Copying answers and steps is strictly forbidden. Evidence of copying results in zero for copied and copier. Working together is encouraged, share ideas not calculations. Explain your steps. This sheet must be printed and attached to your assignment as a cover sheet. The calculations and answers should be written neatly on one-side of paper which is attached and neatly stapled in the upper left corner. Box your answers where appropriate. Please do not fold. Thanks!

These problems cover the material from Chapter 5 of Shores' text. However, it is also important that you study my Lecture Notes on this material. In particular, Chapters 6,9 and §7.8 – 7.9. Each problem is worth 4pts. You can earn 88pts on this assignment.

✓**Problem 1** Shores' 5.2# 2 (can we diagonalize?)

✓**Problem 2** Shores' 5.2# $3a, 4a$ (diagonalize and power)

✓**Problem 3** Shores' 5.2# 7 (jordan block)

✓**Problem 4** Shores' 5.2# 11 (sine and cosine of matrix)

Problem 5 Shores' 5.2# 19 (Fibonacci)

Problem 6 Shores' 5.3# 3 (Markov chain)

Problem 7 Shores' 5.3# 5 (Ergodic Theorem)

left to reader.

✓**Problem 8** Shores' 5.3# 7 (Jordan form list)

✓**Problem 9** Shores' 5.4# 8 (unitary diagonalize and power)

✓**Problem 10** Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$. Complete the following very related tasks:

✓**(a.)** find the eigenvalues and eigenvectors for A

✓**(b.)** solve the system of differential equations $\frac{d\vec{r}}{dt} = A\vec{r}$ where $\vec{r} = (x, y)$.

✓**(c.)** Let $Q(x, y) = x^2 + 4xy + y^2$. Find the formula for Q in terms of eigencoordinates $\bar{x}, \bar{y}$ and analyze what extreme values Q attains on the unit-circle $x^2 + y^2 = 1$.

✓**Problem 11** Let $A = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$. Complete the following very related tasks:

(a.) find the eigenvalues and eigenvectors for A

(b.) solve the system of differential equations $\frac{d\vec{r}}{dt} = A\vec{r}$ where $\vec{r} = (x, y)$.

(c.) Let $Q(x, y) = 6xy$. Find the formula for Q in terms of eigencoordinates $\bar{x}, \bar{y}$ and analyze what extreme values Q attains on the unit-circle $x^2 + y^2 = 1$.

✓**Problem 12** Let $A = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$. Complete the following very related tasks:

(a.) find the eigenvalues and eigenvectors for A

(b.) solve the system of differential equations $\frac{d\vec{r}}{dt} = A\vec{r}$ where $\vec{r} = (x, y, z)$.

(c.) Let $Q(x, y, z) = 5x^2 + 5y^2 + 2z^2 + 2xy + xz + yz$. Find the formula for Q in terms of eigencoordinates $\bar{x}, \bar{y}, \bar{z}$ and analyze what extreme values Q attains on the unit-sphere $x^2 + y^2 + z^2 = 1$.

✓**Problem 13** Let $A \in \mathbb{C}^{n \times n}$ and define for $1 \leq i, j \leq n$

$$\rho_i(A) = \sum_{j=1}^{n} |(\text{row}_i(A))_j| = \sum_{j=1}^{n} |A_{ij}| \qquad \& \qquad \nu_j(A) = \sum_{i=1}^{m} |(\text{col}_j(A))_i| = \sum_{i=1}^{m} |A_{ij}|.$$

Then, using these we define the **row sum** of A denoted $\rho(A)$ and the **column sum** of A denoted $\nu(A)$ as follows:

$$\rho(A) = \max\{\rho_i(A) \mid 1 \leq i \leq n\} \qquad \& \qquad \nu(A) = \max\{\nu_j(A) \mid 1 \leq j \leq m\}$$

Then define the **Gerschgorin disk** C_i for A is the disk in the complex plane with center A_{ii} and radius $r_i = \rho_i(A) - |A_{ii}|$. That is,

$$C_i = \{z \in \mathbb{C} \mid |z - A_{ii}| < r_i\}$$

With all of this terminology we have a few very interesting results about the eigenvalues of a complex matrix. Proofs of these assertions can be found in §5.3 of Insel Spence and Friedberg's excellent text on Linear Algebra:

(i.) every eigenvalue of $A \in \mathbb{C}^{n \times n}$ is contained in a Gerschgorin disk of A,

(ii.) for any eigenvalue λ of $A \in \mathbb{C}^{n \times n}$ we have $|\lambda| \leq \min\{\rho(A), \nu(A)\}$,

Let $A = \begin{bmatrix} 1+2i & 1 \\ 2i & -3 \end{bmatrix}$. Find the following:

(a) find and picture the Gerschgorin disks for A in in the complex plane,

(b) explain why A does not have eigenvalue zero in view of the disks from part (a.).

✓ **Problem 14** A **stochastic** or **transition matrix** is a matrix $A \in \mathbb{R}^{n \times n}$ such that $A_{ij} \geq 0$ and

$$A_{1j} + A_{2j} + \cdots + A_{nj} = 1$$

for each $j = 1, 2, \ldots, n$. In words, a transition matrix is a non-negative matrix where each column's entries sum to 1. A vector with non-negative entries which sum to 1 is called a **probabilty vector**. Thus, a transtion matrix is a square matrix formed by concatenating probability vectors. With the above terminology in mind:

(a.) show the product of a transition matrix and a probablility vector is a probability vector,

(b.) show the product of transition matrices is a transition matrix.

