Quiz 2 **MATH 221 Section:** ________ **Instructor: James Cook**

Name: ______________________ **Date:** ____________

Instructions: Give exact answers unless otherwise directed. Show all work, presenting a clear, logical solution to each problem. (just write the answers on this page, put your work on additional pages written neatly and in order, thanks!) $\boxed{\text{Box answers}}$. Due 2-14-17 start of class.

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(1) Find the solution set for the system of equations given below:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + x_4 &= 10 \\ x_3 + 3x_4 &= 8 \\ x_1 - 2x_2 + x_4 &= 6 \end{aligned}$$

$$\text{rref}\left[\begin{array}{cccc|c} 1 & 2 & 3 & 1 & 10 \\ 0 & 0 & 1 & 3 & 8 \\ 1 & -2 & 0 & 1 & 6 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 0 & -7/2 & -4 \\ 0 & 1 & 0 & -9/4 & -5 \\ 0 & 0 & 1 & 3 & 8 \end{array}\right] \begin{array}{l} \to x_1 = \frac{7}{2}x_4 - 4 \\ \to x_2 = \frac{9}{4}x_4 - 5 \\ \to x_3 = -3x_4 + 8 \end{array}$$

$$\boxed{\text{Sol}^{\text{n}} \text{ Set} = \left\{ \left(\tfrac{7}{2}x_4 - 4, \tfrac{9}{4}x_4 - 5, -3x_4 + 8, x_4\right) \middle| x_4 \in \mathbb{R} \right\}}$$

(2) Let $A = \begin{bmatrix} 2 & 3 & 0 \\ 2 & 9 & 1 \\ 0 & 3 & 1 \end{bmatrix}$. Calculate A^{-1}.

$$\text{rref}\left[\begin{array}{ccc|ccc} 2 & 3 & 0 & 1 & 0 & 0 \\ 2 & 9 & 1 & 0 & 1 & 0 \\ 0 & 3 & 1 & 0 & 0 & 1 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & \\ 0 & 1 & 0 & \underbrace{\begin{array}{ccc} 1 & -1/2 & 1/2 \\ -1/3 & 1/3 & -1/3 \\ 1 & -1 & 2 \end{array}}_{A^{-1}} \\ 0 & 0 & 1 & \end{array}\right]$$

$$A^{-1}Av = A^{-1}(4,5,1) \therefore \quad v = A^{-1}\begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -1/2 & 1/2 \\ -1/3 & 1/3 & -1/3 \\ 1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 5 \\ 1 \end{bmatrix}$$

$$\therefore \boxed{v = (2, 0, 1)}$$

(3) Using A from the previous problem, solve $Av = (4, 5, 1)$.

(4) $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \end{bmatrix}$. Find elementary matrices E_1, E_2 such that $E_2 E_1 A = I$. Calculate A^{-1}.

$$A \xrightarrow{r_3 - 2r_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 - r_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I = \underbrace{E_2 E_1}_{A^{-1}} A$$

$$\underline{E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}}. \qquad \underline{E_2 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}.$$

$$A^{-1} = E_2 E_1 = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 3 & 0 & -1 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = A^{-1}}$$

(5) Find all cubic polynomials (these have formula $f(x) = Ax^3 + Bx^2 + Cx + D$) whose graphs pass through the points $(0,1), (1,3), (-1,8)$. (here I use the term *cubic* to include the case $A = 0$)

$$f(0) = 1 = D$$
$$f(1) = 3 = A + B + C + D$$
$$f(-1) = 8 = -A + B - C + D$$

$$\text{rref} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 3 \\ -1 & 1 & -1 & 1 & 8 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] = \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & -7/2 \\ 0 & 1 & 0 & 0 & 7/2 \\ 0 & 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{aligned} \therefore\ & A = -C - 7/2 \\ \therefore\ & B = 7/2 \\ \therefore\ & D = 1 \end{aligned}$$

$$\boxed{f(x) = -\left(C + \tfrac{7}{2}\right)X^3 + \tfrac{7}{2}X^2 + CX + 1} \quad \text{for any } C \in \mathbb{R}$$

(6) Let $u = [1, 2, 3, 4]$ and $v = [3, 3, 3]^T$. Calculate the following (if possible):

(a) $uu^T + v^T v$ $= \underbrace{[1, 2, 3, 4] \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}}_{(1\times4)(4\times1)} + \underbrace{[3, 3, 3] \begin{bmatrix} 3 \\ 3 \\ 3 \end{bmatrix}}_{(1\times3)(3\times1)} = \underbrace{1 + 2^2 + 3^2 + 4^2 + 3(3^2)}_{\boxed{57}}$

(b) $u^T u + v^T v$

$\underbrace{(4\times1)(1\times4)}_{4\times4}$ $\underbrace{(1\times3)(3\times1)}_{1\times1}$

<u>NOT POSSIBLE TO CALCULATE</u>.

(7) Let $A = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$. Define $f(t) = t^2 - 2t + 5$ and $g(t) = t - 3$ and calculate $f(A)$ and $g(A)$. Also, calculate the product $f(A)g(A)$. Use the algebra you discover to find a formula for A^{-1} without resorting to the sort of calculation you had to do in the first problem of this Quiz.

$$A^2 = AA = \begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & -2 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} -3 & -4 & 0 \\ 4 & -3 & 0 \\ 0 & 0 & 9 \end{bmatrix}$$

$$f(A) = A^2 - 2A + 5I = \begin{bmatrix} -3 & -4 & 0 \\ 4 & -3 & 0 \\ 0 & 0 & 9 \end{bmatrix} - \begin{bmatrix} 2 & -4 & 0 \\ 4 & 2 & 0 \\ 0 & 0 & 6 \end{bmatrix} + \begin{bmatrix} 5 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix} = \boxed{\underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 8 \end{bmatrix}}_{f(A)}}$$

$$g(A) = A - 3I = \boxed{\begin{bmatrix} -2 & -2 & 0 \\ 2 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = g(A)}$$

Thus, $f(A)\,g(A) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

$$f(A)g(A) = (A^2 - 2A + 5I)(A - 3I)$$
$$0 = A^3 - 2A^2 + 5A - 3A^2 + 6A - 15I$$

$$15I = A^3 - 5A^2 + 11A$$
$$I = A\underbrace{\left(\frac{A^2 - 5A + 11I}{15}\right)}_{A^{-1}}$$

$$\boxed{A^{-1} = \frac{1}{15}\begin{bmatrix} 3 & 6 & 0 \\ -6 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}}$$

(8) Let $T(x, y, z) = (x + y - 2z, y + 3z)$. Find the standard matrix of T.

$$T(x,y,z) = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \quad \therefore \quad \boxed{[T] = \begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 3 \end{bmatrix}}$$

(9) Let $S(u, v) = (u + v, u + 3v, 7v, 8u)$. Find the standard matrix of S.

$$S(u,v) = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 0 & 7 \\ 8 & 0 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} \quad \therefore \quad \boxed{[S] = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 0 & 7 \\ 8 & 0 \end{bmatrix}}$$

(10) Let $S \circ T$ denote the composition of the linear transformations in the previous two problems. Find the standard matrix for $S \circ T$.

$$[S \circ T] = [S][T] = \begin{bmatrix} 1 & 1 \\ 1 & 3 \\ 0 & 7 \\ 8 & 0 \end{bmatrix}\begin{bmatrix} 1 & 1 & -2 \\ 0 & 1 & 3 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 2 & 1 \\ 1 & 4 & 7 \\ 0 & 7 & 21 \\ 8 & 8 & -16 \end{bmatrix}}$$

$(4 \times 2)\ (2 \times 3)$