Copying answers and steps is strictly forbidden. Evidence of copying results in zero for copied and copier. Working together is encouraged, share ideas not calculations. Explain your steps. This sheet must be printed and attached to your assignment as a cover sheet. The calculations and answers should be written neatly on one-side of paper which is attached and neatly stapled in the upper left corner. Box your answers where appropriate. Please do not fold. Thanks!

Problem 81 Your signature below indicates you have:

(a.) I have read §4.8 and Chapter 5 of Cook: __________________.

(b.) I have attempted homeworks from Salas and Hille as listed below:________________.

The following homeworks from the text are good rudimentary skill problems. These are not collected or graded. However, they are all usually odd problems thus there are answers given within Salas, Hille and Etgen's text:

§ 15.5 #'s 5, 9, 15, 19, 23, 25, 29, 31, 33, 37, 41, 47, 49

§ 15.6 #'s 1, 3, 9, 13, 19, 25, 36

Problem 82 Let $f(x,y) = \frac{x+y}{\sqrt{x^2+y^2}}$. *Find ∇f in terms of polar coordinates (use the polar frame $\hat{r}, \hat{\theta}$ as* discussed in my notes).

Problem 83 Let $f(x,y,z) = \frac{x+y+z}{x^2+y^2+z^2}$. Find ∇f in terms of spherical coordinates (your answer should use the spherical frame $\rho, \widehat{\phi}, \widehat{\theta}$ as discussed in my notes).

Problem 84 Suppose $\vec{F} = r^2\widehat{r} + \frac{1}{r}\theta\widehat{\theta} + ze^{z^2}\widehat{z}$. Find f such that $\nabla f = \vec{F}$.

Problem 85 Find all critical points of $f(x,y) = 2x - x^2 - y^2$

Problem 86 Find all critical points[1] of $f(x,y) = 3x^2 + xy - y^2 + 5x - 5y + 4$.

Problem 87 Find all critical points of $f(x,y) = xy^{-1} - yx^{-1}$. The answer here is involves an infinite set of points so it would be proper to use set-notation to express the answer.

Problem 88 Find all critical points of $f(x,y,z) = \cos(x^2 + y^2 + z^2)$. The answer here is involves an infinite set of points so it would be proper to use set-notation to express the answer.

Problem 89 Consider $f(x,y) = x^2 + kxy + 4y^2$ where k is a constant. For what value(s) of k is the second derivative test inconclusive?

Problem 90 Consider $f(x,y) = x^2 - y^2$. Find the absolute extrema for f on the rectangle $[-1,1] \times [0,2]$.

Problem 91 Let a, b be constants. Find the absolute maximum value of $f(x,y) = \frac{(ax+by+x)^2}{x^2+y^2+1}$.

Problem 92 Use the method of Lagrange multipliers to find the point on the plane $x + 2y - 3z = 10$ which is closest to the point $(10, 9, 8)$.

[1] *Salas and Hille call these "stationary points" when ∇f exists*

Problem 93 Apply the method of Lagrange multipliers to solve the following problem: Find the distance from $(6,0)$ to the parabola $x^2 = 4y$.

Problem 94 Find the maximum and minimum values for $f(x,y) = x^2 - y^2 - 1$ on the region bounded by the triangle with vertices $(1,1), (3,5)$ and $(6,0)$.

Problem 95 Find the multivariate power series expansion for $f(x,y) = (x+y)e^x \sin(y)$ centered at $(0,0)$ to second order.

Problem 96 Given that $f(x,y) = 3 + x^2 - y^2 + 13x^3 + 42y^3 + \cdots$ decide if $(0,0)$ is a critical point of f and if $(0,0)$ is critical then classify the nature of the critical point as maximum, minimum or saddle.

Problem 97 Given that $f(x,y) = 3 + x + (x-1)^2 + (y-3)^2 + \cdots$ decide if $(1,3)$ is a critical point of f and if $(1,3)$ is critical then classify the nature of the critical point as maximum, minimum or saddle.

Problem 98 Find the multivariate power series expansion of $f(x,y) = \sin(x+y)$ to second order about the point (π, π).

Problem 99 Find the multivariate power series explansion of $f(x,y,z) = \frac{1}{1+x+y+z}$ centered about $(0,0,0)$ to second order.

