Show your work and box answers. Once complete, please staple in upper left corner. Thanks.

Suggested Reading You may find the following helpful resources beyond lecture,

(a.) Chapter 6 of my lecture notes for Math 221

Let us be clear on some notation which is not in Lay, but is in my notes. If $\beta = \{v_1, v_2, \dots, v_n\}$ is a basis for $\mathbb{R}^n$ then for any $x \in \mathbb{R}^n$ we define $[x]_\beta = (y_1, \dots, y_n)$ if and only if $x = y_1v_1 + \cdots + y_nv_n$. Notice,

$$x = y_1v_1 + \cdots + y_nv_n = \underbrace{[v_1|v_2|\cdots|v_n]}_{[\beta]} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = [\beta][x]_\beta \quad \Rightarrow \quad \boxed{[x]_\beta = [\beta]^{-1}x}$$

Likewise, if we consider a linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ then the matrix of T with respect to the β basis is defined by

$$[T]_{\beta,\beta} = [[T(v_1)]_\beta|[T(v_2)]_\beta|\cdots|[T(v_n)]_\beta].$$

In the special case of the standard basis we define

$$[T] = [T(e_1)|T(e_2)|\cdots|T(e_n)].$$

and note $T(x) = [T]x$ for all x. Let's derive how to calculate $[T]_{\beta,\beta}$,

$$\begin{aligned} [T]_{\beta,\beta} &= [[T(v_1)]_\beta|[T(v_2)]_\beta|\cdots|[T(v_n)]_\beta] \\ &= [[\beta]^{-1}T(v_1)|[\beta]^{-1}T(v_2)|\cdots|[\beta]^{-1}T(v_n)] \\ &= [\beta]^{-1}[T(v_1)|T(v_2)|\cdots|T(v_n)] \\ &= [\beta]^{-1}[[T]v_1|[T]v_2|\cdots|[T]v_n] \\ &= [\beta]^{-1}[T][v_1|v_2|\cdots|v_n] \\ &= [\beta]^{-1}[T][\beta] \quad \Rightarrow \quad \boxed{[T]_{\beta,\beta} = [\beta]^{-1}[T][\beta]}. \end{aligned}$$

You will need to use the boxed formulas to solve some of the problems in this mission.

Problem 61: Let $\beta = \{(1, 2), (3, 4)\}$ and suppose $x = (-1, 0)$. Calculate $[x]_\beta$.

Problem 62: Let $\beta = \left\{\frac{1}{3}(1, 2, 2), \frac{1}{\sqrt{2}}(0, 1, -1), \frac{1}{\sqrt{18}}(-4, 1, 1)\right\}$ and define coordinates by $(y_1, y_2, y_3) = [\beta]^{-1}x$ for each $x \in \mathbb{R}^n$. Fun helpful fact, you can check $[\beta]^T[\beta] = I$, this makes β an **orthonormal basis**. Find formulas y_1, y_2, y_3 in terms of x_1, x_2, x_3 and show that:

$$x_1^2 + x_2^2 + x_3^2 = y_1^2 + y_2^2 + y_3^2.$$

Does the analog of the identity hold for $x = (-1, 0)$ and $y = [x]_\beta$ which you calculated in the previous problem ?

Problem 63: Let $\beta = \{(1,0,1),(0,1,1),(1,1,0)\}$. Calculate $[(a,b,c)]_\beta$ and use your result to find c_1, c_2, c_3 for which $c_1(1,0,1)+c_2(0,1,1)+c_3(1,1,0) = (3,5,7)$.

Problem 64: Let $\beta = \{(0,1,0,0),(0,0,1,0),(0,0,0,1),(1,0,0,0)\}$. Calculate $[(a,b,c,d)]_\beta$.

Problem 65: Find a matrix A for which $T(x) = Ax$ for each linear transformation below. Also, calculate $rref(A)$ and determine if the given map is onto, one-to-one or both. Find the rank and nullity for each map.

(a.) $T(x,y) = (x-y, 3x+2y)$

(b.) $T(x,y) = (x+y, 2x+2y, y)$

(c.) $T(x,y,z) = (x+2y+z, 2x+4y-2z, x+2y+z)$

Problem 66: Let $T(x,y) = (x+2y, -x+3y, 3x)$ and $S(u,v) = (u, 2u+v)$.

