MATH 221: SPRING 2017 TEST 2

In proof questions write complete sentences to communicate your point. In computational questions please box your answer. You are allowed a page of notes. Enjoy!

(1.) (10pts) Let $W = \{A \in \mathbb{R}^{3\times 3} \mid A^T = -A\}$. Show that W is a subspace of $\mathbb{R}^{3\times 3}$.

Note $0^T = 0 = -0$ thus $0 \in W \neq \phi$.

Let $A, B \in W$ and suppose $c \in \mathbb{R}$. Consider,

$$(cA+B)^T = cA^T + B^T \quad : \text{prop. of transpoce}$$
$$= -cA - B \quad : A^T = -A,\ B^T = -B \text{ as } A, B \in W,$$
$$= -(cA+B) \Rightarrow cA+B \in W.$$

Thus $A, B \in W$, $c \in \mathbb{R} \Rightarrow cA,\ A+B \in W$ and we conclude $W \leq \mathbb{R}^{3\times 3}$ by subspace test.

(2.) (10pts) Let $W = \{A \in \mathbb{R}^{3\times 3} \mid A^T = -A\}$. Find a basis for W. What is the dimension of W ?

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \Rightarrow A^T = -A \iff \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} = \begin{bmatrix} -a & -b & -c \\ -d & -e & -f \\ -g & -h & -i \end{bmatrix}$$

Thus $a = -a$, $b = -d$, $c = -g$, $e = -e$, $h = -f$, $i = -i$

so $a = e = i = 0$ and

$$A = \begin{bmatrix} 0 & b & c \\ -b & 0 & h \\ -c & -h & 0 \end{bmatrix} = b\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + c\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix} + h\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$\therefore \beta = \{E_{12} - E_{21},\ E_{13} - E_{31},\ E_{23} - E_{32}\}$ serves as basis for W $\therefore \boxed{\dim W = 3}$

(3.) (15pts) Let $\beta = \{1, (x-3), (x-3)^2\}$ form a basis for $P_2(\mathbb{R})$. Find the coordinates of x^2 with respect to β; that is, calculate $[x^2]_\beta$.

$$f(x) = f(3) + f'(3)(x-3) + \tfrac{1}{2}f''(3)(x-3)^2 \quad \text{is easiest way,}$$
$$x^2 = 9 + 2x\Big|_{x=3}(x-3) + \tfrac{1}{2}(2)(x-3)^2$$
$$x^2 = (x-3)^2 + 6(x-3) + 9$$

$$\therefore \boxed{[x^2]_\beta = \begin{bmatrix} 1 \\ 6 \\ 9 \end{bmatrix}}$$

(4.) (10pts) Consider $\beta = \{\overbrace{(1,1,1)}^{V_1}, \overbrace{(1,-1,0)}^{V_2}, \overbrace{(1,1,-2)}^{V_3}\}$ as a subset of $\mathbb{R}^3$. Is β an orthogonal set? Is β an orthonormal set ? Answer these questions. Also, if $v = (5,0,7)$ then find $[v]_\beta$.

$V_1 \cdot V_2 = V_1 \cdot V_3 = V_2 \cdot V_3 = 0 \therefore \beta$ is orthogonal.

$V_1 \cdot V_1 = 3, \quad V_2 \cdot V_2 = 2, \quad V_3 \cdot V_3 = 6 \therefore \beta$ is <u>NOT</u> orthonormal

$$V = \left(\frac{V \cdot V_1}{\|V_1\|^2}\right) V_1 + \left(\frac{V \cdot V_2}{\|V_2\|^2}\right) V_2 + \left(\frac{V \cdot V_3}{\|V_3\|^2}\right) V_3$$

$$= \frac{12}{3} V_1 + \frac{5}{2} V_2 + \left(\frac{5-14}{6}\right) V_3$$

$$\Rightarrow \boxed{[V]_\beta = (4, 5/2, -3/2)}$$

(5.) (15pts) Find the equation of the line $y = mx + b$ which is closest to the points

$$(-2,-4), \quad (-1,0), \quad (0,3), \quad (1,2).$$

$y = mx + c$

$$\begin{aligned} -4 &= -2m + c \\ 0 &= -m + c \\ 3 &= m(0) + c \\ 2 &= m(1) + c \end{aligned} \longrightarrow \underbrace{\begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} m \\ c \end{bmatrix} = \underbrace{\begin{bmatrix} -4 \\ 0 \\ 3 \\ 2 \end{bmatrix}}_{b}$$