Problem 100 *Suppose objective function $f(x,y)$ has an extremum on $g(x,y) = 0$. Show that F de-*fined by $F(x,y,\lambda) = f(x,y) - \lambda g(x,y)$ recovers the extremum as a critical point. From this viewpoint, the adjoining of the multiplier converts the constrained problem in n-dimensions to an unconstrained problem in $(n+1)$-dimensions (you can easily generalize your argument to $n > 2$).

Bonus: An armored government agent decides to investigate a disproportionate use of electricity in a gated estate. Foolishly entering without a warrant he find himself at the mercy of Ron Swanson (at $(1,0,0)$), Dwight Schrute (at $(-1,1,0)$) and Kakashi (in a tree at $(1,1,3)$). Supposing Ron Swanson inflicts damage at a rate of 5 units inversely proportional from the square of his distance to the agent, and Dwight inflicts constant damage at a rate of 3 in a sphere of radius 2. If Kakashi inflicts a damage at a rate of 5 units directly proportional to the square of his distance from his location (because if you flee it only gets worse the further you run as he attacks you retreating) then where should you assume a defensive position as you call for back-up? What location minimizes your damage rate? Assume the ground is level and you have no jet-pack and/or antigravity devices.

PROBLEM 82

$$f(x,y) = \frac{1}{\sqrt{x^2+y^2}}(x+y) \longrightarrow f = \frac{1}{r}(r\cos\theta + r\sin\theta) = \cos\theta + \sin\theta$$

Thus, $\frac{\partial f}{\partial r} = 0$ and $\frac{\partial f}{\partial \theta} = \cos\theta - \sin\theta$ $\therefore$ $\boxed{\nabla f = \frac{1}{r}(\cos\theta - \sin\theta)\hat{\theta}}$

Remark: I used our general result $\nabla f = \hat{r}\frac{\partial f}{\partial r} + \frac{\hat{\theta}}{r}\frac{\partial f}{\partial \theta}$.

PROBLEM 83

$$f(x,y,z) = \frac{x+y+z}{x^2+y^2+z^2} = \frac{\cancel{\rho}\cos\theta\sin\phi + \cancel{\rho}\sin\theta\sin\phi + \cancel{\rho}\cos\phi}{\rho^{\cancel{2}}}$$

$$\frac{\partial f}{\partial \rho} = \frac{-1}{\rho^2}(\cos\theta\sin\phi + \sin\theta\sin\phi + \cos\phi)$$

$$\frac{\partial f}{\partial \phi} = \frac{1}{\rho}(\cos\theta\cos\phi + \sin\theta\cos\phi - \sin\phi)$$

$$\frac{\partial f}{\partial \theta} = \frac{1}{\rho}(-\sin\theta\sin\phi + \cos\theta\sin\phi)$$

$$\nabla f = \hat{\rho}\frac{\partial f}{\partial \rho} + \hat{\phi}\frac{1}{\rho}\frac{\partial f}{\partial \phi} + \hat{\theta}\frac{1}{\rho\sin\phi}\frac{\partial f}{\partial \theta}$$

$$\Rightarrow \boxed{\nabla f = \frac{-\hat{\rho}}{\rho^2}(\cos\theta\sin\phi + \sin\theta\sin\phi + \cos\phi) + \frac{\hat{\phi}}{\rho^2}(\cos\theta\cos\phi + \sin\theta\cos\phi - \sin\phi) + \frac{\hat{\theta}}{\rho^2\sin\phi}(-\sin\theta\sin\phi + \cos\theta\sin\phi)}$$

PROBLEM 84

$$\vec{F} = r^2\hat{r} + \frac{1}{r}\theta\hat{\theta} + ze^{z^2}\hat{z} = \nabla f = \hat{r}\frac{\partial f}{\partial r} + \frac{1}{r}\hat{\theta}\frac{\partial f}{\partial \theta} + \hat{z}\frac{\partial f}{\partial z}$$

Integrating w.r.t r, θ, z the $\hat{r}, \hat{\theta}, \hat{z}$ eq^{n}'s yield,

$$\boxed{f(r,\theta,z) = \frac{1}{3}r^3 + \frac{1}{2}\theta^2 + \frac{1}{2}e^{z^2}}$$

(you can add any constant you like, I took $c = 0$)

PROBLEM 85

$$f(x,y) = 2x - x^2 - y^2$$

$$\nabla f = \langle 2-2x, -2y\rangle$$

critical pt has $\nabla f = \langle 0,0\rangle = \langle 2-2x, -2y\rangle$

$\therefore\ 2-2x = 0 \ \ \& \ -2y = 0 \implies x=1, y=0 \ \therefore \boxed{(1,0)}$ only critical pt. of f.

