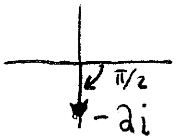
MATH 221: SPRING 2017 TEST 1

In proof questions write complete sentences to communicate your point. In computational questions please box your answer. You are allowed a 3″ × 5″ card. Enjoy!

(1.) (10pts) Let $z = 1 - i$ where i is the imaginary unit with $i^2 = -1$. Calculate the polar form of z^2. *(note: polar form requires the use of $e^{i\theta}$ for appropriate θ in radians)*

$$z^2 = (1-i)(1-i) = 1 - 2i + i^2 = -2i = \boxed{2e^{\frac{-\pi i}{2}}}$$



(2.) (10pts) Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 5 & 6 \\ 3 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix}$. Notice A and B are related by an elementary row operation. Find an elementary matrix E for which $B = EA$. Check your answer by multiplying.

$$A \xrightarrow{r_1 \leftrightarrow r_3} B \quad \therefore \quad B = E_{r_1 \leftrightarrow r_3} A$$

$$B = \underbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}}_{E} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 4 & 5 & 6 \\ 3 & 4 & 5 \\ 1 & 2 & 3 \end{bmatrix} \checkmark$$

(3.) (10pts) Let $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix}$. Express $BX + XB^T = \begin{bmatrix} 9 & 13 \\ 0 & 7 \end{bmatrix}$ as system of equations in x_1, x_2, x_3, x_4. Also, find the augmented coefficient matrix for the system.

$$\begin{bmatrix} 1 & 1 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} + \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 1 & -2 \end{bmatrix} = \begin{bmatrix} 9 & 13 \\ 0 & 7 \end{bmatrix}$$

$$\left[\begin{array}{c|c} x_1 + x_3 & x_2 + x_4 \\ \hline 3x_1 - 2x_3 & 3x_2 - 2x_4 \end{array}\right] + \left[\begin{array}{c|c} x_1 + x_2 & 3x_1 - 2x_2 \\ \hline x_3 + x_4 & 3x_3 - 2x_4 \end{array}\right] = \begin{bmatrix} 9 & 13 \\ 0 & 7 \end{bmatrix}$$

$$\boxed{\begin{aligned} 2x_1 + x_2 + x_3 &= 9 \\ 3x_1 - x_2 + x_4 &= 13 \\ 3x_1 - x_3 + x_4 &= 0 \\ 3x_2 + 3x_3 - 4x_4 &= 7 \end{aligned}} \longrightarrow \boxed{\left[\begin{array}{cccc|c} 2 & 1 & 1 & 0 & 9 \\ 3 & -1 & 0 & 1 & 13 \\ 3 & 0 & -1 & 1 & 0 \\ 0 & 3 & 3 & -4 & 7 \end{array}\right]}$$

(4.) (20pts) Find the solution set of the system below by row-reduction on the augmented coefficient matrix. Use non-pivot variables as parameters in your solution set if needed.

$$x_1 + 2x_2 - x_3 = 7$$
$$3x_1 + 8x_2 + 5x_3 + 6x_4 = 1.$$

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 7 \\ 3 & 8 & 5 & 6 & 1 \end{array}\right] \xrightarrow{r_2 - 3r_1} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 7 \\ 0 & 2 & 8 & 6 & -20 \end{array}\right] \xrightarrow{r_1 - r_2} \left[\begin{array}{cccc|c} 1 & 0 & -9 & -6 & 27 \\ 0 & 2 & 8 & 6 & -20 \end{array}\right]$$

$$\xrightarrow{r_2/2} \left[\begin{array}{cccc|c} 1 & 0 & -9 & -6 & 27 \\ 0 & 1 & 4 & 3 & -10 \end{array}\right] \quad \begin{array}{l} \therefore\ X_1 = 9X_3 + 6X_4 + 27 \\ \therefore\ X_2 = -4X_3 - 3X_4 - 10 \end{array}$$

$$\boxed{\text{Sol}^{\underline{n}}\ \text{Set} = \{(9X_3 + 6X_4 + 27, -4X_3 - 3X_4 - 10, X_3, X_4) \mid X_3, X_4 \in \mathbb{R}\}}$$

(5.) (10pts) Let $F(x, y) = (x^2 + y^2, xy^3)$. Find the formula for the linearization of F at $(3, -1)$. That is, calculate $L_F^{(3,-1)}(x, y)$.

$$J_F = \left[\frac{\partial F}{\partial x} \Big| \frac{\partial F}{\partial y}\right] = \begin{bmatrix} 2x & 2y \\ y^3 & 3xy^2 \end{bmatrix} \quad \therefore \quad J_F(3,-1) = \begin{bmatrix} 6 & -2 \\ -1 & 9 \end{bmatrix}$$

$$\therefore \quad L_F^{(3,-1)}(x,y) = F(3,-1) + J_F(3,-1)\begin{bmatrix} x-3 \\ y+1 \end{bmatrix}$$

$$L_F^{(3,-1)}(x,y) = (3^2+1^2, 3(-1)^3) + \begin{bmatrix} 6 & -2 \\ -1 & 9 \end{bmatrix}\begin{bmatrix} x-3 \\ y+1 \end{bmatrix}$$

$$L_F^{(3,-1)}(x,y) = (10, -3) + (6(x-3) - 2(y+1), -1(x-3) + 9(y+1))$$

$$\boxed{L_F^{(3,-1)}(x,y) = (10 + 6(x-3) - 2(y+1), -3 - (x-3) + 9(y+1))}$$

$$\boxed{L_F^{(3,-1)}(x,y) = (6x - 2y - 10, -x + 9y + 9)} \ \nearrow \text{aka.}$$