Problem 15 Let $A = \begin{bmatrix} 0.9 & 0.02 \\ 0.1 & 0.98 \end{bmatrix}$.

(a.) diagonalize A,

(b.) calculate $\lim_{n\to\infty} A^n$,

(c.) let $x_o = (0.7, 0.3)$. Define $x_n = A^n x_o$ hence $x_1 = Ax_o$ and $x_2 = Ax_1 = AAx_o$ etc. Calculate x_1, x_{10} and x_{100}. What is $\lim_{n\to\infty} x_n$? How does this relate to things you found in (a.)

The vectors you find in (c.) are an example of a **Markov chain**. *Notice A is a transition matrix and x_o is a probability vector.*

✓**Problem 16** Let $A = \begin{bmatrix} 5 & -5 & -5 \\ -1 & 4 & 2 \\ 3 & -5 & -3 \end{bmatrix}$. Find an complex eigenbasis for A. Also, construct a real basis β for which $[\beta]^{-1}A[\beta]$ is in real Jordan form.

✓**Problem 17** Solve $\frac{dx}{dt} = Ax$ where A is the matrix in the previous problem.

✓**Problem 18** Consider $A = J_4(3)$. Find diagonalizable matrix D and a nilpotent matrix N for which $A = D+N$ and $DN = ND$. Calculate e^{tA} with the help of the $A = D+N$ decomposition.

✓**Problem 19** Once more consider $A = J_4(3)$. Let $B = A^2$. What is the Jordan form of B? How is it related to A ?

✓**Problem 20** Let $A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$. Calculate e^{tA} and write the general solution to $\frac{d\vec{r}}{dt} = A\vec{r}$

✓**Problem 21** Let A be diagonalizable. Show that $\det(e^A) = \exp(\text{trace}(A))$.

✓**Problem 22** Suppose $A = J_3(7) \oplus J^{\mathbb{R}}(2+3i)$. Find the general solution to $\frac{d\vec{r}}{dt}$ where $\vec{r} = (x_1, x_2, x_3, x_4, x_5)$.

MATH 221 : EIGENVECTORS & APPLICATIONS

P1 find e-vectors to see if A is diagonalizable, triangular so I can read e-values off diagonal.

(a.) $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \rightarrow \lambda_1 = 2, \ \lambda_2 = 1$ repeated

$$A - 2I = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \Rightarrow \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ is e-vector of } \lambda_1 = 2$$

$$A - I = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$$ Null$(A - I)$ has vectors of form (u, v, w) with $u + w = 0 \ \therefore -w = u$

Hence $(u, v, w) = (-w, v, w) = w(-1, 0, 1) + v(0, 1, 0)$

Thus $\vec{u}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\vec{u}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ are e-vect. with $\lambda_2 = 1$

$\therefore \beta = \{ (1, 0, 0), (-1, 0, 1), (0, 1, 0) \}$ is e-basis of A

Hence A is diagonalizable. Moreover,

$$[\beta]^{-1} A [\beta] = \begin{bmatrix} 2 & & \\ & 1 & \\ & & 1 \end{bmatrix}.$$

(b.) $A = \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \leftarrow A^{-1}$ d.n.e. $\Rightarrow \lambda = 0$ is e-value.

$$\det[A - \lambda I] = \det\begin{bmatrix} 2-\lambda & 1 & 1 \\ 0 & 1-\lambda & 1 \\ 0 & 1 & 1-\lambda \end{bmatrix} = (2-\lambda)[(\lambda-1)^2 - 1]$$
$$= (2-\lambda)[\lambda^2 - 2\lambda]$$
$$= -(\lambda-2)^2 \lambda$$

Hence $\underline{\lambda_1 = 0, \ \lambda_2 = 2}$ repeated.

By inspection $A(0, 1, -1) = 0 \ \therefore (0, 1, -1)$ is $\lambda_1 = 0$ e-vector.

$$A - 2I = \begin{bmatrix} 0 & 1 & 1 \\ 0 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \ \therefore \text{Null}(A - 2I) = \text{span}\{e_1\}$$

Only get one $\lambda_1 = 2$ e-vector (up to LI)

Hence A is not diagonalizable, it has no e-basis.

P1 continued, lower triangular, has e-values on diag.

(c.) $A = \begin{bmatrix} 2 & 0 \\ 3 & 2 \end{bmatrix} \Rightarrow \lambda_1 = 2$ repeated

$A - 2I = \begin{bmatrix} 0 & 0 \\ 3 & 0 \end{bmatrix} \Rightarrow \mathrm{Null}(A-2I) = \mathrm{span}\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

We cannot find e-basis for A $\therefore$ <u>A not diagonalizable</u>

(d.) $A = \begin{bmatrix} 2 & 1 & -1 & -1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 2 \end{bmatrix}$ Triangular $\longrightarrow$ $\lambda_1 = 2$ repeated. $\lambda_2 = 1$ repeated.

$$A - 2I = \begin{bmatrix} 0 & 1 & -1 & -1 \\ 0 & -1 & 0 & 2 \\ 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$X = (X_1, X_2, X_3, X_4) \in \mathrm{Null}(A - 2I)$ has $X_2 = 2X_4$, $X_3 = X_4$

$X = (X_1, 2X_4, X_4, X_4) = X_1(1,0,0,0) + X_4(0,2,1,1)$.

