Please work the problems in the white space provided and clearly box your solutions. You are allowed one 3″ × 5″ notecard. Enjoy! This quiz has 3pts of bonus credit.

Problem 1 Calculate the limit below (if it doesn't exist then explain why)

$$\lim_{(x,y)\to(0,0)} \frac{2xy}{x^2+y^2} = \lim_{r\to 0}\left(\frac{2r^2\sin\theta\cos\theta}{r^2}\right) = \lim_{r\to 0}\left(\sin(2\theta)\right)$$

oops. ⇒ try 2 path test.

Path 1 : $(\theta = 0)$

$$\lim_{(x,0)\to(0,0)}\left(\frac{2xy}{x^2+y^2}\right) = \lim_{x\to 0}\left(\frac{0}{x^2+0}\right) = 0.$$

Path 2 : $\theta = \pi/4$

$$\lim_{(x,x)\to(0,0)}\left(\frac{2xy}{x^2+y^2}\right) = \lim_{x\to 0}\left(\frac{2x^2}{2x^2}\right) = 1.$$

thus the limit does not exist as we have two path limits which disagree.

Problem 2 Let $f(x,y) = x^3 + xy^2$. Find the rate of change in f at $(1,2)$ in the $\langle 3,4\rangle$ direction.

$$\nabla f = \langle 3x^2 + y^2,\ 2xy\rangle$$

$$(\nabla f)(1,2) = \langle 3+4,\ 2(1)(2)\rangle = \langle 7,4\rangle$$

$$\hat{u} = \widehat{\langle 3,4\rangle} = \frac{1}{\sqrt{3^2+4^2}}\langle 3,4\rangle = \langle 3/5,\ 4/5\rangle$$

$$(D_{\hat{u}} f)(1,2) = \langle 7,4\rangle \cdot \left\langle \frac{3}{5}, \frac{4}{5}\right\rangle = \frac{21}{5} + \frac{16}{5} = \boxed{\frac{37}{5}}$$

Problem 3 Explain how to express $\frac{\partial f}{\partial x}$ at (x_o, y_o) as a directional derivative of f.

$$\frac{\partial f}{\partial x}(x_o, y_o) = (D_{\hat{x}} f)(x_o, y_o)$$ by definition

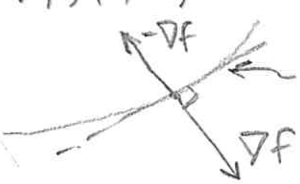
Problem 4 Explain which directions the function $f(x,y) = x^3 + y\sin(y)$ is locally constant at $(1,\pi)$.

$$\nabla f = \langle 3x^2,\ \sin y + y\cos y\rangle$$

$\langle 3,-\pi\rangle \cdot \langle a,b\rangle = 0$

$3a - \pi b = 0 \therefore b = \frac{3a}{\pi}$?

$$(\nabla f)(1,\pi) = \langle 3, -\pi\rangle \leftarrow \text{max-rate of change.}$$



directions $\pm\langle \pi, 3\rangle$ gives a $\hat{u} = \pm\widehat{\langle \pi, 3\rangle}$ for which $D_{\hat{u}} f(1,\pi) = 0$.

Problem 5 Calculate ∇f for $f(x,y,z) = x^y + \cos(z^2)$

$$\nabla f = \left\langle \frac{\partial}{\partial x}(x^y), \frac{\partial}{\partial y}(x^y), \frac{\partial}{\partial z}(\cos(z^2)) \right\rangle \quad \text{(omitted trivial terms)}$$

$$= \langle y x^{y-1}, \ln(x) x^y, -2z \sin(z^2) \rangle$$

Problem 6 Let $z = \cos(xy^2)$. Furthermore, $x = u^2 + v$ and $y = ve^v$. Calculate $\frac{\partial z}{\partial v}$ and $\frac{\partial z}{\partial u}$.