(a.) find standard matrices of T and S,

(b.) calculate $T \circ S$ by working through $(T \circ S)(u,v) = T(S(u,v))$ and find the standard matrix for $T \circ S$.

(c.) show $[T \circ S] = [T][S]$

Remark: this is why we defined matrix multiplication as we did, at least this is a common motivation in many texts. There are others, some of which date to antiquity

Problem 67: Suppose $T(x) = Ax+b$ where $A \in \mathbb{R}^{n\times n}$ and $A^TA = I$ and $b \in \mathbb{R}^n$. The distance between points in $\mathbb{R}^n$ is given by $d(P,Q) = \sqrt{(Q-P)^T(Q-P)}$. Show that[1]

$$d(P,Q) = d(T(P),T(Q)).$$

Problem 68: Construct the standard matrix of the linear transformations described below:

(a.) $T: \mathbb{R}^3 \to \mathbb{R}^2$ where $T(e_1) = (1,3)$, $T(e_2) = (4,-7)$ and $T(e_3) = (-5,4)$

(b.) $T: \mathbb{R}^2 \to \mathbb{R}^2$ rotates points about origin through $-\pi/4$. Note, $T(e_1) = (1/\sqrt{2}, -1/\sqrt{2})$.

Problem 69: Construct the standard matrix of the linear transformations described below:

(a.) $T: \mathbb{R}^2 \to \mathbb{R}^2$ where T is a horizontal shear transformation which leaves e_1 unchanged and maps $e_2 \mapsto e_2 + 3e_1$.

(b.) $T: \mathbb{R}^2 \to \mathbb{R}^2$ given by composition of a horizontal shear with $e_2 \mapsto e_2 - 2e_1$ and $e_1 \mapsto e_1$ followed by a reflection through the line $x_2 = -x_1$

Problem 70: Consider $T(x) = Ax$ where $A = \begin{bmatrix} 9 & 7 & -13 \\ 7 & 9 & -13 \\ -13 & -13 & 29 \end{bmatrix}$.

[1] *Remark: a transformation like T is known as a* **rigid motion** *if $det(A) = 1$. These are the transformations which preserve the shape of rigid objects. The determinant has to be one in order that the transformation not turn things inside out. It is a far more difficult task, but it can be shown that maps such as T are the* **only** *functions on $\mathbb{R}^n$ which preserve the Euclidean distance between points.*

(a) show T is onto and one-to-one map on $\mathbb{R}^3$

(b) find the standard matrix for T^{-1}.

Problem 71: Let $\beta = \left\{\frac{1}{\sqrt{3}}(1,1,1), \frac{1}{\sqrt{2}}(1,-1,0), \frac{1}{\sqrt{6}}(1,1,-2)\right\}$. For T given in the previous problem, calculate both $[T]_{\beta,\beta}$ and $[T^{-1}]_{\beta,\beta}$.

Problem 72: Calculate $det([T])$ and $det([T]_{\beta,\beta})$ as well as trace($[T]$) and trace($[T]_{\beta,\beta}$). Do you see any patterns ?

Problem 73: Find all transformations on $\mathbb{R}^3$ for which $T(1,2,3) = (1,0,0)$ and $T(1,0,1) = (0,1,0)$. The answer should be a set of linear transformations indexed by some parameter.

Problem 74: Find the formula for the linear transformation on $\mathbb{R}^3$ for which $T(1,1,1) = (3,4,8)$ and $T(0,1,1) = (1,0,1)$ and $T(0,0,1) = (-1,2,0)$.

Problem 75: Let S be the tetrahedron with vertices e_1, e_2, e_3 and the origin. Suppose T is a linear transformation which maps $T(e_1) = v_1$, $T(e_2) = v_2$, $T(e_3) = v_3$. Say S' is the tetrahedron with vertices v_1, v_2, v_3 and the origin.

(a.) Find $[T]$

(b.) Calculate the volume of S using the triple product determinant identity

(c.) Calculate the volume of S' using the triple product determinant identity

(we shoudl see that $Vol(S') = \det(T)Vol(S)$.)