Show your work and box answers. Once complete, please staple in upper left corner. Thanks.

Suggested Reading You may find the following helpful resources beyond lecture,

(a.) Chapter 7 amd 8 of my lecture notes for Math 221

Problem 106: Let $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Calculate e^{tJ} and express your answer in terms of $\cosh t$ and $\sinh t$ as well as I and J.

Problem 107: Suppose $M = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$. Calculate e^M directly from the power series definition of the matrix exponential. *Hint: convergence is not an issue here.*

Problem 108: Let $A = \begin{bmatrix} 0 & 3 \\ -3 & 0 \end{bmatrix}$. Calculate e^{tA} and solve $\frac{dx}{dt} = Ax$.

Problem 109: Let $A = \lambda I + N$ where $\lambda \in \mathbb{R}$ and $N = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ and I is the usual 3×3 identity matrix. Notice I and N commute. Calculate e^{tA}.

Problem 110: Let $\beta = \{v_1, v_2, v_3, v_4, v_5\}$ be a basis such that

$$T(v_1) = 7v_1, \quad T(v_2) = 7v_2 + v_1, \quad T(v_3) = 7v_3 + v_2$$

and

$$T(v_4) = 11v_4, \quad T(v_5) = 11v_5 + v_4.$$

Calculate $[T]_{\beta,\beta}$ and explain why T is not diagonalizable. Classify each vector in β as an eigenvector or generalized eigenvector of a particular order.

Problem 111: If $A = [T]_{\beta,\beta}$ as given in the previous problem then solve $\frac{dx}{dt} = Ax$ where $x = (x_1, x_2, x_3, x_4, x_5)$ using the matrix exponential technique as shown in lecture.

Problem 112: One place we can anticipate the need for something more than eigenvectors is in the case of the differential equation $y'' = 0$ where $y' = \frac{dy}{dt}$. The solution is obtained by twice integrating to find $y = c_1 + c_2 t$. But, what does this have to do with systems of first order differential equations ? Well, let us make a **reduction of order** by introducing

$$x_1 = y \qquad \& \qquad x_2 = y'$$

then $x_1' = y' = x_2$ whereas $x_2' = y'' = 0$ hence we face:

$$\begin{aligned} \tfrac{dx_1}{dt} &= x_2 \\ \tfrac{dx_2}{dt} &= 0 \end{aligned}$$

That is, we face $\frac{dx}{dt} = Ax$ where $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Show A is not diagonalizable by showing there are not enough linearly independent eigenvectors to form an eigenbasis for A.

Remark: notice the general solution $y = c_1 + c_2 t$ gives us $y' = c_2$ and hence $x_1 = c_1 + c_2 t$ and $x_2 = c_2$ thus the general solution has the following form in terms of our reduced variables:

$$x = \begin{bmatrix} c_1 + c_2 t \\ c_2 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 \begin{bmatrix} t \\ 1 \end{bmatrix}.$$

We can understand the solution with c_1 as its coefficient as an eigensolution stemming from $\lambda = 0$ which makes $e^{\lambda t} = e^0 = 1$, however the term with coefficient c_2 is not something which was we could cipher with mere eigenvectors. It requires a deeper magic.

Problem 113: Let us work through an analysis similar to the previous problem. Except this time let's look at the family of differential equations of the form $y'' - 2ay' + a^2 y = 0$ where $a \in \mathbb{R}$.

(a) show $y_1 = e^{at}$ and $y_2 = te^{at}$ serve as solutions to the DEqn.

(b) let $x_1 = y$ and $x_2 = y'$ and rewrite the given second order differential equation as $\frac{dx}{dt} = Ax$ where $x = (x_1, x_2)$

(c) find an eigenvalue and eigenvector of A

(d) given $y = c_1 e^{at} + c_2 te^{at}$ is the general solution to $y'' - 2ay' + a^2 y = 0$ find the corresponding solution to $\frac{dx}{dt} = Ax$. Which part of the vector solution is an eigensolution and which part is not ?

Problem 114: Consider $A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$. Show $(A - 3I)e_1 = 0$ and $(A - 3I)e_2 = e_1$. Find the general solution of $\frac{dx}{dt} = Ax$ using the magic formula with $\lambda = 3$. How does your result compare the previous problem ?

Problem 115: If we faced a problem with a spring under a force tuned to the natural frequency of the spring then we would find the system has a pure resonance. Reduction of order for such a problem leads to $\frac{dx}{dt} = Ax$ where A has a complex eigenvector $v_1 = a_1 + ib_1$ and a generalized complex eigenvector $v_2 = a_2 + ib_2$ where $a_1, b_1, a_2, b_2 \in \mathbb{R}^4$ and there exists $\omega > 0$ for which

$$Av_1 = i\omega v_1 \qquad \& \qquad Av_2 = i\omega v_2 + v_1$$

Let $\beta = \{a_1, b_1, a_2, b_2\}$ serve as a basis for $\mathbb{R}^4$ and define $T(x) = Ax$.

(a.) show v_1, v_2 is a 2-chain of complex eigenvectors for A with $\lambda = i\omega$.

(b.) Calculate $[T]_{\beta,\beta}$.

(c.) find the real solution of $\frac{dx}{dt} = Ax$ in terms of the given vectors and ω.

Problem 116: Find the singular values of $A = \begin{bmatrix} \sqrt{6} & 1 \\ 0 & \sqrt{6} \end{bmatrix}$. Note: the singular values of A are the square roots of the eigenvalues of $A^T A$. We denote these by $\sigma_1, \sigma_2, \dots, \sigma_n$ arranged by $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$.

Problem 117: Find a singular value decomposition (SVD) for $A = \begin{bmatrix} 2 & -1 \\ 2 & 2 \end{bmatrix}$.

Problem 118: Let $A = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}$. Find a SVD for A.

Problem 119: Suppose A is square and invertible. Find a singular value decomposition of A^{-1}.

Problem 120: If $U = [u_1|u_2|\cdots|u_m]$ and $V = [v_1|v_2|\cdots|v_n]$ give $A = U\Sigma V^T$ where

$$\Sigma = Diag(\sigma_1, \sigma_2, \ldots, \sigma_r, 0, \ldots, 0).$$

then show $A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_r u_r v_r^T$.