$\therefore$ <u>$(1,0,0,0)$ & $(0,2,1,1)$ are $\lambda_1 = 2$ e-vectors</u>.

$$A - I = \begin{bmatrix} 1 & 1 & -1 & -1 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \longrightarrow \begin{array}{l} X_1 = -X_2 + X_3 \\ X_4 = 0 \end{array}$$

$X \in \mathrm{Null}(A - I) \Rightarrow X = (-X_2 + X_3, X_2, X_3, 0)$
$= X_2(-1, 1, 0, 0) + X_3(1, 0, 1, 0)$

$\therefore$ $(-1,1,0,0)$, $(1,0,1,0)$ e-vect. with e-value $\lambda_2 = 1$

$$\boxed{\text{Hence } \beta = \{(1,0,0,0), (0,2,1,1), (-1,1,0,0), (1,0,1,0)\} \text{ is an e-basis for } A \text{ and } [\beta]^{-1}A[\beta] = \begin{bmatrix} 2 & & & \\ & 2 & & \\ & & 1 & \\ & & & 1 \end{bmatrix}}$$

P2 § 5.2 # 3a, 4a

(3a) $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ has $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = 3$

$$A - 2I = \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \longrightarrow \vec{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \ \lambda_1 = 2$$

$$A - I = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 2 \end{bmatrix} \longrightarrow \vec{u}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \ \lambda_2 = 1$$

$$A - 3I = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \longrightarrow \vec{u}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \ \lambda_3 = 3$$

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ with } P^{-1} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

used tech. to find.

We can calculate $P^{-1}AP = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = D$

(4a.) $A^k = AA \cdots AA = (PDP^{-1})(PDP^{-1}) \cdots (PDP^{-1})(PDP^{-1}) = PD^kP^{-1}$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2^k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3^k \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2^k & 0 & -2^k \\ 0 & 1 & 0 \\ 0 & 0 & 3^k \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 2^k & 0 & -2^k + 3^k \\ 0 & 1 & 0 \\ 0 & 0 & 3^k \end{bmatrix}}$$

P3 § 5.2 #7

Show $J_2(\lambda) = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ is not diagonalizable

for any scalar λ. Observe,

$$\det\left(J_2(\lambda) - xI\right) = \det\begin{bmatrix} \lambda - x & 1 \\ 0 & \lambda - x \end{bmatrix} = (\lambda - x)^2$$

Thus $x = \lambda$ repeated. Eigenvalues are $\lambda_1 = \lambda$ repeated.

$$J_2(\lambda) - \lambda I = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad \text{has e-vectors } (x, 0)$$

Thus no e-basis for $J_2(\lambda)$ exists $\therefore$ $J_2(\lambda)$ not diagonalizable.

$$J_2(\lambda)\, J_2(\lambda) = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} \lambda^2 & \lambda + \lambda \\ 0 & \lambda^2 \end{bmatrix} = \begin{bmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{bmatrix}$$

$$J_2(\lambda)^3 = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}\begin{bmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{bmatrix} = \begin{bmatrix} \lambda^3 & 3\lambda \\ 0 & \lambda^3 \end{bmatrix}$$

$$\vdots$$

$$\boxed{\left(J_2(\lambda)\right)^k = \begin{bmatrix} \lambda^k & k\lambda \\ 0 & \lambda^k \end{bmatrix}}$$

§ 5.2 #11

P4 $\sin\left(\frac{\pi A}{6}\right)$ and $\cos\left(\frac{\pi A}{6}\right)$ where $A = \begin{bmatrix} 2 & 4 \\ 0 & -3 \end{bmatrix}$

Notice, $A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $(A + 3I) = \begin{bmatrix} 5 & 4 \\ 0 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} -4 \\ 5 \end{bmatrix}$ has $\lambda_2 = -3$.

$\beta = \{(1,0), (-4,5)\}$ is e-basis for A, $[\beta] = \begin{bmatrix} 1 & -4 \\ 0 & 5 \end{bmatrix}$

gives $[\beta]^{-1} = \frac{1}{5}\begin{bmatrix} 5 & 4 \\ 0 & 1 \end{bmatrix}$ thus

$$A^n = [\beta]\begin{bmatrix} 2 & 0 \\ 0 & -3 \end{bmatrix}^n [\beta]^{-1} = \frac{1}{5}\begin{bmatrix} 1 & -4 \\ 0 & 5 \end{bmatrix}\begin{bmatrix} 2^n & 0 \\ 0 & (-3)^n \end{bmatrix}\begin{bmatrix} 5 & 4 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{5}\begin{bmatrix} 1 & -4 \\ 0 & 5 \end{bmatrix}\left[\begin{array}{c|c} 5 \cdot 2^n & 4 \cdot 2^n \\ 0 & (-3)^n \end{array}\right]$$

$$= \frac{1}{5}\left[\begin{array}{c|c} 5 \cdot 2^n & 4 \cdot 2^n - 4(-3)^n \\ 0 & 5 \cdot (-3)^n \end{array}\right] = \left[\begin{array}{c|c} 2^n & \frac{4}{5}\left(2^n - (-3)^n\right) \\ 0 & (-3)^n \end{array}\right]$$

