Calculus III and Differential Forms

I'll give a brief advertisement here. You can look in my ma 430 notes or Colley for more details. In short differential forms unify the concepts of calculus III in a slick elegant fashion. <u>Fundamental Correspodance</u>:

$$\vec{F} = \langle P, Q, R \rangle \longrightarrow \omega_{\vec{F}} = P\,dx + Q\,dy + R\,dz$$

$$\vec{F} = \langle P, Q, R \rangle \longrightarrow \Phi_{\vec{F}} = P\,dy\wedge dz + Q\,dz\wedge dx + R\,dx\wedge dy$$

we say $\omega_{\vec{F}}$ is the one-form corresponding to $\vec{F}$ while $\Phi_{\vec{F}}$ is the two-form corresponding to $\vec{F}$, $\underline{\omega_{\vec{A}}\wedge\omega_{\vec{B}} = \Phi_{\vec{A}\times\vec{B}}}$

the wedge product generalizes the cross product.

<u>Exterior Derivative</u>:

usual	corresponds	Output	Corresponds to
f	function f	df	$\omega_{\nabla f} = df$
$\vec{F}$	$\omega_{\vec{F}}$	$d\omega_{\vec{F}}$	$\Phi_{\nabla\times\vec{F}} = d\omega_{\vec{F}}$
$\vec{G}$	$\Phi_{\vec{G}}$	$d\Phi_{\vec{G}}$	$d\Phi_{\vec{G}} = (\nabla\cdot\vec{G})dx\wedge dy\wedge dz$

the single operation of exterior differentiation reproduces the gradient, curl and divergence!

Integration of Forms:

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \omega_{\vec{F}}$$

$$\iint_S \vec{F}\cdot d\vec{S} = \int_S \Phi_{\vec{F}}$$

$$\iiint_E f\,dV = \int_E f\,dx\wedge dy\wedge dz$$

- Notice a p-form can only be integrated over a p-dim'l space. In contrast we integrate vector fields along a line or accross a surface.

GENERALIZED STOKE'S THEOREM

$$\boxed{\int_M d\beta = \int_{\partial M} \beta}$$

this encodes the FTC, GREEN's, STOKES and DIVERGENCE Th$\stackrel{=}{}$ in one unified frame work.

$$\omega_{\nabla f} = df$$

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \omega_{\vec{F}}$$

$$\int_C df = \int_{\partial C = \{a,b\}} f$$

$$\boxed{\int_C \nabla f \cdot dr = f(b) - f(a)}$$

FTC for line integrals

$$d\omega_{\vec{F}} = \Phi_{\nabla\times\vec{F}}$$

$$\iint_S \vec{F}\cdot d\vec{S} = \int_S \Phi_{\vec{F}}$$

$$\int_S d\omega_{\vec{F}} = \int_{\partial S} \omega_{\vec{F}}$$

$$\int_S \Phi_{\nabla\times\vec{F}} = \int_S d\omega_{\vec{F}} = \int_{\partial S} \omega_{\vec{F}}$$

$$\boxed{\iint_S (\nabla\times\vec{F})\cdot d\vec{S} = \int_{\partial S} \vec{F}\cdot d\vec{r}}$$

Stoke's Th^{m}

$$d\Phi_{\vec{F}} = (\nabla\cdot\vec{F})\,dx\wedge dy\wedge dz$$

$$\iiint_E f\,dV = \int_E f\,dx\wedge dy\wedge dz$$

$$\int_E d\Phi_{\vec{F}} = \int_{\partial E} \Phi_{\vec{F}}$$

$$\boxed{\iiint_E (\nabla\cdot\vec{F})\,dV = \iint_{\partial E} \vec{F}\cdot d\vec{S}}$$

Divergence or Gauss' Th^{m}.

the true beauty of the theory of differential forms is that they unify the 3-d calculus and better yet are defined in higher dimensions. Differential forms replace vector fields as the object of primary physical interest in modern physical theory.