Normal Eq^{n}s give least squares fit, $A^T A \begin{bmatrix} m \\ c \end{bmatrix} = A^T \begin{bmatrix} -4 \\ 0 \\ 3 \\ 2 \end{bmatrix}$

$$\begin{bmatrix} -2 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} -2 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ 0 \\ 3 \\ 2 \end{bmatrix}$$

$$\begin{bmatrix} 6 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} 10 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} m \\ c \end{bmatrix} = \frac{1}{24-4} \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} 10 \\ 1 \end{bmatrix} = \frac{1}{20} \begin{bmatrix} 42 \\ 26 \end{bmatrix} = \begin{bmatrix} 21/10 \\ 13/10 \end{bmatrix}$$

$$\therefore \boxed{y = \frac{21}{10} x + \frac{13}{10} = 2.1x + 1.3}$$

(6.) (15pts) Let $T(a+bx+cx^2) = \begin{bmatrix} a+b & b-3c \\ 4a+5c & 13c \end{bmatrix}$. Find the matrix $[T]_{\beta,\gamma}$ where we define basis $\beta = \{1, x, x^2\}$ to $P_2(\mathbb{R})$ and $\gamma = \{E_{11}, E_{12}, E_{21}, E_{22}\}$ to $\mathbb{R}^{2\times 2}$.

$$\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -3 \\ 4 & 0 & 5 \\ 0 & 0 & 13 \end{bmatrix} \underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{[v]_\beta} = \underbrace{\begin{bmatrix} a+b \\ b-3c \\ 4a+5c \\ 13c \end{bmatrix}}_{[T(v)]_\gamma}$$

$$[T]_{\beta,\gamma}[v]_\beta = [T(v)]_\gamma$$

$$v = a + bx + cx^2$$

$$\therefore \boxed{[T]_{\beta,\gamma} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & -3 \\ 4 & 0 & 5 \\ 0 & 0 & 13 \end{bmatrix}}$$

(7.) (20pts) Suppose $x, y, z \in V$ an inner product space with inner product of v, w denoted by $\langle v, w \rangle$. You are given the following data:

$$\langle x, x\rangle = 4 \qquad \langle y, y\rangle = 1 \qquad \langle z, z\rangle = 3 \qquad \& \qquad \langle x, y\rangle = 2 \qquad \langle x, z\rangle = 0 \qquad \langle y, z\rangle = 0$$

(a.) Show $\{x, y, z\}$ is a linearly independent set of vectors in V. (sorry)

(b.) Find an orthonormal basis for span$\{x, y, z\}$ (was supposed to be calculation free Gram-Schmidt problem ☹)

(a.) $c_1x + c_2y + c_3z = 0$

$$\Rightarrow \langle c_1x + c_2y + c_3z, x\rangle = c_1\langle x,x\rangle + c_2\langle y,x\rangle + c_3\overset{0}{\cancel{\langle z,x\rangle}}$$

$$\Rightarrow \underline{4c_1 + 2c_2 = 0}.$$

Also, $\langle y, c_1x + c_2y + c_3z\rangle = c_1\langle y,x\rangle + c_2\langle y,y\rangle + c_3\underset{0}{\cancel{\langle y,z\rangle}}$

$$\Rightarrow \underline{2c_1 + c_2 = 0}. \quad \therefore c_2 = -2c_1$$

Hence, $4c_1 + 2c_2 = 4c_1 + 2(-2c_1) = 0$.

Oh, so, clear $\{x,z\}$ and $\{y,z\}$ are LI

Towards →← suppose $x = ky$ then $\langle x,x\rangle = k^2\langle y,y\rangle$

$4 = k^2(1) \therefore \underline{k = \pm 2}$.

and $\langle x,y\rangle = \langle \cancel{x} \pm 2y, y\rangle = \pm 2 = 2 \therefore \underline{k = 2}$.

Oh noes! Consider $x = (2,0,0)$, $y = (1,0,0)$, $z = (0,\sqrt{3},0)$

then $\{x,y,z\}$ fits data and yet $\{x,y,z\}$ not LI!

(8.) (20pts) Let $A = \begin{bmatrix} 1 & 1 & 0 & 2 \\ 2 & 1 & 1 & 3 \\ 1 & 0 & -1 & 1 \end{bmatrix}$. Find the following:

(a.) a basis for the column space of A.
(b.) a basis for the null space of A.

$$\begin{bmatrix} 1 & 1 & 0 & 2 \\ 2 & 1 & 1 & 3 \\ 1 & 0 & -1 & 1 \end{bmatrix} \xrightarrow[r_3 - r_1]{r_2 - 2r_1} \begin{bmatrix} 1 & 1 & 0 & 2 \\ 0 & -1 & 1 & -1 \\ 0 & -1 & -1 & -1 \end{bmatrix} \xrightarrow[r_3 - r_2]{r_1 + r_2} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & -1 \\ 0 & 0 & -2 & 0 \end{bmatrix}$$

$$\xrightarrow[r_2/-1]{r_3/-2} \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(a.) $\beta = \{(1,2,1), (1,1,0), (0,1,-1)\}$ is basis for $Col(A)$ as is any set of LI 3-vectors as $Col(A) = \mathbb{R}^3$ here.