$$\frac{\partial z}{\partial u} = -\sin(xy^2) \frac{\partial}{\partial u}(xy^2)$$
$$= -\sin(xy^2)\left[y^2 \frac{\partial x}{\partial u} + 2xy \frac{\partial y}{\partial u}\right]$$
$$= -\sin(xy^2)\left[y^2(2u)\right]$$
$$= \boxed{-2uy^2 \sin(xy^2)}$$

$$\frac{\partial z}{\partial v} = -\sin(xy^2)\left[y^2 \frac{\partial x}{\partial v} + 2xy \frac{\partial y}{\partial v}\right]$$
$$= -\sin(xy^2)\left[y^2 + 2xy(e^v + ve^v)\right]$$
$$= \boxed{-(y^2 + 2xye^v(1+v))\sin(xy^2)}$$

Problem 7 Consider the surface which is the solution set of the equation $x^2 + y^3 + z^4 = 10$. Find the equation of the tangent plane to this surface at $(-1, 2, 1)$.

$$\nabla F = \nabla(x^2 + y^3 + z^4) = \langle 2x, 3y^2, 4z^3 \rangle$$
$$(\nabla F)(-1,2,1) = \langle -2, 3(4), 4(1) \rangle = \langle -2, 12, 4 \rangle$$
$$\boxed{-2(x+1) + 12(y-2) + 4(z-1) = 0}$$

Problem 8 Let $\vec{r}(s,t) = \langle s\cos(t), s\sin(t), s \rangle$ be the parametrization of a surface M. Find the normal vector field to M.

$$\frac{\partial \vec{r}}{\partial s} = \langle \cos t, \sin t, 1 \rangle$$

$$\frac{\partial \vec{r}}{\partial t} = \langle -s\sin t, s\cos t, 0 \rangle$$

$$\vec{N}(s,t) = \frac{\partial \vec{r}}{\partial s} \times \frac{\partial \vec{r}}{\partial t} = \det \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \cos t & \sin t & 1 \\ -s\sin t & s\cos t & 0 \end{bmatrix}$$

$$= \hat{x}(-s\cos t) - \hat{y}(s\sin t) + \hat{z}(s\cos^2 t + s\sin^2 t)$$

$$= \boxed{\langle -s\cos t, -s\sin t, s \rangle}$$

Problem 9 You are stuck on an ellipitical surface $x^2+y^2+3z^2=5$. At the point $(1,1,1)$ you measure your x-velocity to be $\underbrace{2ft/s}_{dx/dt}$ and your y-velocity is $\underbrace{-3ft/s}_{dy/dt}$. Assuming units of ft and s find $\frac{dz}{dt}$.

$$2x\frac{dx}{dt} + 2y\frac{dy}{dt} + 6z\frac{dz}{dt} = 0$$

$$\frac{dz}{dt} = \frac{-2x\frac{dx}{dt} - 2y\frac{dy}{dt}}{6z} = \left[\frac{-2(1)(2)-2(1)(-3)}{6(1)}\right]\frac{ft}{s} = \boxed{\frac{1}{3}\frac{ft}{s}}$$

Problem 10 Ron Swanson is holding a barbeque of epic size in a heavily misty park. At the same time a bagpipe club is playing at an embarrassing volume. You can't see or hear in effect. The smell of meat is measured to roughly follow $M(x,y) = x^2 - 2xy$ at $(1,1)$. In which direction should you travel to join the feast?

$$\nabla M = \langle 2x - 2y, -2x \rangle$$

$$(\nabla M)(1,1) = \langle 0, -2 \rangle$$

$\Rightarrow$ $\boxed{\text{Head due South for the meat.}}$
(in $-\hat{y}$ direction)

