Conservation of Energy, Work-Energy Theorem

In §17.3 Th$^{\underline{m}}$(5) and Th$^{\underline{m}}$(6) describe conditions that gaurantee $\vec{F}$ is conservative. We discussed it in the 3-d case on 370 - 371 where we said $\vec{F}$ conservative $\Longleftrightarrow \nabla \times \vec{F} = 0$ provided dom($\vec{F}$) is simply connected. We can shed a little more light on this discussion with the help of Stokes' Th$^{\underline{m}}$, but that must wait a bit. For now we consider the connection between conservative forces and the conservation of energy.

- Let $\vec{r}(t)$ for $a \leq t \leq b$ describe the position of mass m as it travels along some path C. Further suppose that a conservative force $\vec{F}$ does work on m. We <u>define</u> the <u>potential energy function</u> to be U such that

$$\vec{F} = -\nabla U \quad \left(\begin{array}{l}\text{in contrast to } \vec{F} = \nabla f \text{ where} \\ \text{we don't care about the sign}\end{array}\right)$$

Lets see why the <u>total energy</u> $E = K + U$ is conserved and also why physicists put in the minus sign for $\vec{F} = -\nabla U$.

$$W \equiv \int_C \vec{F} \cdot d\vec{r} = -\int_C (\nabla U) \cdot d\vec{r} = -\left[U(\vec{r}(b)) - U(\vec{r}(a))\right]$$

$$= \int_C m\vec{a} \cdot d\vec{r} \quad : \text{ Newton's Law holds on } C.$$

$$= \int_a^b m \frac{d^2\vec{r}}{dt^2} \cdot \frac{d\vec{r}}{dt}\, dt$$

$$\frac{d}{dt}\left[\vec{r}' \cdot \vec{r}'\right] = \vec{r}'' \cdot \vec{r}' + \vec{r}' \cdot \vec{r}''$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}\left[\vec{r}' \cdot \vec{r}'\right] = \vec{r}'' \cdot \vec{r}$$

$$\Rightarrow \frac{d}{dt}\left[\frac{m}{2}\frac{d\vec{r}}{dt} \cdot \frac{d\vec{r}}{dt}\right] = m\frac{d^2\vec{r}}{dt^2} \cdot \frac{d\vec{r}}{dt}.$$

$$= \int_a^b \frac{d}{dt}\left[\frac{1}{2}m \frac{d\vec{r}}{dt} \cdot \frac{d\vec{r}}{dt}\right] dt$$

$$= \frac{1}{2}m|\vec{v}(b)|^2 - \frac{1}{2}m|\vec{v}(a)|^2 \quad : \text{ identify this is the difference in Kinetic Energy from time } b \text{ to time } a.$$

$$= K(b) - K(a)$$

We find, $K(b) - K(a) = -U(b) + U(a)$

$$\therefore \; U(a) + K(a) = U(b) + K(b)$$

$$\therefore \; \underline{E(a) = E(b)}$$

The total energy is conserved when the net force on some mass m is conservative.

WORK ENERGY THEOREM

Notice that our previous result had two lines of logic. One part held because $\vec{F} = -\nabla U$, of course if $\vec{F}$ is not conservative we cannot hope the total energy for m is conserved. However, the other line of logic depended only on Newton's Law $\vec{F} = m\vec{a}$ (assuming $m = \text{constant}$). The Work Energy Th^{m} States that

$$\boxed{W = K(b) - K(a)}$$

this holds even for frictional forces etc....

$\boxed{E150}$ The Electric field $\vec{E}$ has $\vec{E} = -\nabla V$ in Electrostatics. The force on a charge q due to $\vec{E}$ is $\vec{F} = q\vec{E}$ then the potential energy $U = qV$ and

$$\vec{F} = -\nabla U = q(-\nabla V) = q\vec{E}$$

the electric potential V is the potential energy per unit charge. In the case $\vec{E} = (kQ/r^2)\hat{r}$ one has $V = -kQ/r$ if we assume that "$\mathcal{O} = \infty$". then

$$U(r) = \frac{-kqQ}{r}$$

And the conservation of energy for the charge q with mass m is

$$E_i = \frac{1}{2}mV_i^2 - \frac{kqQ}{r_i} = \frac{1}{2}mV_f^2 - \frac{kqQ}{r_f} = E_f$$

Other Electric Fields give different potential energy functions But always we can calculate the potential from

$$V(\vec{r}) = -\int_{\mathcal{O}}^{\vec{r}} \vec{E}\cdot d\vec{l}$$

and if you don't pick a zero for the potential (called $\mathcal{O}$) you can still calculate differences in the potential.