(6.) (10pts) Let $A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 1 & 1 \end{bmatrix}$. Find all $x \in \mathbb{C}$ for which $A - xI$ is **not** invertible. Here I denotes the 3×3 identity matrix.

$$\text{Need } \det(A - xI) = 0 \longrightarrow \det\begin{bmatrix} 3-x & 0 & 0 \\ 0 & 2-x & -1 \\ 0 & 1 & 1-x \end{bmatrix} = 0$$

$$\longrightarrow (3-x)\det\begin{bmatrix} 2-x & -1 \\ 1 & 1-x \end{bmatrix} = 0$$

$$\therefore \quad (x-3)[(x-1)(x-2) + 1] = 0$$

$$(x-3)(x^2 - 3x + 3) = 0$$

$$0 = x^2 - 3x + 3 = \left(x - \tfrac{3}{2}\right)^2 + 3 - \tfrac{9}{4}$$
$$= \left(x - \tfrac{3}{2}\right)^2 + \tfrac{3}{4}$$

Hence, $\boxed{x = 3 \text{ and } x = \frac{3 \pm i\sqrt{3}}{2}}$

(7.) (20pts) Find every cubic polynomial whose graph includes the points $(0, 1)$ and $(1, 3)$ and $(-1, 5)$. Also, find the unique quadratic polynomial whose graph intersects the given points.

$$f(x) = Ax^3 + Bx^2 + Cx + D$$

$$f(0) = \underline{D = 1}.$$

$$+\begin{pmatrix} f(1) = A + B + C + 1 = 3 \\ f(-1) = -A + B - C + 1 = 5 \end{pmatrix}$$

$$2B + 2 = 8 \quad \therefore \quad 2B = 6 \quad \therefore \quad \underline{B = 3}.$$

Hence, $\begin{matrix} A + C = 3 - 1 - 3 = -1 \\ -A - C = 5 - 1 - 3 = 1 \end{matrix}$ > same eq^{n}. $\underline{A = -1 - C}$.

$$\therefore \boxed{f(x) = (-1 - c)x^3 + 3x^2 + Cx + 1 \quad \text{for } C \in \mathbb{R}}$$

Choose $C = -1$ to obtain $\underline{f_{\text{quadratic}}(x) = 3x^2 - x + 1}$ → $f_q(0) = 1$ ✓ → $f_q(1) = 3$ ✓ → $f_q(-1) = 5$ ✓

(8.) (20pts) Find A^{-1} given $A = \begin{bmatrix} 2 & 1 & 8 \\ 1 & 0 & 4 \\ 1 & 1 & 5 \end{bmatrix}$. Solve $Av = (1, 2, 2)$ for v via multiplication by A^{-1}.

$$\left[\begin{array}{ccc|ccc} 2 & 1 & 8 & 1 & 0 & 0 \\ 1 & 0 & 4 & 0 & 1 & 0 \\ 1 & 1 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow{r_1 \leftrightarrow r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 0 & 1 & 0 \\ 2 & 1 & 8 & 1 & 0 & 0 \\ 1 & 1 & 5 & 0 & 0 & 1 \end{array}\right] \xrightarrow[r_3 - r_1]{r_2 - 2r_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & 1 & 1 & 0 & -1 & 1 \end{array}\right]$$

$$\xrightarrow{r_3 - r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 4 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right] \xrightarrow{r_1 - 4r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 4 & -3 & -4 \\ 0 & 1 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -1 & 1 & 1 \end{array}\right]$$

Let's check it, $\begin{bmatrix} 2 & 1 & 8 \\ 1 & 0 & 4 \\ 1 & 1 & 5 \end{bmatrix}\begin{bmatrix} 4 & -3 & -4 \\ 1 & -2 & 0 \\ -1 & 1 & 1 \end{bmatrix} \overset{\checkmark}{=} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ A^{-1}

$$Av = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \Rightarrow v = A^{-1}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 & -3 & -4 \\ 1 & -2 & 0 \\ -1 & 1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 - 6 - 8 \\ 1 - 4 \\ -1 + 2 + 2 \end{bmatrix}$$

$$\therefore \boxed{V = (-10, -3, 3).}$$

(9.) (20pts) Remember, we have many properties to use in addition to the cofactor formulae,

(a.) If $v = [1, 1, 1]^T$ and $A = vv^T$ calculate $\det(A)$.

$$VV^T = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}[1,1,1] = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \quad \therefore \quad \det(A) = \det\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} = \boxed{0.}$$

