Conservative Vector Fields Revisited Once More

We have seen the physical and mathematical significance of the potential function, path independence etc... Lets make a list of facts. The following are equivalent, assuming $\text{dom}(\vec{F})$ is simply connected.

(1) $\vec{F}$ is conservative

(2) $\exists f$ such that $\vec{F} = \nabla f$

(3) $\vec{F}$ is path independent, $\int_{C_1} \vec{F} \cdot d\vec{r} = \int_{C_2} \vec{F} \cdot d\vec{r}$ for all paths C_1 & C_2 with same initial & terminal points

(4) $\text{dom}(\vec{F})$ simply connected & $\nabla \times \vec{F} = 0$

(5) $\oint_C \vec{F} \cdot d\vec{r} = 0$ for all closed paths C.

If we drop the demand of $\text{dom}(\vec{F})$ being simply connected then we'll not be able to assume $\nabla \times \vec{F} = 0 \Rightarrow \vec{F} = \nabla f$. Lets see how Stoke's Th^{m} connects these statements. Assume $\text{dom}(\vec{F})$ is simply connected, consider closed path $C = \partial S$ where S is some surface that takes C as its consistently oriented boundary.

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \oint_C \vec{F} \cdot d\vec{r}$$

If (5) holds then $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = 0$ for all surfaces which implies $\nabla \times \vec{F} = 0$, thus $(5) \Rightarrow (4)$. The other implications we've argued earlier. Notice that we need $\text{dom}(\vec{F})$ to be simply connect in order that $\oint_C \vec{F} \cdot d\vec{r}$ doesn't get caught on any holes, we wouldn't have $\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = 0$ $\forall$ surfaces around a point in $\text{dom}(\vec{F})$, we'd have to worry about the holes in $\text{dom}(\vec{F})$ and ultimately that spoils the implication $\nabla \times \vec{F} = 0 \Rightarrow \vec{F}$ conservative.

($\vec{F}$ conservative, $\nabla f = \vec{F} \Rightarrow \nabla \times \vec{F} = 0$ is always true)