P4 continued

$$\sin\left(\frac{\pi A}{6}\right) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!}\left(\frac{\pi}{6}\right)^{2j+1} A^{2j+1}$$

(-3)

$$= \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!}\left(\frac{\pi}{6}\right)^{2j+1} \left[\begin{array}{c|c} 2^{2j+1} & \frac{4}{5}\left(2^{2j+1} - (-3)^{2j+1}\right) \\ \hline 0 & (-3)^{2j+1} \end{array}\right]$$

$$= \left[\begin{array}{c|c} \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!}\left(2\cdot\left(\frac{\pi}{6}\right)\right)^{2j+1} & \text{similar} \\ \hline 0 & \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j+1)!}\left(-3\cdot\frac{\pi}{6}\right)^{2j+1} \end{array}\right]$$

$$= \left[\begin{array}{c|c} \sin\left(\pi/3\right) & \frac{4}{5}\left[\sin\left(2\cdot\frac{\pi}{6}\right) - \sin\left(-3\cdot\frac{\pi}{6}\right)\right] \\ \hline 0 & \sin\left(-3\cdot\frac{\pi}{6}\right) \end{array}\right]$$

$$= \boxed{\left[\begin{array}{c|c} \sqrt{3}/2 & \frac{4}{5}\left(\frac{\sqrt{3}}{2}+1\right) \\ \hline 0 & -1 \end{array}\right]}$$

(Shores' §5.2 #11)

Likewise, $\cos\left(\frac{\pi}{6}A\right) = \sum_{j=0}^{\infty} \frac{(-1)^j}{(2j)!}\left(\frac{\pi}{6}\right)^{2j} A^{2j}$

simplifies to $\boxed{\cos\left(\frac{\pi A}{6}\right) = \begin{bmatrix} 1/2 & 2/5 \\ 0 & 0 \end{bmatrix}}$

why? Since $\cos\left(\frac{\pi}{2}\right) = 0$, $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$ and the pattern for sine's matrix powers is same.

(P8) What are the possible Jordan forms for a 5×5 matrix with e-values $2, 2, 3, 3, 3$?

$$\begin{bmatrix} 2 & & & & \\ & 2 & & & \\ & & 3 & & \\ & & & 3 & \\ & & & & 3 \end{bmatrix}, \quad \begin{bmatrix} 2 & & & & \\ & 2 & & & \\ & & 3 & 1 & \\ & & & 3 & \\ & & & & 3 \end{bmatrix}, \quad \begin{bmatrix} 2 & & & & \\ & 2 & & & \\ & & 3 & 1 & \\ & & & 3 & 1 \\ & & & & 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & & & \\ & 2 & & & \\ & & 3 & & \\ & & & 3 & \\ & & & & 3 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & & & \\ & 2 & & & \\ & & 3 & 1 & \\ & & & 3 & \\ & & & & 3 \end{bmatrix}, \quad \begin{bmatrix} 2 & 1 & & & \\ & 2 & & & \\ & & 3 & 1 & \\ & & & 3 & 1 \\ & & & & 3 \end{bmatrix}$$

aha,

$J_1(2) \oplus J_1(2) \oplus J_1(3) \oplus J_1(3) \oplus J_1(3)$

$J_1(2) \oplus J_1(2) \oplus J_2(3) \oplus J_1(3)$

$J_1(2) \oplus J_1(2) \oplus J_3(3)$

$J_2(2) \oplus J_1(3) \oplus J_1(3) \oplus J_1(3)$

$J_2(2) \oplus J_2(3) \oplus J_1(3)$

$J_2(2) \oplus J_3(3)$.

p9

$$A = \begin{bmatrix} 3 & i \\ -i & 3 \end{bmatrix}$$

$$\det(A-\lambda I) = \det\begin{bmatrix} 3-\lambda & i \\ -i & 3-\lambda \end{bmatrix} = (\lambda-3)^2 + i^2$$
$$= \lambda^2 - 6\lambda + 9 - 1$$
$$= \lambda^2 - 6\lambda + 8$$
$$= (\lambda-4)(\lambda-2)$$

$A - 2I = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \Rightarrow$ $(u,v) \in Null(A-2I)$ has $u + iv = 0$

Hence $(u,v) = c(-iv, v)$

set $\underline{u_1 = \frac{1}{\sqrt{2}}(-i, 1)}$ (normalize so $\|u_1\| = 1$)

$A - 4I = \begin{bmatrix} -1 & i \\ -i & -1 \end{bmatrix} \Rightarrow$ $(u,v) \in Null(A-4I)$ has $u = iv$

$(u,v) = c(iv, v)$ thus,

set $\underline{u_2 = \frac{1}{\sqrt{2}}(i, 1)}$.