(b.) $x = (x_1, x_2, x_3, x_4) \in Null(A) \Rightarrow \begin{matrix} x_1 = -x_4 \\ x_2 = -x_4 \\ x_3 = 0 \end{matrix}$

$\therefore\ x = (-x_4, -x_4, 0, x_4)$

$\Rightarrow \boxed{\gamma = \{(-1,-1,0,1)\} \text{ is basis for } Null(A)}$

(9.) (10pts) If $A \in \mathbb{R}^{n \times n}$ and $v, w \in \mathbb{R}^n$ then show that $(A^T v) \cdot w = v \cdot Aw$.

$$\begin{aligned} (A^T v) \cdot w &= (A^T v)^T w \\ &= v^T (A^T)^T w \\ &= v^T A w \\ &= v \cdot (Aw). \end{aligned}$$

– (this was a homework) –

(10.) (10pts) If $z = \langle 1+i, 2, 3 \rangle$ and $w = \langle i, 2+i, -i \rangle$ then calculate $\langle z, w \rangle$ where $\langle , \rangle$ denotes the usual inner product on $\mathbb{C}^3$.

$$\begin{aligned} \langle z, w \rangle &= w^\dagger z \\ &= [-i, 2-i, i] \begin{bmatrix} 1+i \\ 2 \\ 3 \end{bmatrix} \\ &= -i(1+i) + (2-i)(2) + i(3) \\ &= \underline{-i} + 1 + 4 \underline{- 2i + 3i} = \boxed{5.} \end{aligned}$$

(11.) (10pts) Let $\vec{A}, \vec{B}$ be orthogonal vectors in $\mathbb{R}^n$. Show that $||\vec{A}+\vec{B}||^2 = ||\vec{A}||^2 + ||\vec{B}||^2$.

$$
\begin{aligned}
||\vec{A}+\vec{B}||^2 &= (\vec{A}+\vec{B}) \cdot (\vec{A}+\vec{B}) \\
&= \vec{A}\cdot\vec{A} + \vec{A}\cdot\vec{B} + \vec{B}\cdot\vec{A} + \vec{B}\cdot\vec{B} \\
&= ||\vec{A}||^2 + ||\vec{B}||^2
\end{aligned}
$$

as $\vec{A}\cdot\vec{B} = \vec{B}\cdot\vec{A} = 0$ and $||A||^2 = \vec{A}\cdot\vec{A}$, $||\vec{B}||^2 = \vec{B}\cdot\vec{B}$.

(12.) (15pts) Let $\langle A, B\rangle = \text{trace}(AB^T)$. If $A_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ and $A_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ then let $W = \text{span}\{A_1, A_2\}$. Find the matrix in W which is closest to $B = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$. In other words, calculate the projection of B onto the W-subspace. *Hint: find an orthonormal basis for W to get started.*

$$\langle A_1, A_1\rangle = 4 \quad \therefore \quad \hat{A}_1 = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = A_1''$$

$$A_2' = A_2 - \langle A_2, A_1''\rangle A_1'' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \frac{1}{4}\left\langle \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right\rangle \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$A_2' = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \frac{1}{4}(2)\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{bmatrix}$$

$$A_2' = \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \quad \text{has} \quad ||A_2'|| = \frac{1}{2}\sqrt{1^2+1^2+1^2+1^2} = 1.$$

Hence $\Big\{ \underbrace{\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{A_1''}, \underbrace{\frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}}_{A_2''} \Big\}$ is orthonormal basis for W

$$
\begin{aligned}
\text{Proj}_W(B) &= B - \langle B, A_1''\rangle A_1'' + \langle B, A_2''\rangle A_2'' \\
&= \frac{1}{4}\left\langle \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\right\rangle \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} + \frac{1}{4}\left\langle \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}\right\rangle \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \\
&= \frac{3}{4}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} + \frac{3}{4}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} +6/4 & 0 \\ 0 & -6/4 \end{bmatrix} \\
&= \boxed{\frac{3}{2}\begin{bmatrix} +1 & 0 \\ 0 & -1 \end{bmatrix}}
\end{aligned}
$$