PROBLEM 86

$$f(x,y) = 3x^2 + xy - y^2 + 5x - 5y + 4$$

$\nabla f = \langle 6x+y+5, x-2y-5\rangle = \langle 0,0\rangle$ for critical pt.

$$* \begin{cases} 6x + y = -5 \\ x - 2y = 5 \end{cases} \longrightarrow \begin{bmatrix} 6 & 1 \\ 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -5 \\ 5 \end{bmatrix}$$

multiplied by $\begin{bmatrix} 6 & 1 \\ 1 & -2 \end{bmatrix}^{-1} \longrightarrow \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{-12-1}\begin{bmatrix} -2 & -1 \\ -1 & 6 \end{bmatrix}\begin{bmatrix} -5 \\ 5 \end{bmatrix} = \frac{-1}{13}\begin{bmatrix} 5 \\ 35 \end{bmatrix}$

thus $\boxed{\left(-5/13, \ -35/13\right) \text{ is the only critical pt.}}$

<u>Remark</u>: there are dozens of methods to do the algebra to solve (*) you just need to master one for <u>this</u> course.

PROBLEM 87

$$f(x,y) = xy^{-1} + yx^{-1}$$

$$\frac{\partial f}{\partial x} = y^{-1} - \frac{y}{x^2}$$

$$\frac{\partial f}{\partial y} = \frac{-x}{y^2} + x^{-1}$$

$$\nabla f = 0 \implies \underbrace{\frac{1}{y} - \frac{y}{x^2} = 0}_{\underline{x^2 = y^2}} \quad \text{and} \quad \frac{-x}{y^2} + \frac{1}{x} = 0$$

$$\frac{x}{y^2} = \frac{1}{x} \hookrightarrow \underline{x^2 = y^2}.$$

Thus $\{(x,y) \mid y = \pm x\}$ is the set of critical pts. for f.

<u>Remark</u>: I unfortunately wrote $f(x,y) = xy^{-1} - yx^{-1}$ and that results in $\nabla f = \langle 0,0\rangle \iff x^2+y^2 = 0 \iff \underline{(0,0)}$.

PROBLEM 88

$f(x,y,z) = \cos(x^2+y^2+z^2)$ find all critical pts.

$\nabla f = -\sin(x^2+y^2+z^2)\,\nabla(x^2+y^2+z^2)$: chain-rule

$\nabla f = -\sin(x^2+y^2+z^2)\langle 2x, 2y, 2z\rangle = \langle 0,0,0\rangle$

Either $\langle 2x, 2y, 2z\rangle = 0$ or $\underbrace{\sin(x^2+y^2+z^2) = 0}$

Sol^{n}'s are $x^2+y^2+z^2 = n\pi$ for $n \in \mathbb{Z}$. However only $n = 0,1,2,3,\dots$ are nontrivial.

$$\boxed{\{(x,y,z) \mid x^2+y^2+z^2 = n\pi, \text{ for } n \in \mathbb{N}\cup\{0\}\}}$$

PROBLEM 89

$f(x,y) = x^2 + kxy + 4y^2$ where k is constant.
For what value(s) of k is the 2nd derivative test inconclusive.

$\nabla f = \langle 2x + ky, 8y + kx\rangle$

Thus $\underbrace{2x + ky = 0}$ and $\underbrace{8y + kx = 0}$ for critical pt.

$ky = -2x \qquad kx = -8y$

$kxy = -2x^2 \qquad kxy = -8y^2$

$k = \frac{-2x}{y} \qquad k = \frac{-8y}{x} \longrightarrow \frac{2x}{y} = \frac{8y}{x}$

$\longrightarrow 2x^2 = 8y^2$

$\longrightarrow x^2 = 4y^2$

We have critical pts along $x = \pm 2y$ or $y = \pm \frac{1}{2}x$ if you prefer.