If $\beta = \{\frac{1}{\sqrt{2}}(-i,1), \frac{1}{\sqrt{2}}(i,1)\}$ then $\langle u_1, u_2 \rangle = \frac{i^2+1}{2} = 0$

and thus $[\beta]^{-1} = \overline{[\beta]}^T = \overline{\frac{1}{\sqrt{2}}\begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}}^T = \frac{1}{\sqrt{2}}\begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix}$

$[\beta]^{-1} A [\beta] = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}$ hence $A^n = [\beta]\begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}^n [\beta]^{-1}$

$$A^n = [\beta]\begin{bmatrix} 2^n & 0 \\ 0 & 4^n \end{bmatrix}\frac{1}{\sqrt{2}}\begin{bmatrix} i & 1 \\ -i & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -i & i \\ 1 & 1 \end{bmatrix}\begin{bmatrix} i2^n & 2^n \\ -i4^n & 4^n \end{bmatrix}$$

$$\Rightarrow \boxed{A^n = \frac{1}{2}\begin{bmatrix} 2^n + 4^n & i(4^n - 2^n) \\ i(2^n - 4^n) & 2^n + 4^n \end{bmatrix}}$$

$\boxed{P10}$ $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$.

(a.) $\det\begin{bmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{bmatrix} = (\lambda-1)^2 - 4 = (\lambda-3)(\lambda+1) = 0 \therefore \boxed{\begin{matrix}\lambda_1 = -1 \\ \lambda_2 = 3\end{matrix}}$

$\underline{\lambda_1 = -1}$ $(A+I)\vec{u}_1 = \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix}\vec{u}_1 = 0 \Rightarrow \boxed{\vec{u}_1 = (1,-1)}$

$\therefore \underbrace{E_1' = \text{span}\{(+1,-1)\}}$.

any non zero vector in here is $\lambda_1 = -1$ e-vector.

$\underline{\lambda_2 = 3}$ $(A-3I)\vec{u}_2 = \begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix}\vec{u}_2 \Rightarrow \boxed{\vec{u}_2 = c(1,1) \text{ for any } c \neq 0}$

(b.) $\dfrac{d\vec{r}}{dt} = A\vec{r}$ has sol$^{\underline{n}}$ $\boxed{\vec{r}(t) = c_1 e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

(c.) $Q(\bar{x},\bar{y}) = -\bar{x}^2 + 3\bar{y}^2$

on the unit-circle $\bar{x}^2 + \bar{y}^2 = 1$ we find $\boxed{Q_{max} = 3}$ at $\bar{x} = 0, \bar{y} = \pm 1$ and $\boxed{Q_{min} = -1}$ at $\bar{x} = \pm 1, \bar{y} = 0$

Remark: to derive this formula, notice $\beta = \{\frac{1}{\sqrt{2}}(1,-1), \frac{1}{\sqrt{2}}(1,1)\}$ is orthonormal e-basis for A. Let $\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = [\beta]^{-1}\begin{bmatrix} x \\ y \end{bmatrix}$ then

$$\begin{bmatrix} x \\ y \end{bmatrix} = [\beta]\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} \bar{x} \\ \bar{y} \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} \bar{x}+\bar{y} \\ -\bar{x}+\bar{y} \end{bmatrix}$$ hence

$x = \frac{1}{\sqrt{2}}(\bar{x}+\bar{y})$ and $y = \frac{1}{\sqrt{2}}(-\bar{x}+\bar{y})$ thus,

$$Q(x,y) = x^2 + 4xy + y^2 = \frac{1}{2}(\bar{x}+\bar{y})^2 + \frac{4}{2}(\bar{x}+\bar{y})(-\bar{x}+\bar{y}) + \frac{1}{2}(-\bar{x}+\bar{y})^2$$

$$= \frac{1}{2}(\bar{x}^2 + \cancel{2\bar{x}\bar{y}} + \bar{y}^2 - 4\bar{x}^2 + 4\bar{y}^2 + \bar{x}^2 - \cancel{2\bar{x}\bar{y}} + \bar{y}^2)$$

$$= \frac{1}{2}(-2\bar{x}^2 + 6\bar{y}^2)$$

$$= \underline{-\bar{x}^2 + 3\bar{y}^2}.$$

P11 $A = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix}$

(a.) $\det \begin{bmatrix} -\lambda & 3 \\ 3 & -\lambda \end{bmatrix} = \lambda^2 - 9 = (\lambda+3)(\lambda-3) = 0 \therefore \underline{\lambda_1 = -3,\ \lambda_2 = 3}$.

$\lambda_1 = -3$ | $(A + 3I)\vec{u}_1 = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix}\vec{u}_1 \Rightarrow \underline{\vec{u}_1 = c(1,-1) \text{ for } c \neq 0}$.

$\lambda_2 = 3$ | $(A - 3I)\vec{u}_2 = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix}\vec{u}_2 \Rightarrow \underline{\vec{u}_2 = c(1,1) \text{ for } c \neq 0}$.

(b.) $\dfrac{d\vec{r}}{dt} = A\vec{r}$ has sol$^{\underline{n}}$ $\boxed{\vec{r}(t) = c_1 e^{-3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}}$

(c.) $Q = -3\bar{x}^2 + 3\bar{y}^2$ since $\lambda_1 = -3$ & $\lambda_2 = 3$

moreover, $Q_{min} = -3$ and $Q_{max} = 3$ on the unit circle.

