Show your work and box answers. Once complete, please staple in upper left corner. Thanks.

Suggested Reading You may find the following helpful resources beyond lecture,

(a.) Chapters 2, 3 and 4 of my lecture notes for Math 221

Problem 16: Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 3 & 0 \end{bmatrix}$. Calculate $AB - BA$. Do A and B commute ?

Problem 17: Let $v = [1, 2, 3, 4] \in \mathbb{R}^{1\times 4}$ and $w = (5, 6, 7, 8) = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix} \in \mathbb{R}^{4\times 1}$. Calculate vw and wv.

Problem 18: A matrix is said to be **antisymmetric** if $A^T = -A$. In terms of components, that means $A_{ji} = -A_{ij}$ for all i, j. Consider the 3×3 matrix $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$. If $\text{Col}_1(A) = \begin{bmatrix} 0 \\ 2 \\ 7 \end{bmatrix}$ and $\text{Row}_3(A) = [7, 3, 0]$ then find A.

Problem 19: Let $N = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. Calculate N^2 and N^3 and show $N^k = 0$ for $k \geq 4$.

Problem 20: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 5 & 6 & 7 \end{bmatrix}$ and $M = \begin{bmatrix} 1 & 0 \\ 3 & -1 \\ 4 & 1 \\ 5 & 2 \end{bmatrix}$. If possible calculate the matrix quantities below. If not, explain why such a calculation does not fall under our definition of matrix addition or multiplication.

(a) $A + M$

(b) A^2

(c) AM

(d) MA

(e) $AA^T + 2MM^T$

(f) $A^TA + 2M^TM$

Problem 21: Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ a & b & c & d \end{bmatrix}$ where a, b, c, d are constants. Let e_i denote the i-th standard basis vector for $\mathbb{R}^4$ where $i = 1, 2, 3, 4$. For example, $e_2 = (0, 1, 0, 0) \in \mathbb{R}^{4\times 1} = \mathbb{R}^4$. Also, let $\bar{e}_j$ denote the j-th standard basis element for $\mathbb{R}^3 = \mathbb{R}^{3\times 1}$. For example, $\bar{e}_2 = (0,1,0)$. With this notation in mind, calculate:

(a) Ae_2

(b) $\bar{e}_3^T A$

(c) $A[e_1|e_2]$

(d) $\left[\begin{array}{c} \bar{e}_2^T \\ \hline \bar{e}_3^T \end{array}\right] A$

(e) $\bar{e}_3^T A e_2$

Problem 22: Let $X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$ and $A = \begin{bmatrix} 1 & -2 \\ 3 & 1 \end{bmatrix}$. Solve $AX = I$ where $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is the 2×2 identity matrix. Does your solution also solve $XA = I$?

Problem 23: Given $x = (x_1, x_2, x_3)$ and $A = \begin{bmatrix} 1 & 3 & -5 \\ 1 & 4 & -8 \\ -3 & -7 & 9 \end{bmatrix}$, solve $Ax = 0$ and express the solution in parametric vector form.

Problem 24: Let $v_1 = (1,2,2)$ and $v_2 = (0,1,-1)$.

(a.) what condition must be given for a, b, c for $(a,b,c) \in \text{span}(v_1, v_2)$?

(b.) what condition is needed for $\{v_1, v_2, (a,b,c)\}$ to by a linearly independent (LI) set ?

Problem 25: Solve the chemical reaction $B_2S_3 + H_2O \to H_3BO_3 + H_2S$ via linear algebra.

Problem 26: Suppose $Av = b$ and $Aw = b$ for a given matrix A and column vector b. If $v \neq w$ then show that the columns of A are linearly dependent.

Problem 27: Let $A = \begin{bmatrix} 1 & 2 & 3 \\ a & b & c \\ 3 & 2 & 1 \end{bmatrix}$ and $R_1 = \begin{bmatrix} a & b & c \\ 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$ and $R_2 = \begin{bmatrix} 1 & 2 & 3 \\ a & b & c \\ 2 & 0 & -2 \end{bmatrix}$. Notice both R_1 and R_2 are obtained by performing an elementary row operation on A. Find 3×3 matrices E_1 and E_2 for which $R_1 = E_1 A$ and $R_2 = E_2 A$.

Problem 28: If $A \to R$ under a row-operation then there exists an elementary matrix E for which $R = EA$. Find elementary matrices $E_1, E_2, \dots, E_k$ for which $\text{rref}(A) = E_k E_{k-1} \dots E_2 E_1 A$ for

(a) $k = 3$ will suffice for $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 3 & 4 \end{bmatrix}$

(b) $k = 2$ will suffice for $A = \begin{bmatrix} 0 & 1 \\ 1 & 2 \\ 0 & 0 \end{bmatrix}$

Problem 29: We defined the matrix product of $A \in \mathbb{R}^{m\times p}$ with $B \in \mathbb{R}^{p\times n}$ by $(AB)_{ij} = \sum_{k=1}^{p} A_{ik}B_{kj}$ for all $1 \le i \le m$ and $1 \le j \le n$. In the case $m = n$ we note the matrix $AB \in \mathbb{R}^{n\times n}$. Let us define the **trace** of a square matrix M to be the sum of its diagonals. To be precise,

$$\text{trace}(M) = \sum_{i=1}^{n} M_{ii}.$$

(a) Let I be the $n \times n$ identity matrix. Calculate trace(I).

(b) Suppose $A \in \mathbb{R}^{n\times p}$ and $B \in \mathbb{R}^{p\times n}$. Show trace$(AB)$ = trace(BA).

Problem 30: Let $A = \begin{bmatrix} 1 & 0 & 1 & 1 \\ -1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix}$. Use Matlab, to calculate $(AA^T)^{100}$ and $(A^TA)^{100}$.