Notice, it doesn't matter if you didn't find the critical pts as f_{xx}, f_{yy}, f_{xy} are all independent of the location of the critical pt.

$f_{xx} = 2$

$f_{yy} = 8$

$f_{xy} = k$

$$\underbrace{D = f_{xx}f_{yy} - f_{xy}^2}_{\text{technically we only consider this at critical pts., but, that's easy to do}} = 16 - k^2 = 0 \Rightarrow \text{inconclusive}$$

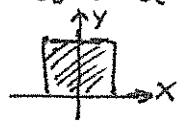
$\therefore \boxed{k = \pm 4}$

PROBLEM 89

Remark: it is usually important to realize the 2nd derivative test applies at critical points. It is a happy accident that you could do this problem while ignoring the actual details of the critical pts.

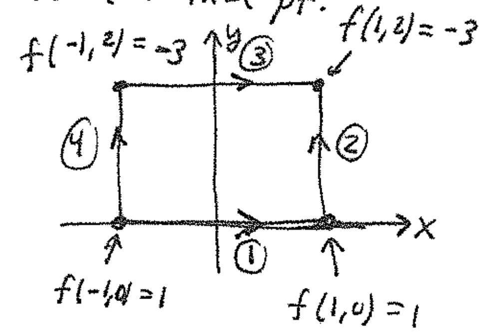
PROBLEM 90

Consider $f(x,y) = x^2 - y^2$. Find absolute extrema on $[-1,1] \times [0,2] = R$



$\nabla f = \langle 2x, -2y \rangle = \langle 0,0 \rangle \Rightarrow (0,0)$ local critical pt.

we'll cover $(0,0)$ on our study of ∂R.



① $-1 \leq x \leq 1, \quad y = 0$

$g_1(x) = x^2 - 0^2 = x^2$

$\dfrac{dg_1}{dx} = 0 = 2x \Rightarrow x = 0$ critical pt.

apply closed interval test on $[-1,1]$ for $g_1(x)$. We consider $g_1(-1) = 1, \quad g_1(0) = 0, \quad g_1(1) = 1$

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② $x = 1, \quad 0 \leq y \leq 2$

$g_2(y) = 1 - y^2$

$g_2'(y) = -2y$

$g_2(0) = 1 = f(1,0)$

$g_2(2) = -3 = f(1,2)$

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④ $x = -1, \quad 0 \leq y \leq 2$

$g_4(y) = 1 - y^2$

$g_4'(y) = -2y$

$g_4(0) = 1 = f(-1,0)$

$g_4(2) = -3 = f(-1,2)$

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③ $-1 \leq x \leq 1, \quad y = 2$

$g_3(x) = x^2 - 4$

$g_3'(x) = 2x \Rightarrow x = 0$ critical

$g_3(-1) = -3 = f(-1,2)$

$g_3(0) = -4 = f(0,2)$

$g_3(1) = -3 = f(1,2)$

The absolute max/min on R must come from $\partial R = \{①, ②, ③, ④\}$.

Examining our work we see

$f(0,2) = -4$	minimum
$f(\pm 1, 0) = 1$	maximum.

PROBLEM 91

$$f(x,y) = \frac{(ax+by+x)^2}{x^2+y^2+1}$$ find absolute max. of $f(x,y)$ as given.

$$\frac{\partial f}{\partial x} = \frac{2(a+1)(ax+by+x)\overset{(x^2+y^2+1)}{\wedge} - 2x(ax+by+x)^2}{(x^2+y^2+1)^2}$$

$$\frac{\partial f}{\partial x} = \frac{2(a+1)(ax+by+x)(x^2+y^2+1) - 2x(ax+by+x)^2}{(x^2+y^2+1)^2}$$

$$\frac{\partial f}{\partial x} = \frac{2(ax+by+x)}{(x^2+y^2+1)^2}\Big[(a+1)(x^2+y^2+1) - x(ax+by+x)\Big]$$

$$\frac{\partial f}{\partial x} = \frac{2(ax+by+x)}{(x^2+y^2+1)^2}\Big[\underbrace{(a+1-a-1)}_{0}x^2 + (a+1)y^2 + (a+1) - bxy\Big]$$

$$\frac{\partial f}{\partial x} = \frac{2(ax+by+x)\big((a+1)y^2 - bxy + a+1\big)}{(x^2+y^2+1)^2}$$