P12 $A = \begin{bmatrix} 5 & 4 & 2 \\ 4 & 5 & 2 \\ 2 & 2 & 2 \end{bmatrix}$

(a.) $\det(A - \lambda I) = -(\lambda^3 - 12\lambda^2 + 21\lambda - 10) = -(\lambda-1)^2(\lambda-10)$

thus $\lambda_1 = 1$ repeated and $\lambda_2 = 10$

$A - I = \begin{bmatrix} 4 & 4 & 2 \\ 4 & 4 & 2 \\ 2 & 2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 1/2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$x \in \text{Null}(A - I) \Rightarrow x = (-x_2 - \frac{x_3}{2}, x_2, x_3)$

$\Rightarrow \boxed{x = \underbrace{x_2(-1,1,0) + \tfrac{x_3}{2}(-1,0,2)}_{\text{typical } \lambda_1 = 1 \text{ e-vector assuming } x_2 x_3 \neq 0.}}$

$\text{Null}(A - 10I) = \text{span}\{(2,2,1)\}$

$\therefore$ $\boxed{\vec{u}_2 = c(2,2,1) \text{ typical e-vector for } \lambda_2 = 10}$

(b.) $\boxed{\vec{r}(t) = c_1 e^{t}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} + c_3 e^{10t}\begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}}$

(c.) $\boxed{Q = \bar{x}^2 + \bar{y}^2 + 10\bar{z}^2}$ and $\underbrace{\boxed{Q_{min} = 1} \text{ and } \boxed{Q_{max} = 10}}_{\text{Th}^{\underline{m}} \text{ says look at e-values to find min/max on } S^2.}$

$\boxed{P13}$

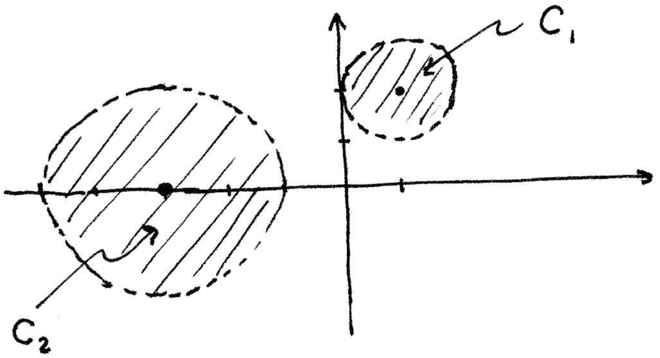
(a.)

$$A = \begin{bmatrix} 1+2i & 1 \\ 2i & -3 \end{bmatrix}$$

$$\rho_1(A) = |1+2i| + |1| = \sqrt{5}+1.$$
$$\rho_2(A) = |2i| + |-3| = 2+3 = 5.$$

$$C_1 = \{ z \in \mathbb{C} \mid |z-1-2i| < r_1 \} \qquad r_1 = \rho_1(A) - |A_{11}|$$
$$= \sqrt{5}+1 - \sqrt{5}$$
$$= \{ z \in \mathbb{C} \mid |z-1-2i| < 1 \}$$

$$C_2 = \{ z \in \mathbb{C} \mid |z+3| < r_2 \} \qquad r_2 = \rho_2(A) - |A_{22}|$$
$$= 3+2-3$$
$$= 2$$
$$= \{ z \in \mathbb{C} \mid |z+3| < 2 \}$$



(b.) A has no $\lambda = 0$ since $z=0$ is not found in any of the Gerschgorin disks.

[P14] Suppose A is a transition matrix, that is each column of A has entries which are nonnegative and sum to one. Also suppose $x = (x_1, \dots, x_n)$ is probability vector; $x_1 + \cdots + x_n = 1$,

(a.) $(Ax)_j = \sum_{i=1}^{n} A_{ji} x_i \leftarrow j^{\text{th}}$ component of Ax

$$\begin{aligned}
\sum_{j=1}^{n} (Ax)_j &= \sum_{j=1}^{n}\left(\sum_{i=1}^{n} A_{ji} x_i\right) \\
&= \sum_{i=1}^{n}\left(\sum_{j=1}^{n} A_{ji}\right) x_i \\
&= \sum_{i=1}^{n} x_i \\
&= 1.
\end{aligned}$$

$$\sum_{j=1}^{n} A_{ji} = \sum_{j=1}^{n} (\text{col}_i(A))_j = 1.$$

$\therefore$ Ax is probability vector.

(b.) Let A, B be probability matrices. Note, for $B = [B_1 | B_2 | \cdots | B_n]$ we have $B_1, B_2, \dots, B_n$ prob. vectors hence by part (a.)

$$\begin{aligned}
AB &= A[B_1 | B_2 | \cdots | B_n] \\
&= \underbrace{[AB_1 | AB_2 | \cdots | AB_n]}_{\text{each column a prob. vect. by (a.)}}
\end{aligned}$$

$\Rightarrow AB$ a prob. matrix.

P15 $\quad A = \begin{bmatrix} 0.9 & 0.02 \\ 0.1 & 0.98 \end{bmatrix}$

(a.) Eigenvalues of A are $\lambda_1 = 1$ with $V_1 = (+1, 5)$
$\lambda_2 = 0.88$ with $V_2 = (-1, 1)$

$\beta = \{(1,5), (-1,1)\}$ is e-basis for A and

$[\beta]^{-1} A [\beta] = \begin{bmatrix} 1 & 0 \\ 0 & 0.88 \end{bmatrix} = D$ (A-diagonalized).

