Show your work and box answers. Once complete, please staple in upper left corner. Thanks.

Suggested Reading You may find the following helpful resources beyond lecture,

(a.) Chapter 5 of my lecture notes for Math 221

Problem 46: Calculate $\det(A)$ where $A = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 2 & 2 \\ 5 & 3 & 1 \end{bmatrix}$

Problem 47: Calculate $\det(B)$ where $B = \begin{bmatrix} 2 & 2 & 0 & 2 & 2 \\ 0 & 2 & 0 & -1 & 0 \\ 2 & 2 & 0 & 0 & 3 \\ 7 & 7 & 2 & 7 & 7 \\ 5 & 3 & 0 & 0 & 0 \end{bmatrix}$

Problem 48: Let A, B be as given in the previous problems. If $M = \left[\begin{array}{c|c} 2A & 0 \\ \hline 0 & 3B \end{array}\right]$ then calculate $\det(M)$ via application of properties of determinants given in the lecture notes and the results of the previous pair of problems.

Problem 49: For which values of x is the matrix $M = \begin{bmatrix} x & 2 & 2 \\ 1 & 1 & 1 \\ 7 & 5 & 3 \end{bmatrix}$ invertible?

Problem 50: Solve $\alpha x + 3y = 7$ and $5x - \beta y = 6$ by Cramer's rule. Comment on needed conditions on α, β for the solution to exist.

Problem 51: Let A be a matrix which is similar to $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$. In other words, suppose there exists an invertible matrix P for which $B = P^{-1}AP$. Calculate $\det(A)$ and $\text{trace}(A)$.

Problem 52: Let $A = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 7 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$. Calculate $\det(A)$ using properties of the determinant based on row-reductions. Almost certainly using Laplace's expansion by minors here is a really bad idea.

Problem 53: Find the volume of the paralell piped with edges $(1, 2, 3), (2, 3, 3), (-1, -2, 0)$.

Problem 54: Let $A, B \in \mathbb{R}^{4\times 4}$ and $\det(A) = -1$ and $\det(B) = 2$. Calculate,

(a.) $\det(AB)$

(b.) $\det(B^5)$

(c.) $\det(2A)$

(d.) $\det(A^TA)$

(e.) $\det(B^{-1}AB)$

Problem 55: Find y as a function of x given that $\det\begin{bmatrix} 1 & x & y \\ 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \end{bmatrix} = 0$. What two points are on the line ? Does this make sense in terms of known properties of the determinant ?

Problem 56: Consider $\left\{ \begin{array}{l} 2x_1 + x_2 + x_3 = 4 \\ -x_1 + 2x_3 = 2 \\ 3x_1 + x_2 + 3x_3 = -2 \end{array} \right\}$. Solve this system via Cramer's rule.

Problem 57: Is $(a, b, c) \in \text{span}\{(1, 2, 3), (0, 1, 1)\}$? Use determinants and the theory of linear algebra we have discussed to answer this question.

Problem 58: Suppose you have a square matrix A for which the matrix equation $A^TJA = J$ holds for some invertible matrix J. Find the possible values for $det(A)$.

Problem 59: **The cross product:** For all $a, b \in \mathbb{R}^3$ we define

$$T(a, b) = \sum_{j=1}^{3} \big(\det[a|b|e_j]\big) e_j.$$

Show $a \bullet T(a, b) = 0$ and $b \bullet T(a, b) = 0$ and for any $c \in \mathbb{R}^3$ we have $T(a, b) \bullet c = \det[a|b|c]$.

Problem 60: A natural candidate for the cross product in $\mathbb{R}^4$ is given by extending the formula in the previous problem: for all $a, b, c \in \mathbb{R}^4$ we define

$$T(a, b, c) = \sum_{j=1}^{4} \big(\det[a|b|c|e_j]\big) e_j$$

Show: $a \bullet T(a, b, c) = 0$ and $b \bullet T(a, b, c) = 0$ and $c \bullet T(a, b, c) = 0$.
I should mention, the equations above tell us a, b, c are perpendicular to $T(a, b, c)$ and we can prove that implies $\{a, b, c, T(a, b, c)\}$ is linearly independent provided $T(a, b, c) \neq 0$. In other words, if you want a fourth vector which is outside the span of $a, b, c \in \mathbb{R}^4$ then $T(a, b, c)$ is a nice choice. It is the normal to the hypervolume spanned by a, b, c.