Next, through much the same calculation,

$$\frac{\partial f}{\partial y} = \frac{2(ax+by+x)\big(b(x^2+1) - (a+1)xy\big)}{(x^2+y^2+1)^2}$$

Observe $\nabla f = 0$ if $a=-1$, $b=0$. That choice makes $f(x,y) = \frac{(-x+x)^2}{x^2+y^2+1} = 0$. In this rather silly case, $f=0$ on all of $\mathbb{R}^2$ and the absolute max of zero is attained everywhere. So, β $(a,b) \neq (-1,0)$.

If $b \neq 0$ then $ax+x+by = 0 \Rightarrow y = \frac{-(a+1)x}{b}$ so there is a whole line of critical points, once again

$$f\left(x, \frac{-(a+1)x}{b}\right) = \frac{1}{x^2+\left(\frac{-(a+1)x}{b}\right)^2+1}\left(ax+ -\left(\frac{ax+x}{b}\right)b+x\right) = 0.$$

These are clearly not maximum values.

PROBLEM 91 continued: need that $\downarrow$this part$\downarrow$ be zero

$$\nabla f = \underbrace{\frac{2(ax+by+x)}{(x^2+y^2+1)^2}} \overbrace{\left\langle (a+1)y^2 - bxy + a + 1,\ b(x^2+1) - (a+1)xy \right\rangle}$$

as the initial factor zero $\Rightarrow f = 0$, which is not maximal.
Consider, for $(-1,0) \neq (a,b)$, $\nabla f = 0$ yields

$$\left.\begin{aligned} (a+1)y^2 - bxy + a + 1 &= 0 \\ b(x^2+1) - (a+1)xy &= 0 \end{aligned}\right\} (*)$$

Notice, $a = -1$ simplifies $(*)$ to

$$-bxy = 0$$
$$b(x^2+1) = 0 \Rightarrow b = 0$$

, but we've assumed $a=-1$ and $b=0$ not the case considered here.

Thus, assume $(a+1) \neq 0$ wlog.

To solve $(*)$ suppose $y = 0$ then $a+1 = 0 \Rightarrow a = -1$.
oh, so $y \neq 0$ hence take 2^{nd} Eq^{n} and multiply by y wlog,

$$b(x^2+1)y - x(a+1)y^2 = 0$$
$$\Rightarrow by(x^2+1) - x[bxy - a - 1] = 0 \quad \text{by } 1^{st} \text{ eq}^{n} \text{ in } (*)$$
$$\Rightarrow y(\cancel{bx^2} + b - \cancel{bx^2}) = -(a+1)x$$
$$\Rightarrow y = \frac{-(a+1)x}{b}$$

Thus $(*)$ has no sol^{n}'s, except those we already found to make $f(x,y) = 0$. I expect, the answer is that $f(x,y)$ has no absolute max, If we analyze $f(x,y)$ for $r >> 0$ then

$$f = \frac{r^2}{r^2+1}(a\cos\theta + b\sin\theta + \cos\theta)^2 \longrightarrow ((a+1)\cos\theta + b\sin\theta) = f$$

as $r \to \infty$.

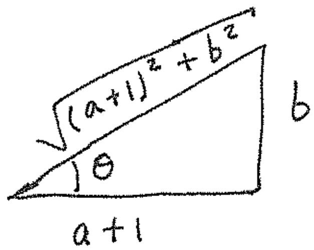
PROBLEM 9

$g(\theta) = (a+1)\cos\theta + b\sin\theta$

$$\frac{dg}{d\theta} = -(a+1)\sin\theta + b\cos\theta = 0 \quad \text{for critical } \theta$$

$$(a+1)\sin\theta = b\cos\theta$$

$$\Rightarrow \tan\theta = \frac{b}{a+1} \quad \therefore \quad \theta = \tan^{-1}\left(\frac{b}{a+1}\right)$$



$$\cos\theta = \frac{a+1}{\sqrt{(a+1)^2+b^2}} \quad \& \quad \sin\theta = \frac{b}{\sqrt{(a+1)^2+b^2}}$$