(b.) $$\lim_{n\to\infty} (A^n) = \lim_{n\to\infty} \left([\beta] D^n [\beta]^{-1} \right) \quad \left([\beta] = \begin{bmatrix} 1 & -1 \\ 5 & 1 \end{bmatrix} \right)$$

$$= [\beta] \left(\lim_{n\to\infty} \begin{bmatrix} 1^n & 0 \\ 0 & (0.88)^n \end{bmatrix} \right) [\beta]^{-1}$$

$$= [\beta] \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} [\beta]^{-1}$$

$$= \begin{bmatrix} 1 & 0 \\ 5 & 0 \end{bmatrix} \frac{1}{6} \begin{bmatrix} 1 & 1 \\ -5 & 1 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} 1 & 1 \\ 5 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 1/6 & 1/6 \\ 5/6 & 5/6 \end{bmatrix}.$$

(c.) $X_0 = \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix}$ let $X_n = A^n X_0$

$$X_1 = \begin{bmatrix} 0.9 & 0.02 \\ 0.1 & 0.98 \end{bmatrix} \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 0.636 \\ 0.364 \end{bmatrix}.$$

$$X_{10} = \begin{bmatrix} 0.9 & 0.02 \\ 0.1 & 0.98 \end{bmatrix}^{10} \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 0.315 \\ 0.684 \end{bmatrix}.$$

$$X_{100} = \begin{bmatrix} 0.9 & 0.02 \\ 0.1 & 0.98 \end{bmatrix}^{100} \begin{bmatrix} 0.7 \\ 0.3 \end{bmatrix} = \begin{bmatrix} 0.1667 \\ 0.8333 \end{bmatrix}.$$

Apparently $\lim_{n\to\infty} X_n \doteq \begin{bmatrix} 0.1667 \\ 0.8333 \end{bmatrix} \cong \begin{bmatrix} 1/6 \\ 5/6 \end{bmatrix}$

The columns of $\lim_{n\to\infty} A^n$ are the steady state sol$^{\underline{n}}$ to the problem.

(I used the website linked on course page)

P16

$$A = \begin{bmatrix} 5 & -5 & -5 \\ -1 & 4 & 2 \\ 3 & -5 & -3 \end{bmatrix} \text{ has } \begin{array}{l} \lambda_1 = 2 \text{ with } \vec{u}_1 = (0, -1, 1) \\ \lambda_2 = 2+i \text{ with } \vec{u}_2 = (5, -2-i, 5) \\ \lambda_3 = 2-i \text{ with } \vec{u}_3 = (5, -2+i, 5) \end{array}$$

Thus $\gamma = \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -2-i \\ 5 \end{bmatrix}, \begin{bmatrix} 5 \\ -2+i \\ 5 \end{bmatrix} \right\}$ is a complex eigenbasis.

Hence construct $\beta = \left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 5 \\ -2 \\ 5 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ for which

$$\boxed{[\beta]^{-1} A [\beta] = \left[\begin{array}{c|cc} 2 & 0 & 0 \\ \hline 0 & 2 & 1 \\ 0 & -1 & 2 \end{array}\right]} \leftarrow \underline{\text{Real Jordan Form}}$$

In contrast, $[\gamma]^{-1} A [\gamma] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2+i & 0 \\ 0 & 0 & 2-i \end{bmatrix}$

A is diagonalized over $\mathbb{C}$ but not over $\mathbb{R}$.

P17 We find real eigensol^{n} to $\frac{dx}{dt} = Ax$ of $X_1 = e^{2t} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ and a complex eigensol^{n} of

$$\begin{aligned} Z = e^{(2+i)t} \begin{bmatrix} 5 \\ -2-i \\ 5 \end{bmatrix} &= e^{2t}(\cos t + i \sin t)\left(\begin{bmatrix} 5 \\ -2 \\ 5 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right) \\ &= e^{2t}\left(\cos t \begin{bmatrix} 5 \\ -2 \\ 5 \end{bmatrix} - \sin t \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right) + i e^{2t} \left(\sin t \begin{bmatrix} 5 \\ -2 \\ 5 \end{bmatrix} + \cos t \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right) \end{aligned}$$

Then $x = c_1 X_1 + c_2 \,\mathrm{Re}(Z) + c_3 \,\mathrm{Im}(Z)$ provides the general sol^{n} to $\frac{dx}{dt} = Ax$. Explicitly,

$$\boxed{x = c_1 e^{2t} \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} 5\cos t \\ -2\cos t + \sin t \\ 5\cos t \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} 5 \sin t \\ -2\sin t - \cos t \\ 5 \sin t \end{bmatrix}}$$

P18

$$A = J_4(3) = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} = \underbrace{\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}}_{D} + \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}}_{N}$$

Since $N^3 = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ and $N^4 = 0$ we have N is nilpotent of degree 4. Moreover D is diagonal hence diagonalizable. Moreoverover $DN = ND$.

