We find

$$\boxed{A = U\Sigma V^T = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{-2}{3} & \frac{-1}{3} & \frac{2}{3} \\ \frac{-2}{3} & \frac{2}{3} & \frac{-1}{3} \end{bmatrix}\begin{bmatrix} \sqrt{90} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & -3/\sqrt{10} \end{bmatrix}}$$

P119 Lay § 7.4 # 17

Suppose A is square and invertible. Find a singular value decomposition of A^{-1}

Since $\det(A^TA) = \lambda_1 \lambda_2 \cdots \lambda_n \neq 0 \Rightarrow \lambda_i > 0 \quad \forall i = 1, 2, \ldots, n$

(we know eigenvalues of A^TA are non-negative so $\mho$)

Thus $n = r$, we have n-singular values

$\sigma_1 = \sqrt{\lambda_1}, \ \sigma_2 = \sqrt{\lambda_2}, \ \ldots, \ \sigma_n = \sqrt{\lambda_n}$.

More to the point, it follows the SVD for A

has form $A = U \Sigma V^T$ where $\Sigma = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_n \end{bmatrix}$

thus, as $U^TU = I$ and $V^TV = I$

where U, V are both $n \times n$ matrices,

$$\begin{aligned} A^{-1} &= (U \Sigma V^T)^{-1} \\ &= (V^T)^{-1} \Sigma^{-1} (U)^{-1} \\ &= \boxed{V \begin{pmatrix} 1/\sigma_1 & & & \\ & 1/\sigma_2 & & \\ & & \ddots & \\ & & & 1/\sigma_n \end{pmatrix} U^T} \end{aligned}$$

$\boxed{P120}$ §7.4 # 23

If $U = [u_1 | u_2 | \cdots | u_m]$ and $V = [v_1 | v_2 | \cdots | v_n]$ then show $A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_r u_r v_r^T$

We assume $A = U \Sigma V^T$ where $\sigma_1, \ldots, \sigma_r$ are singular values of A thus

$$\Sigma = \begin{bmatrix} \sigma_1 & & & & \\ & \ddots & & & \\ & & \sigma_r & & \\ & & & 0 & \\ & & & & \ddots \\ & & & & & 0 \end{bmatrix} =$$

Consider that

$$\Sigma V^T = \begin{bmatrix} \sigma_1 e_1^T \\ \hline \sigma_2 e_2^T \\ \hline \vdots \\ \hline \sigma_r e_r^T \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix} V^T = \begin{bmatrix} \sigma_1 e_1^T V^T \\ \hline \vdots \\ \hline \sigma_r e_r^T V^T \\ \hline 0 \\ \hline \vdots \\ \hline 0 \end{bmatrix} = \begin{bmatrix} \sigma_1 v_1^T \\ \hline \vdots \\ \hline \sigma_r v_r^T \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix}$$

Hence,

$$A = [u_1 | u_2 | \cdots | u_m] \begin{bmatrix} \sigma_1 v_1^T \\ \hline \vdots \\ \hline \sigma_r v_r^T \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix}$$

See Th^m 10 on pg. 137 of Lay.

$$= u_1(\sigma_1 v_1^T) + u_2(\sigma_2 v_2^T) + \cdots + u_r(\sigma_r v_r^T) + 0 + \cdots + 0$$

$$= \boxed{\sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_r u_r v_r^T}$$