(b.) Calculate $\det(B)$ where $B = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 6 & 1 & 8 & 0 \\ 2 & 0 & 3 & 0 \\ 7 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{C_1 \leftrightarrow C_4} \begin{bmatrix} 1 & 2 & 3 & 1 \\ 0 & 1 & 8 & 6 \\ 0 & 0 & 3 & 2 \\ 0 & 0 & 0 & 7 \end{bmatrix} \Rightarrow \boxed{\det(B) = -21}$

(c.) Let A, B be as given in the previous parts. If $M = \left[\begin{array}{c|c} 2A & 0 \\ \hline 0 & -B^T \end{array}\right]$ then calculate $\det(M)$ via application of properties of determinants and the results of the previous pair of problems.

$$\det(M) = \det(2A)\det(-B^T)$$
$$= 2^3 \det(A)\,(-1)^4 \det(B^T) = 8(0)(-1)^4 \det(B) = \boxed{0.}$$

(10.) (10pts) Given that $T(1, 1) = (2, 3)$ and $T(1, -1) = (4, 5)$. If T is a linear transformation on $\mathbb{R}^2$ then find the formula for $T(x, y)$.

It suffices to find $[T] = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$T(1,1) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} a+b \\ c+d \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix} \quad \begin{aligned} &\to a+b = 2 \quad ① \\ &\to c+d = 3 \quad ② \end{aligned}$$

$$T(1,-1) = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a-b \\ c-d \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \end{bmatrix} \quad \begin{aligned} &\to a-b = 4 \quad ③ \\ &\to c-d = 5 \quad ④ \end{aligned}$$

① + ③ : $2a = 6 \therefore \underline{a = 3} \Rightarrow b = 2 - 3 = -1 \therefore \underline{b = -1}.$

② + ④ : $2c = 8 \therefore \underline{c = 4}. \Rightarrow d = 3 - 4 = -1 \therefore \underline{d = -1}.$

$$[T] = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix} \longrightarrow T(x,y) = \begin{bmatrix} 3 & -1 \\ 4 & -1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$\therefore \boxed{T(x,y) = (3x - y,\ 4x - y)}$$

(11.) (10pts) Let $A = \begin{bmatrix} -1 & 2 & 2 \\ 0 & 3 & 2 \\ 0 & 0 & 5 \end{bmatrix}$. Let $f(x) = (x+1)(x-3)(x-5)$. Calculate $f(A)$. Use your calculation to find a formula for A^{-1} in terms of A.

$$f(A) = (A+I)(A-3I)(A-5I)$$

$$= \begin{bmatrix} 0 & 2 & 2 \\ 0 & 4 & 2 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} -4 & 2 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} -6 & 2 & 2 \\ 0 & -2 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2 & 2 \\ 0 & 4 & 2 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 24 & -12 & -4 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$= \underline{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}.$$

$$(A+I)(A^2 - 8A + 15I) = 0$$

$$A^3 - \underline{8A^2} + \underline{15A} + \underline{A^2} - \underline{8A} + 15I = 0$$

$$I = \frac{1}{15}\left(-A^3 + 7A^2 - 7A\right)$$

$$I = A\left(\frac{1}{15}(-A^2 + 7A - 7I)\right)$$

$$\therefore \boxed{A^{-1} = \frac{-1}{15}\left(A^2 - 7A + 7I\right)}$$

(12.) (10pts) Let A be a 4×4 matrix for which $\underline{Ae_2 = e_1 \text{ and } Ae_1 = e_2}$. Suppose $A(0,0,3,4) = e_3$ and $A(0,0,-4,3) = e_4$. Find A^{-1} and find A.

$$A\,[y_1 | y_2 | y_3 | y_4] = I = [e_1 | e_2 | e_3 | e_4]$$

$Ay_1 = e_1$ ah ha. take $\underline{y_1 = e_2}$.
$Ay_2 = e_2$ ah ha. take $\underline{y_2 = e_1}$ } given
$Ay_3 = e_3$
$Ay_4 = e_4$ } it remains to find y_3 & y_4, wait, nope.

these are also given $y_3 = (0,0,3,4)$
$y_4 = (0,0,-4,3)$

$$\therefore\ A^{-1} = \left[\begin{array}{cc|cc} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 3 & -4 \\ 0 & 0 & 4 & 3 \end{array}\right] = \left[\begin{array}{c|c} I_2 & 0 \\ \hline 0 & B \end{array}\right] \qquad B^{-1} = \frac{1}{9+16}\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix} = \frac{1}{25}\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$$

$$A = (A^{-1})^{-1} = \left[\begin{array}{c|c} I_2^{-1} & 0 \\ \hline 0 & B^{-1} \end{array}\right] = \left[\begin{array}{cc|c} 1 & 0 & 0\ 0 \\ 0 & 1 & 0\ 0 \\ \hline 0 & 0 & \multirow{2}{*}{$\frac{1}{25}\begin{bmatrix} 3 & 4 \\ -4 & 3 \end{bmatrix}$} \\ 0 & 0 & \end{array}\right] = \underbrace{\boxed{\frac{1}{25}\begin{bmatrix} 25 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 \\ 0 & 0 & 3 & 4 \\ 0 & 0 & -4 & 3 \end{bmatrix}}}_{A}$$