$$\begin{aligned} e^{tA} &= e^{t(D+N)} \\ &= e^{tD} e^{tN} \\ &= e^{3t} I \left(I + tN + \tfrac{1}{2}t^2 N + \tfrac{1}{6} t^3 N^3 + \cdots \right) \\ &= \boxed{e^{3t} \begin{bmatrix} 1 & t & t^2/2 & t^3/6 \\ 0 & 1 & t & t^2/2 \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{bmatrix}} \end{aligned}$$

P19

$$B = (J_4(3))^2 = \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 9 & 6 & 1 & 0 \\ 0 & 9 & 6 & 1 \\ 0 & 0 & 9 & 6 \\ 0 & 0 & 0 & 9 \end{bmatrix}$$

Hence $\lambda_1 = 9$ with multiplicity (algebraic) of 4.

$$B - 9I = \begin{bmatrix} 0 & 6 & 1 & 0 \\ 0 & 0 & 6 & 1 \\ 0 & 0 & 0 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \vec{u}_1 = c(1,0,0,0)$$

only eigenvector

Thus B has Jordan form $J_4(9)$ as we explained in Lecture 4/27/17. In fact we found most of $\beta = \{\vec{u}_1, \vec{u}_2, \vec{u}_3, \vec{u}_4\}$ for which $[\beta]^{-1} B [\beta] = J_4(9)$.

P20

$$A = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & & \\ & 2 & \\ & & 2 \end{bmatrix}}_{D} + \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}}_{N \text{ with } N^3 = 0,\ N^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}$$

$$e^{tA} = e^{t(D+N)} = e^{tD} e^{tN}$$

$$= e^{2t} I \left(I + tN + \tfrac{1}{2} t^2 N^2 + \cdots \right)$$

$$= \underbrace{\boxed{e^{2t} \begin{bmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{bmatrix}}}$$

each column is sol$^{\underline{n}}$ to $\dfrac{dx}{dt} = Ax$ since $\dfrac{d}{dt}\left(e^{tA}\right) = Ae^{tA}$.

Thus, $\boxed{\vec{x} = c_1 e^{2t} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + c_2 e^{2t} \begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} t^2/2 \\ t \\ 1 \end{bmatrix}}$

P21 Let $D = P^{-1}AP$ where $D = \text{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$

observe $e^{P^{-1}AP} = I + P^{-1}AP + \frac{1}{2} P^{-1}AP\,P^{-1}AP + \cdots$

$$= P^{-1}\left(I + A + \tfrac{1}{2}A^2 + \cdots\right) P$$

$$= P^{-1} e^A P$$

Thus, $e^D = e^{P^{-1}AP} = P^{-1} e^A P$

$$\Rightarrow \det(e^D) = \det \begin{bmatrix} e^{\lambda_1} & & & \\ & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{bmatrix} = \det\left[P^{-1} e^A P\right) = \det(e^A)$$

$$\Rightarrow e^{\lambda_1} e^{\lambda_2} \cdots e^{\lambda_n} = e^{\lambda_1 + \lambda_2 + \cdots + \lambda_n} = \det(e^A)$$

But $\text{trace}(A) = \text{trace}(PDP^{-1}) = \text{trace}(DP^{-1}P) = \text{trace}(D) = \lambda_1 + \cdots + \lambda_n$

$$\therefore \boxed{e^{\text{trace}(A)} = \det(e^A)}$$

P22

$A = J_3(7) \oplus J^{\mathbb{R}}(2+3i)$

$$= \left[\begin{array}{ccc|cc} 7 & 1 & 0 & & \\ 0 & 7 & 1 & & 0 \\ 0 & 0 & 7 & & \\ \hline & 0 & & 2 & 3 \\ & & & -3 & 2 \end{array}\right]$$

Observe,

$(A - 7I)e_1 = 0$

$(A - 7I)e_2 = e_1 \longrightarrow (A - 7I)^2 e_2 = 0$

$(A - 7I)e_3 = e_2 \longrightarrow (A-7I)^2 e_3 = e_1 \longrightarrow (A - 7I)^3 e_3 = 0.$

Working backwards from our Th^{m} in class,

$A(e_4 + ie_5) = (2+3i)(e_4 + ie_5)$

The complex-e-vector gives $Z = e^{(2+3i)t}(e_4 + ie_5)$ complex sol^{n}.

Then

$$e^{tA} = e^{7t}\left(I + t(A-7I) + \frac{t^2}{2}(A-7I)^2 + \frac{t^3}{3!}(A-7I)^3 + \cdots\right)$$

Hence,

$e^{tA}e_1 = e^{7t}e_1$

$e^{tA}e_2 = e^{7t}(e_2 + te_1)$

$e^{tA}e_3 = e^{7t}(e_3 + te_2 + \frac{1}{2}t^2 e_1)$

In total, once again selecting $Re(Z)$, $Im(Z)$ as real sol^{n}'s to $\frac{dx}{dt} = Ax$ we find,

$$\boxed{\begin{aligned} X = {} & c_1 e^{7t}e_1 + c_2 e^{7t}(e_2 + te_1) + c_3 e^{7t}(e_3 + te_2 + \tfrac{1}{2}t^2 e_1) \\ & + c_4 e^{2t}\left((\cos 3t)e_4 - (\sin 3t)e_5\right) + c_5 e^{2t}\left((\sin 3t)e_4 + (\cos 3t)e_5\right) \end{aligned}}$$