MATH 231 QUIZ 1

Please work the problems in the white space provided and clearly box your solutions. You are allowed one 3″ × 5″ notecard. Enjoy! This quiz has 3pts of bonus credit.

Problem 1 (0.5 pt) Parametrize the line-segment from $P_1 = (3, 0, 7)$ to $P_2 = (10, 10, 10)$. Also, find the distance between P_1 and P_2.

$$\vec{r}(t) = P_1 + t(P_2 - P_1) = (3,0,7) + t(7,10,3) = \boxed{\langle 3+7t,\ 10t,\ 7+3t \rangle}$$

$$d(P_1,P_2) = \| P_2 - P_1 \| = \sqrt{7^2+10^2+3^2} = \boxed{\sqrt{158} \approx 12.57}$$

Problem 2 (0.5 pt) Find an integral which represents the arclength function starting at $t = 0$ for $\vec{\gamma}(t) = \langle t^2 \sin(t), t, \frac{1}{2t+3} \rangle$. **DO NOT ATTEMPT THE INTEGRAL.**

$$\frac{d\vec{\gamma}}{dt} = \left\langle 2t\sin t + t^2\cos t,\ 1,\ \frac{-2}{(2t+3)^2} \right\rangle = \vec{\gamma}\,'(t)$$

$$s(t) = \int_0^t \| \vec{\gamma}\,'(u) \| \, du = \boxed{\int_0^t \sqrt{\left[2u\sin u + u^2\cos(u)\right]^2 + 1 + \frac{4}{(2u+3)^4}}\, du}$$

Problem 3 (0.5 pt) Where does the helix $x = 7\cos t,\ y = 7\sin t,\ z = 3t$ intersect the $z = \pi$ plane ?

$z = \pi = 3t$ at intersection we find $\underline{t = \pi/3}$.

thus $x = 7\cos\frac{\pi}{3} = 7/2$, $y = 7\sin(\pi/3) = 7\sqrt{3}/2$ $\therefore \boxed{\left(\frac{7}{2}, \frac{7\sqrt{3}}{2}, \pi\right)}$

Problem 4 (0.5 pt) Suppose $A = 2$, $B = 3$ and $\vec{A} \bullet \vec{B} = 3$. Find the angle between $\vec{A}$ and $\vec{B}$.

$\vec{A} \cdot \vec{B} = AB\cos\theta \hookrightarrow \cos\theta = \frac{\vec{A}\cdot\vec{B}}{AB} = \frac{3}{2\cdot 3} = \frac{1}{2}$ $\therefore \boxed{\theta = \frac{\pi}{3}}$

Problem 5 (0.5 pt) Find the spherical coordinates of the point $(1, 1, \sqrt{7})$

$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{1+1+7} = \sqrt{9} = 3.$

$\theta = \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}(1) = \pi/4$ (point falls in Quad. I of xy)

$z = \rho\cos\phi \rightarrow \phi = \cos^{-1}\left[\frac{z}{\rho}\right] = \cos^{-1}\left[\frac{\sqrt{7}}{3}\right] = 28.13°$ or 0.4909 radians. $\therefore \boxed{(\rho, \phi, \theta) \cong \left(3, 0.4909, \frac{\pi}{4}\right)}$

Problem 6 (1 pt) Find the point on the plane $x + 2y + 3z = 6$ which is closest to $(3, 5, 7)$.

Look for intersection of line $\vec{r}(t) = (3,5,7) + t\langle 1,2,3\rangle$ and the plane.

$x = 3+t$
$y = 5+2t$
$z = 7+3t$

$x + 2y + 3z = 6$

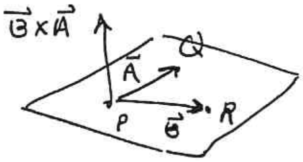
$3+t+10+4t+21+9t = 6$

$14t + 34 = 6$

$14t = -28$

$\underline{t = -2}$ $\therefore \vec{r}(-2) = (3,5,7) - 2\langle 1,2,3\rangle = \boxed{(1,1,1)}$

Problem 7 (0.5 pt) Let $P = (2,0,3)$, $Q = (1,-3,0)$ and $R = (0,0,1)$. Find a parametrization of the plane which contains the points P, Q, R.



$$\vec{r}(s,t) = P + s(Q-P) + t(R-P)$$
$$= \boxed{(2,0,3) + s\underbrace{\langle -1,-3,-3\rangle}_{\vec{A}} + t\underbrace{\langle -2,0,-2\rangle}_{\vec{B}}}$$

Problem 8 (0.5 pt) Find the Cartesian equation of the plane which contains the points P, Q, R of the previous problem.