$$f = \frac{r^2}{r^2+1}\left(\frac{(a+1)^2}{\sqrt{(a+1)^2+b^2}} + \frac{b^2}{\sqrt{(a+1)^2+b^2}}\right)$$

$$f = \frac{r^2}{r^2+1}\sqrt{(a+1)^2+b^2}$$

$$\boxed{\text{So, } f(x,y) \longrightarrow \sqrt{(a+1)^2+b^2} \text{ asymptotically as } r \to \infty \text{ along } \theta = \tan^{-1}\left(\frac{b}{a+1}\right)}$$

(this is the best answer I can give here)

PROBLEM 92

Use Lagrange multipliers to find the point on $x+2y-3z=10$ which is closest to $(10, 9, 8)$

Objective: $f(x,y,z) = (x-10)^2 + (y-9)^2 + (z-8)^2$

Constraint: $g(x,y,z) = x + 2y - 3z = 10$

Consider, $\nabla f = \lambda \nabla g$,

$$\langle 2(x-10),\ 2(y-9),\ 2(z-8)\rangle = \lambda\langle 1, 2, -3\rangle$$

$$\lambda = 2(x-10) = y-9 = \frac{2(z-8)}{-3}$$

$$\Rightarrow \quad 2x-20 = y-9 = -\frac{2}{3}(z-8)$$

$$\Rightarrow \quad \underbrace{6x-60 = 3y-27} = -2z+16$$

$$y = \frac{6x-60+27}{3} = 2x-11$$

$$z = \frac{6x-60-16}{-2} = -3x+38$$

Hence, $x + 2(2x-11) - 3(-3x+38) = 10$

$$x + 4x + 9x = 10 + 22 + 114$$

$$14x = 146$$

$$x = \frac{146}{14}, \quad y = 2\left(\frac{146}{14}\right) - 11, \quad z = -3\left(\frac{146}{14}\right) + 38$$

$$\boxed{\left(\frac{73}{7}, \frac{69}{7}, \frac{47}{7}\right)}$$

PROBLEM 93

Find $\underbrace{\text{distance from } (6,0)}$ to the parabola $\underbrace{x^2 = 4y}$

objective $f(x,y) = (x-6)^2 + y^2$ constraint: $g(x,y) = x^2 - 4y$

$$\nabla f = \lambda \nabla g$$

$$\langle 2(x-6),\ 2y \rangle = \lambda \langle 2x, -4 \rangle$$

Notice $6^2 = 36 \neq 4(0)$ thus $(6,0)$ not on the parabola

Consider, $x \neq 0$,

$$\lambda = \frac{2(x-6)}{2x} = \frac{2y}{-4} \longrightarrow y = \frac{-2(x-6)}{x}$$

