LIMITS & CONTINUITY FOR FUNCTIONS OF MANY VARIABLES

To do it correctly we should give ε-δ formulation to make the limiting process precise. Take Advanced Calculus if you're interested. We will continue in the heuristic tradition of calculus I & II, all the elementary functions as well as their sums/differences/quotients/composites/roots are continuous where they are defined. No division by zero or $\sqrt{\text{negative}}$ # allowed. For $a \in \text{dom}(f)$

$$\boxed{\lim_{x \to a} f(x) = f(a) \iff f \text{ continuous at } a}$$

CONTINUITY & LIMITS

If $f: \mathbb{R}^2 \longrightarrow \mathbb{R}$ then f is continuous at (a,b) iff

$$\boxed{\lim_{(x,y)\to(a,b)} (f(x,y)) = f(a,b). \iff f \text{ continuous at } (a,b). \iff f \text{ is class } C^0 \text{ at } (a,b)}$$

Now this begs an obvious question, what do I mean by $(x,y) \longrightarrow (a,b)$? In the case $x \longrightarrow a$ recall we req$^{\underline{d}}$ $x \longrightarrow a^+$ <u>and</u> $x \longrightarrow a^-$. For two or more variables we need that $(x,y) \longrightarrow (a,b)$ from <u>all directions and paths</u>. If the value of $f(x,y)$ is approaching the same value for every direction then we call that limiting value $\lim_{(x,y)\to(a,b)} f(x,y)$. Just as $a \notin \text{dom}(f)$ necessarily we may also have $(a,b) \notin \text{dom}(f)$. In fact its only really interesting in those cases, otherwise why bother with the limit?

$\boxed{\text{E46}}$ Let $f(x,y) = x^2 + \sqrt{y} + \tan^{-1}(x) + 3$, find limit at $(0,1)$

$$\lim_{(x,y)\to(0,1)} \left(x^2 + \sqrt{y} + \tan^{-1}(x) + 3\right) = \underbrace{0 + \sqrt{1} + \tan^{-1}(0) + 3}_{\text{all the functions involved are well behaved at } x=0,\ y=1.} = \boxed{4}$$

TWO PATH TEST FOR NONEXISTENCE OF A LIMIT:
If a function $f(x,y)$ has limit L_1 along one path $(x,y) \xrightarrow{P_1} (a,b)$ and $L_2 \neq L_1$ along another path P_2 also approaching (a,b) then

$$\lim_{(x,y)\to(a,b)} f(x,y) \quad \text{does not exist}$$

However if $L_1 = L_2$ then we can't conclude anything imediately, remember we have to approach the same value for all paths.

$\boxed{E47}$ Consider $f(x,y) = \begin{cases} 2xy/(x^2+y^2) & (x,y) \neq 0 \\ 0 & (x,y) = 0 \end{cases}$
this function is continuous everywhere except the origin, lets see why.
Approach $(0,0)$ along the line $Y = mX$. As $(x,y) = (x,mx) \to (0,0)$,

$$\frac{2xy}{x^2+y^2} \longrightarrow \frac{2mx^2}{x^2+m^2x^2} \longrightarrow \frac{2m}{1+m^2}$$

Clearly for different lines we obtain different limits thus the limit d.n.e.

- Usually if you try $(x,y) = (x,0)$ or $(x,y) = (0,y)$ or $(x,y) = (x,x)$ then compare it will expose the limit's non-existance. There are more subtle cases,

$\boxed{E48}$ Consider $f(x,y) = \dfrac{2x^2y}{x^4+y^2}$. Show $f \to 0$ as $(x,y) \to 0$.
First we try approaching via the line $Y = mx$, $(m \neq 0)$

$$\frac{2x^2y}{x^4+y^2} \longrightarrow \frac{2mx^3}{x^4+m^2x^2} \longrightarrow \frac{2mx}{x^2+m^2} \longrightarrow \frac{2mx}{m^2} \to 0$$

On all nonhorizontal lines we find limit zero. This function is particularly sneaky. We consider a parabolic path $y = kx^2$

$$\frac{2x^2y}{x^4+y^2} \longrightarrow \frac{2kx^4}{x^4+k^2x^4} = \frac{2kx^4}{x^4(1+k^2)} \longrightarrow \frac{2k}{1+k^2} \neq 0 \quad \text{for } k \neq 0.$$

Thus the limit does not exist by the two path test.

Remark: I find it remarkable that we can prove many two-dimensional limits do exist. Its hard to imagine infinitely different paths. Obviously there is something subtle here, to really do it right study another text. We will content ourselves with this brief intro. and now move on to more practicle topics. You can see Stewart for more examples. I've borrowed these from the excellent calculus text by Thomas (10^{th} ed.) (§ 11.2)