Show your work and box answers. Once complete, please staple in upper left corner. Thanks.

Suggested Reading You may find the following helpful resources beyond lecture,

(a.) Chapter 7 and 8 of my lecture notes for Math 221

Problem 76: Consider $A : \mathbb{R} \to \mathbb{R}^{n\times n}$ where $A_{ij} : \mathbb{R} \to \mathbb{R}$ is a smooth function for each component function. We defined $\left(\frac{dA}{dt}\right)_{ij} = \frac{d}{dt}[A_{ij}]$. In other words, the derivative of a matrix is done component-wise. Calculate $\frac{d}{dt}(A^2)$ and $\frac{d}{dt}(A^3)$.

Problem 77: Suppose $\frac{dx}{dt} = x + 2y$ and $\frac{dy}{dt} = 2x + y$. Find the general solution using the eigenvector method we derived in lecture.

Problem 78: Suppose $\left\{ \begin{array}{c} \frac{dx}{dt} = 9x + 7y - 13z \\ \frac{dy}{dt} = 7x + 9y - 13z \\ \frac{dz}{dt} = -13z - 13y + 29z \end{array} \right\}$. Find the general solution using the eigenvector method we derived in lecture.
Fun fact: $\lambda^3 - 47\lambda^2 + 216\lambda - 252$ has $\lambda = 2$ as a zero.

Problem 79: Eigenvector problems:

(a.) Let $A = \begin{bmatrix} 10 & -9 \\ 4 & -2 \end{bmatrix}$. Find a basis for the eigenspace with eigenvalue $\lambda = 4$

(b.) Let $A = \begin{bmatrix} 7 & 4 \\ -3 & -1 \end{bmatrix}$ find bases for the eigenspaces of A. Note $\lambda_1 = 1$ and $\lambda_2 = 5$.

Problem 80: Eigenvector problems:

(a.) Let $A = \begin{bmatrix} 1 & 0 & -1 \\ 1 & -3 & 0 \\ 4 & -13 & 1 \end{bmatrix}$. Find a basis for the eigenspace with eigenvalue $\lambda = -2$

(b.) Let $A = \begin{bmatrix} 3 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix}$. Find a basis for the eigenspace with eigenvalue $\lambda = 4$

Problem 81: Let $A = \begin{bmatrix} 5 & 5 & 5 \\ 5 & 5 & 5 \\ 5 & 5 & 5 \end{bmatrix}$.

(a.) Find a basis for eigenspace with $\lambda = 0$. Hint: guess.

(b.) Calculate $\det(xI - A)$ and determine the other eigenvalue of A.

(c.) Check that $\text{trace}(A) = \lambda_1 + \lambda_2 + \lambda_3$ (here $\lambda_1 = \lambda_2 = 0$)

Problem 82: Let λ be the eigenvalue of an invertible matrix A. Prove A^{-1} has eigenvalue $1/\lambda$.

Problem 83: Suppose $Au = \lambda u$ and $Av = \mu v$ for $u, v \neq 0$ and define

$$x_k = c_1 \lambda^k u + c_2 \mu^k v$$

for $k = 0, 1, 2, \dots$ where $c_1, c_2 \in \mathbb{R}$. Show that $Ax_k = x_{k+1}$ for $k = 0, 1, 2, \dots$.

Problem 84: For each matrix below find the factored form of the characteristic equation and state the eigenvalues for A

$$(\mathbf{a}.)\ A = \begin{bmatrix} 5 & -2 & 3 \\ 0 & 1 & 0 \\ 6 & 7 & -2 \end{bmatrix}, \quad (\mathbf{b}.)\ A = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 8 & -4 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 1 & -5 & 2 & 1 \end{bmatrix}, \quad (\mathbf{c}.)\ A = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 \\ -5 & 1 & 0 & 0 & 0 \\ 3 & 8 & 0 & 0 & 0 \\ 0 & -7 & 2 & 1 & 0 \\ -4 & 1 & 9 & -2 & 3 \end{bmatrix}.$$

Problem 85: Consider the stochastic matrix $A = \begin{bmatrix} 0.5 & 0.2 & 0.3 \\ 0.3 & 0.8 & 0.3 \\ 0.2 & 0 & 0.4 \end{bmatrix}$. Let $v_1 = (0.3, 0.6, 0.1)$ and $v_2 = (1, -3, 2)$ and $v_3 = (-1, 0, 1)$ and $w = (1, 1, 1)$.

(a.) Show v_1, v_2, v_3 are eigenvectors of A.

(b.) Suppose $x_o = (\alpha, \beta, \gamma)$ where $\alpha + \beta + \gamma = 1$ and $\alpha, \beta, \gamma \geq 0$. If $x_o = c_1 v_1 + c_2 v_2 + c_3 v_3$ then derive why $c_1 = 1$. *Hint: multiply by* w^T

(c.) Define $x_k = A^k x_o$. Show $x_k \to v_1$ as $k \to \infty$.

Problem 86: Observe $A = \begin{bmatrix} -2 & 12 \\ -1 & 5 \end{bmatrix} = \begin{bmatrix} 3 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & 4 \\ 1 & -3 \end{bmatrix} = PDP^{-1}$. Calculate A^k explicitly as a 2×2 matrix. Simplify where possible.

Problem 87: Let $A = \begin{bmatrix} 7 & 4 & 16 \\ 2 & 5 & 8 \\ -2 & -2 & -5 \end{bmatrix}$. Find P for which $P^{-1}AP$ is a diagonal matrix. Verify your claim by multiplying out $P^{-1}AP$. You can use technology to find P and calculate P^{-1}.

Problem 88: Solve $\frac{dx}{dt} = 5x - 2y$ and $\frac{dy}{dt} = x + 3y$ via the eigenvector technique.

Problem 89: We saw $x_k = A^k x_o$ where $x_o = c_1 u_1 + c_2 u_2$ and $Au_1 = \lambda_1 u_1$ and $Au_2 = \lambda_2 u_2$ yields $x_k = c_1(\lambda_1)^k + c_2(\lambda_2)^k$. Classify the behavior of x_k as $k \to \infty$.

(a.) $A = \begin{bmatrix} 1.7 & -0.3 \\ -1.2 & 0.8 \end{bmatrix}$.

(b.) $A = \begin{bmatrix} 0.4 & 0.5 \\ -0.4 & 1.3 \end{bmatrix}$.

Problem 90: Let $A = \begin{bmatrix} 30 & 64 & 23 \\ -11 & -23 & -9 \\ 6 & 15 & 4 \end{bmatrix}$. Find the general solution of $\frac{d\vec{r}}{dt} = A\vec{r}$. Please use technology to find the necessary eigenvectors and/or complex eigenvectors and then use the theory for differential equations which was discussed in lecture.