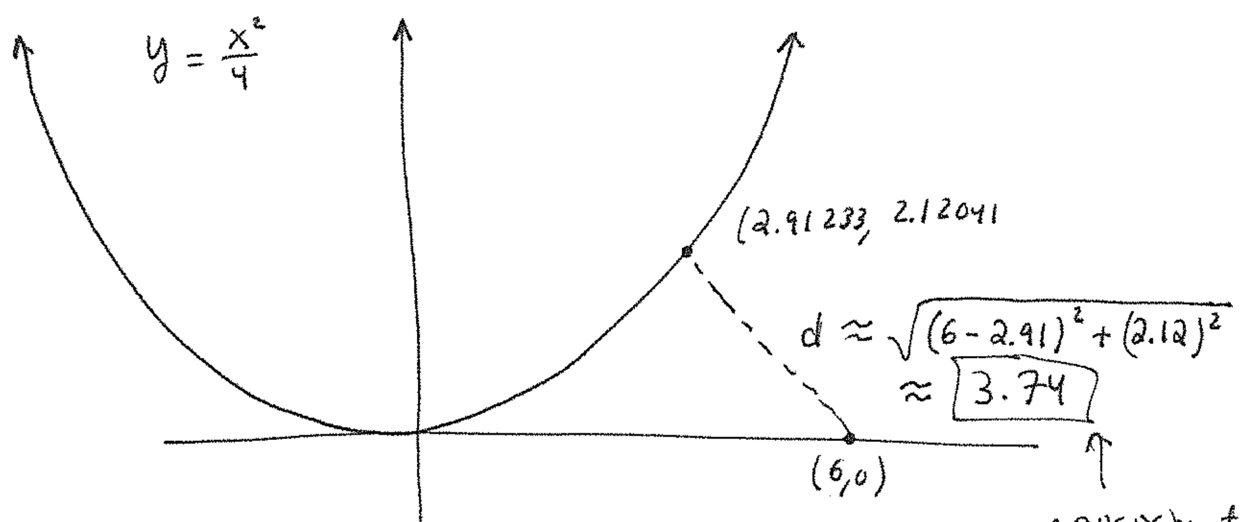
Now, plug-into constraint,

$$0 = x^2 - 4\left(\frac{-2(x-6)}{x}\right) \longrightarrow x^2 + \frac{8(x-6)}{x} = 0$$

$$\longrightarrow x^3 + 8(x-6) = 0$$

$$\longrightarrow x^3 + 8x - 48 = 0$$

An exact sol^{n} may be found, but I'll content myself with an approximate sol^{n} here, $x \approx 2.91233$ which yields $y \approx 2.12041$ $\therefore$ $\boxed{(2.91233,\ 2.12041) \approx \text{closest point}}$



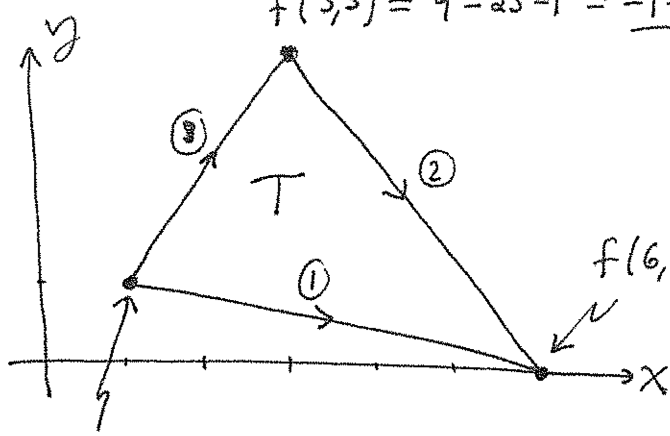
approximate distance from (6,0) to the parabola.

(Def^{n}/ the distance between an extended object and a point is the smallest distance between the given point and points on the object)

$\boxed{\text{PROBLEM 94}}$ find extrema of $f(x,y) = x^2 - y^2 - 1$ on the region bounded by triangle with corners $(1,1)$, $(3,5)$, $(6,0)$

Notice $\nabla f = \langle 2x, -2y \rangle = \langle 0, 0 \rangle \Rightarrow (0,0)$ only local min/max location. (not inside our Δ)

$f(3,5) = 9 - 25 - 1 = \underline{-17}$.



$f(6,0) = 36 - 0^2 - 1 = \underline{35}$.

$f(1,1) = 1 - 1 - 1 = \underline{-1}$.

Oh, so, we've already analyzed endpts. We may now simply search for any local min/max in ∂T

① $\vec{r}_1(t) = (1,1) + t\langle 5, -1 \rangle = \langle 1+5t, 1-t \rangle$ for $0 \le t \le 1$

$g_1(t) = f(\vec{r}_1(t)) = (1+5t)^2 - (1-t)^2 - 1$

$\frac{dg_1}{dt} = 48t + 12 \Rightarrow t = -0.25 \notin [0,1]$. $\therefore$ no local min/max on ① except at endpts.

② $\vec{r}_2(t) = (3,5) + t\langle 3, -5 \rangle = \langle 3+3t, 5-5t \rangle$ for $0 \le t \le 1$.

$g_2(t) = f(\vec{r}_2(t)) = (3+3t)^2 - (5-5t)^2 - 1$

$\frac{dg_2}{dt} = 68 - 32t = 0 \Rightarrow t = \frac{68}{32} \notin [0,1] \therefore$ no min max on ② except at endpts.

③ $\vec{r}_3(t) = (1,1) + t\langle 2, 4 \rangle = \langle 1+2t, 1+4t \rangle$ for $0 \le t \le 1$.