$\vec{A}$ & $\vec{B}$ are tangent to the plane

$$\vec{B}\times\vec{A} = \det\begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ -2 & 0 & -2 \\ -1 & -3 & -3 \end{bmatrix} = \hat{x}(-6) - \hat{y}(6-2) + \hat{z}(6) = \underbrace{\langle -6,-4,6\rangle}_{\vec{n}}$$

$\vec{n}$ normal to plane.

We should check, $\vec{n}\cdot\vec{A} = 0$ and $\vec{n}\cdot\vec{B} = 0$, yep ✓.

Thus, $\boxed{-6(x-2) - 4y + 6(z-3) = 0}$

Problem 9 (1 pt) Find the volume of the parallel-piped which has edges which line up with the vectors $\vec{A} = \langle 2,0,3\rangle$, $\vec{B} = \langle 1,-3,0\rangle$ and $\vec{C} = \langle 0,0,1\rangle$.

$$Vol(\vec{A},\vec{B},\vec{C}) = \left|\det\begin{bmatrix} 2 & 1 & 0 \\ 0 & -3 & 0 \\ 3 & 0 & 1 \end{bmatrix}\right| = |2(-3) - 1(0) + 0(9)| = |-6| = \boxed{6.}$$

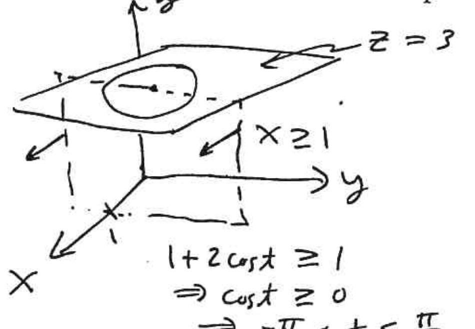
Or, use $|\vec{A}\cdot(\vec{B}\times\vec{C})|$.

Problem 10 (1 pt) Let $\vec{r}(t) = \langle 1, t^2, te^t\rangle$. Find the parametrization of the tangent line to the given curve at $(1,1,e)$. ⇐ at $t=1$ since $(1,t^2,te^t) = (1,1,e)$

$\Rightarrow t^2 = 1 \Rightarrow t = \pm 1$
and $te^t = e \Rightarrow t = 1$ } $\underline{t=1}$.

$$\frac{d\vec{r}}{dt} = \langle 0, 2t, e^t + te^t\rangle$$
$$\vec{r}\,'(1) = \langle 0, 2, e\cancel{\ }+e\rangle = \langle 0,2,2e\rangle$$
$$\boxed{\vec{l}(t) = (1,1,e) + t\langle 0,2,2e\rangle = \langle 1, 1+2t, e+2et\rangle}$$

Problem 11 (1 pt) Consider a circle centered at $(1,1,3)$ which lies in a plane parallel to the xy-plane. If the circle has radius 2 then parametrize the part of the circle with $x \geq 1$. (include the domain of the parameter in your solution)



$$\boxed{\vec{r}(t) = \langle 1 + 2\cos t, 1 + 2\sin t, 3\rangle \text{ for } -\tfrac{\pi}{2} \leq t \leq \tfrac{\pi}{2}}$$

$1 + 2\cos t \geq 1$
$\Rightarrow \cos t \geq 0$
$\Rightarrow -\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$.

Problem 12 (0.5 pt) Find the projection of $\vec{A} = \langle 1,2,2\rangle$ in the $\vec{B} = \langle 1,1,0\rangle$-direction.

$$\text{Proj}_{\vec{B}}(\vec{A}) = (\vec{A}\cdot\hat{B})\hat{B} = \left(\frac{\vec{A}\cdot\vec{B}}{\vec{B}\cdot\vec{B}}\right)\vec{B} = \frac{1+2}{1+1}\langle 1,1,0\rangle = \boxed{\langle 3/2, 3/2, 0\rangle}$$

Problem 13 (1 pt) Let $\vec{A}, \vec{B}$ be constant vectors. Let $\vec{F}(t) = \cos t\,\vec{A} + \sin t\,\vec{B}$. Calculate $\int \vec{F}(t)dt$ and also calculate $d\vec{F}/dt$.

$$\begin{aligned}\int \vec{F}(t)dt &= \int(\cos t\,\vec{A} + \sin t\,\vec{B})dt \\ &= \left(\int \cos t\,dt\right)\vec{A} + \left(\int \sin t\,dt\right)\vec{B} \\ &= (\sin t + c_1)\vec{A} + (-\cos t + c_2)\vec{B} \\ &= \boxed{\sin t\,\vec{A} - \cos t\,\vec{B} + \vec{C}}\end{aligned}$$

$$\begin{aligned}&\frac{d}{dt}\left[\cos t\,\vec{A} + \sin t\,\vec{B}\right] = \\ &= \frac{d}{dt}(\cos t)\vec{A} + \frac{d}{dt}(\sin t)\vec{B} \\ &= \boxed{-\sin t\,\vec{A} + \cos t\,\vec{B}}\end{aligned}$$

