THE DIVERGENCE OF A VECTOR FIELD

Again we use the vector of operators ∇ to define,

Def^{n}/ Let $\vec{F}$ be a vector field with differentiable component fncts,

$$\text{div}(\vec{F}) \equiv \nabla \cdot \vec{F} \equiv \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \partial_1 F_1 + \partial_2 F_2 + \partial_3 F_3$$

where $\vec{F} = \langle P, Q, R \rangle = \langle F_1, F_2, F_3 \rangle$.

E131 Let $\vec{F} = \langle x, y, z \rangle$ then

$$\nabla \cdot \vec{F} = \frac{\partial x}{\partial x} + \frac{\partial y}{\partial y} + \frac{\partial z}{\partial z} = 3.$$

E132 Let $\vec{G} = \langle -y, x, 0 \rangle$ then

$$\nabla \cdot \vec{G} = \frac{\partial}{\partial x}(-y) + \frac{\partial}{\partial y}(x) = 0.$$

<u>Langauge</u>: the vector fields considered in E127, E128, E131, E132 are examples of <u>irrotational</u> and <u>incompressible</u> vector fields. Speficalby,

$\vec{F} = \langle x, y, z \rangle$ has $\nabla \cdot \vec{F} = 3$ & $\underline{\nabla \times \vec{F} = 0}$, <u>$\vec{F}$ is irrotational</u>

$\vec{G} = \langle -y, x, 0 \rangle$ has $\nabla \times \vec{G} = 2\hat{k}$ & $\underline{\nabla \cdot \vec{G} = 0}$, <u>$\vec{G}$ is incompressible</u>

<u>Remark</u>: the terminology "irrotational" and "incompressible" stem from early applications of vector fields to fluid flow. If $\vec{V}$ is the velocity field of a fluid then $\nabla \times \vec{V} \neq 0 \Rightarrow$ a little paddle wheel will spin at the location where $\nabla \times \vec{V} \neq 0$. If $\nabla \cdot \vec{V} \neq 0$ that indicates the fluid is flowing in or out of the infinitesimal volume where $\nabla \cdot \vec{V}$ is non zero. Electromagnetism shares many analogies to fluid flow, for example,

$$\nabla \cdot \vec{J} = -\frac{\partial \rho}{\partial t}$$

flow of current in/out of dV — change in charge density in dV

this eq^{n} expresses the <u>conservation of charge</u> locally.

Th^{m}/ If $\vec{F} = \langle P, Q, R \rangle$ is a vector field such that P, Q, R have continuous 2^{nd} order partial derivatives then

$$\text{div}(\text{curl}(\vec{F})) = \nabla \cdot (\nabla \times \vec{F}) = 0$$

<u>Proof:</u> follows from Clauriaut's Th^{m} again, use $\vec{F} = \langle F_1, F_2, F_3 \rangle$

$$\begin{aligned} \nabla \cdot (\nabla \times \vec{F}) &= \nabla \cdot \langle \partial_2 F_3 - \partial_3 F_2,\ \partial_3 F_1 - \partial_1 F_3,\ \partial_1 F_2 - \partial_2 F_1 \rangle \\ &= \underset{(1)}{\cancel{\partial_1\partial_2 F_3}} - \underset{(2)}{\cancel{\partial_1\partial_3 F_2}} + \underset{(3)}{\cancel{\partial_2\partial_3 F_1}} - \underset{(1)}{\cancel{\partial_2\partial_1 F_3}} + \underset{(2)}{\cancel{\partial_3\partial_1 F_2}} - \underset{(3)}{\cancel{\partial_3\partial_2 F_1}} \\ &= 0 // . \end{aligned}$$

$\boxed{E133}$ Is it possible that $\vec{F} = \langle xz, xyz, -y^2 \rangle = \nabla \times \vec{G}$ for some $\vec{G}$? Notice if this were true then $\nabla \cdot \vec{F} = \nabla \cdot (\nabla \times \vec{G}) = 0$ however $\nabla \cdot \vec{F} \neq 0$ as we may easily calculate,

$$\nabla \cdot \vec{F} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(xyz) + \frac{\partial}{\partial z}(-y^2) = z + xz \neq 0.$$

$\therefore$ $\boxed{\text{No such } \vec{G} \text{ exists}}$

$\boxed{E134}$ One of Maxwells Eq^{n}'s can be written as $\nabla \cdot \vec{B} = 0$, find another vector field which automatically solves this eq^{n}. The so-called <u>vector potential</u> $\vec{A}$ does the job, it is req^{d} that

$$\vec{B} = \nabla \times \vec{A} \qquad \text{thus} \qquad \nabla \cdot \vec{B} = \nabla \cdot (\nabla \times \vec{A}) = 0.$$

Notice that $\vec{A}$ is not unique since $\nabla \times \nabla g = 0$ we'll find the same magnetic field $\vec{B}$ from $\vec{A}$ or $\vec{A} + \nabla g$. This is <u>gauge symmetry</u>.

<u>Laplace Operator</u>

$$\nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

this can be applied to real-valued functions or vectors.

$\boxed{E135}$ Gauss Law states $\nabla \cdot \vec{E} = \rho/\epsilon_0$ and in electrostatics the electric field is given by the potential V according to $\vec{E} = -\nabla V$ thus Gauss Law becomes $\nabla \cdot (-\nabla V) = \rho/\epsilon_0$ this gives $\nabla^2 V = -\rho/\epsilon_0$.

$\boxed{E136}$ In §13.5 #36 (or 17.5 #38 in Calculus Ed. 6) we encounter the Laplace Operator acting on vector fields

$$\nabla^2 \vec{E} = \frac{1}{c^2}\frac{\partial^2 \vec{E}}{\partial t^2} \quad \& \quad \nabla^2 \vec{B} = \frac{1}{c^2}\frac{\partial^2 \vec{B}}{\partial t^2}$$

the sol^{n}'s to these eq^{n}'s are electromagnetic waves, that is <u>light</u>.