$g_3(t) = f(\vec{r}_3(t)) = (1+2t)^2 - (1+4t)^2 - 1$

$\frac{dg_3}{dt} = -4(6t+1) = 0 \Rightarrow t = -\frac{1}{6} \notin [0,1] \therefore$ no min/max except at endpts.

In summary, the corners are where the extrema are found for this problem; $\boxed{f(6,0) = 35 \text{ max}, \ f(3,5) = -17 \text{ min on the region } T}$

PROBLEM 95 Find power series expansion to 2^{nd} order about $(0,0)$ for $f(x,y) = (x+y)e^x \sin(y)$

$$= (x+y)(1+x+\tfrac{1}{2}x^2+\cdots)(y-\tfrac{1}{6}y^3+\cdots)$$

$$= (x+y+x^2+yx+\cdots)(y-\tfrac{1}{6}y^3+\cdots)$$

$$= \boxed{xy+y^2+\cdots}$$

PROBLEM 96

$f(x,y) = 3+x^2-y^2+13x^3+42y^3+\cdots$
analyze the nature of the critical pt. $(0,0)$.

Observe, $f_{xx}=2$, $f_{yy}=-2$, $f_{xy}=0$ hence

$D=-4<0$ $\therefore$ ~~[illegible]~~ saddle point.

Of course, we could simply observe x^2 increases above 0 near $(0,0)$ whereas $-y^2$ decreases below 0 near $(0,0)$ hence $f(x,y)$ takes values below & above 3 near $(0,0)$ $\therefore$ it's a saddle point.

PROBLEM 97

$$f(x,y) = 3+x+(x-1)^2+(y-3)^2+\cdots$$
$$= 4+(x-1)+(x-1)^2+(y-3)^2+\cdots$$

$f_x(1,3)=1$ $\therefore$ $\nabla f(1,3) = \langle 1,0\rangle \neq \vec{0}$
$f_y(1,3)=0$

$\therefore$ $(1,3)$ not a critical pt.

PROBLEM 98 $f(x,y)=\sin(x+y)$ expand about (π,π) to order two, Observe $f(\pi,\pi)=\sin(2\pi)=0$ and $f_{xx}=f_{yy}=-f$ whereas $f_x=f_y=\cos(x+y)$ hence $f_x(\pi,\pi)=f_y(\pi,\pi)=\cos(2\pi)=1$. Thus, using the 2^{nd} order Taylor expansion we derived in Lecture (and, see page 254 etc.) note $f_{xx}(\pi,\pi)=f_{yy}(\pi,\pi)=0$,

$$f(x,y) = 1\cdot(x-\pi)+1\cdot(y-\pi)+\cdots = x+y-2\pi+\cdots$$
$$= \boxed{(x-\pi)+(y-\pi)+\cdots}$$

PROBLEM 99

$$f(x,y,z) = \frac{1}{1+x+y+z}$$ expand to 2^{nd} order about $(0,0,0)$

$$= \frac{1}{1-(-(x+y+z))}$$ geometric series!

$$= \sum_{n=0}^{\infty} \left(-(x+y+z)\right)^n$$

$$= 1 - (x+y+z) + (x+y+z)^2 + \cdots$$

$$= 1 - x - y - z + (x+y)^2 + 2(x+y)z + z^2 + \cdots$$

$$= 1 - x - y - z + x^2 + 2xy + y^2 + 2xz + 2yz + z^2 + \cdots$$

$$= \boxed{1 - x - y - z + x^2 + y^2 + z^2 + 2xy + 2xz + 2yz + \cdots}$$

PROBLEM 100

Suppose $f(x,y)$ is objective fnct and it has extremum on $g(x,y)=0$. Let $F(x,y,\lambda) = f(x,y) - \lambda g(x,y)$. Observe

$$\nabla F = \langle F_x, F_y, F_\lambda \rangle$$

$$= \langle f_x - \lambda g_x, f_y - \lambda g_y, -g \rangle$$

Hence $\nabla F = \langle 0,0,0 \rangle \longrightarrow \begin{cases} \nabla f = \lambda \nabla g \\ g = 0 \end{cases}$

Remark: adding a dimension has converted a constrained problem to a local problem.

BONUS PROBLEM: may discuss in Lecture.