Problem 14 (2 pt) Let $\vec{G} = \cos(t)\hat{A} + \sin(t)\hat{B}$ where $\vec{A}, \vec{B}$ are nonzero constant vectors. If G is constant then determine the angle between $\vec{A}$ and $\vec{B}$.

$$\begin{aligned}G^2 = \vec{G}\cdot\vec{G} &= (\cos t\,\hat{A} + \sin t\,\hat{B})\cdot(\cos t\,\hat{A} + \sin t\,\hat{B}) \\ &= \cos^2 t\,\hat{A}\cdot\hat{A} + \sin t\cos t\,\hat{A}\cdot\hat{B} + \sin t\cos t\,\hat{B}\cdot\hat{A} + \sin^2 t\,\hat{B}\cdot\hat{B} \\ &= [\cos^2 t + \sin^2 t] + 2\sin t\cos t\,\hat{A}\cdot\hat{B} \\ &= 1 + \sin(2t)\,\hat{A}\cdot\hat{B}\end{aligned}$$

$$\frac{d}{dt}(G^2) = 2G\frac{dG}{dt} = 0 = 2\underbrace{\cos(2t)}_{\text{nonzero for many } t!}\hat{A}\cdot\hat{B}$$

hence $\hat{A}\cdot\hat{B} = 0$

$\Rightarrow \vec{A}\cdot\vec{B} = 0$

$\Rightarrow \boxed{\theta = 90°.}$

Problem 15 (2 pt) Suppose $\vec{v}(t) = \langle 2t, -\sin t, \cos t\rangle$ is the velocity of a ninja hound which is at $(0,1,0)$ at time $t=0$. Calculate the position $\vec{r}(t)$ and acceleration $\vec{a}$ at time t. Also, calculate the tangential and normal components of $\vec{a}$ (find a_T and a_N).

$$\vec{v}(t) = \langle 2t, -\sin t, \cos t\rangle = \frac{d\vec{r}}{dt} \Rightarrow \vec{r}(t) = \langle t^2, \cos t, \sin t\rangle + \vec{C}$$

But, $\vec{r}(0) = (0,1,0) = (0,1,0) + \vec{C} \quad \therefore \vec{C} = 0.$

Thus $\boxed{\vec{r}(t) = \langle t^2, \cos t, \sin t\rangle}$ ← position.

$\vec{a} = \dfrac{d\vec{v}}{dt} = \boxed{\langle 2, -\cos t, -\sin t\rangle}$ ← acceleration.

$$T = \frac{1}{v}\vec{v} = \frac{1}{\sqrt{4t^2+1}}\langle 2t, -\sin t, \cos t\rangle$$

$\vec{a} = a_T T + a_N N$
$a^2 = a_T^2 + a_N^2$

$$a_T = \vec{a}\cdot T = \frac{1}{\sqrt{4t^2+1}}\left(4t + \cancel{\sin t\cos t} - \cancel{\sin t\cos t}\right) = \boxed{\frac{4t}{\sqrt{4t^2+1}} = a_T}$$

$$a_N = \sqrt{a^2 - a_T^2} = \sqrt{5 - \frac{16t^2}{4t^2+1}} = \sqrt{\frac{20t^2+5-16t^2}{4t^2+1}} = \boxed{\sqrt{\frac{4t^2+5}{4t^2+1}} = a_N}$$

Problem 16 A surface S has parametric equations as given below:

$$x = \sinh\beta\cos t, \qquad y = \sinh\beta\sin t, \qquad z = \cosh t.$$

Find the Cartesian equation of S and identify the surface.

Problem 17 Suppose $\vec{A} \bullet \vec{B} = 0$ for all $\vec{B}$. Show $\vec{A} = 0$.

Problem 18 Find all vectors $\vec{A}$ for which $\vec{A} \bullet \widehat{z} = 3$ and $\vec{A} \times \widehat{z} = 0$. If there are many solutions then characterize them in your solution.

Problem 19 Show $\dfrac{d}{dt}[f\vec{A}] = \dfrac{df}{dt}\vec{A} + f\dfrac{d\vec{A}}{dt}$.

Problem 20 Find the parametrization of the curve of intersection of the plane $x - 2y + 3z = 1$ and the cone $\phi = \pi/6$.

Problem 21 Let $\vec{A}, \vec{B}$ be nonzero, non-colinear vectors. Let C be a curve parametrized by:

$$\vec{\gamma}(t) = \vec{r}_o + f(t)\vec{A} + g(t)\vec{B}$$

for $t \in \mathbb{R}$ where $f, g : \mathbb{R} \to \mathbb{R}$ are smooth functions. Find the torsion of C.