Problem 11 Suppose $w = x^2 + y^3 + z^4$ and $xyz = 1$. Calculate $\left(\frac{\partial w}{\partial x}\right)_y$ and $\left(\frac{\partial w}{\partial x}\right)_z$

$$dw = 2x\,dx + 3y^2dy + 4z^3dz$$

$$yz\,dx + xz\,dy + xy\,dz = 0 \rightarrow dz = -\left(\frac{yz\,dx + xz\,dy}{xy}\right)$$

$$\Rightarrow dw = 2x\,dx + 3y^2dy + 4z^3\left(-z\left(\frac{dx}{x} + \frac{dy}{y}\right)\right)$$

$$dw = \underbrace{\left(2x - \frac{4z^4}{x}\right)}_{\left(\frac{\partial w}{\partial x}\right)_y}dx + \underbrace{\left(3y^2 - \frac{4z^4}{y}\right)}_{\left(\frac{\partial w}{\partial y}\right)_x}dy$$

answers are coefficients of dx

Problem 12 If $dw = (2x^2+3y)dx + (e^z + \cos(y) - 4)dy + dz$ then calculate $\frac{\partial w}{\partial x}$, $\frac{\partial w}{\partial y}$ and $\frac{\partial w}{\partial z}$.

$$\boxed{\begin{aligned} \left.\frac{\partial w}{\partial x}\right|_{y,z} &= 2x^2 + 3y \\ \left.\frac{\partial w}{\partial y}\right|_{x,z} &= e^z + \cos(y) - 4 \\ \left.\frac{\partial w}{\partial z}\right|_{x,y} &= 1. \end{aligned}}$$

oops, I misread it,

$$\left.\frac{\partial w}{\partial x}\right|_z = 2x + 3y^2\left.\frac{\partial y}{\partial x}\right|_z$$

$$= 2x + 3y^2\left.\frac{\partial}{\partial x}\right|_z\left(\frac{1}{xz}\right)$$

$$= \boxed{2x - 3y^2\frac{x}{(xz)^2}}$$

(other answer formats also reasonable)

Problem 13 Let $\omega = \overbrace{(2xy^2 + e^x)}^{M}dx + \overbrace{(2x^2y + y\cos(y^2))}^{N}dy$. Show ω is a closed form on $\mathbb{R}^2$ and show it is exact by finding a function F for which $\omega = dF$.

$$\frac{\partial M}{\partial y} = 4xy \quad \text{and} \quad \frac{\partial N}{\partial x} = 4xy \quad \therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \ \forall (x,y) \in \mathbb{R}^2$$

Hence ω closed on $\mathbb{R}^2$.

Solve
$$① \ \frac{\partial F}{\partial x} = 2xy^2 + e^x$$
$$② \ \frac{\partial F}{\partial y} = 2x^2y + y\cos(y^2)$$

Integrate ① to obtain $F(x,y) = x^2y^2 + e^x + C_1(y)$.
Substitute into ② to find $\frac{\partial F}{\partial y} = \cancel{2x^2y} + \frac{dC_1}{dy} = \cancel{2x^2y} + y\cos y^2$

Hence $\frac{dC_1}{dy} = y\cos y^2$. Let $u = y^2$ and so $\frac{du}{2} = y\,dy$

thus $\int y\cos y^2\,dy = \frac{1}{2}\int \cos(u)\,du = \frac{+\sin u}{2} \Rightarrow \boxed{F(x,y) = x^2y^2 + e^x + \frac{\sin(y^2)}{2}}$

Problem 14 Solve $\underbrace{(2xy^2 + e^x)dx + (2x^2y + y\cos(y^2))dy}_{} = 0$.

$dF = 0$ has Sol$^{\underline{n}}$ $F(x,y) = c$

$$\therefore \boxed{x^2y^2 + e^x - \tfrac{1}{2}\sin(y^2) = c}$$

Problem 15 Let $\frac{dy}{dx} + Py = Q$ be a differential equation where P, Q are continuous functions of x. Show that the corresponding Pfaffian equation is not exact. Then, assume $I = I(x)$ and find a formula for I for which $I\frac{dy}{dx} + IPy = IQ$ is an exact equation.

$$dy + Py\,dx = Q\,dx \longrightarrow \boxed{\underbrace{(Q - Py)}_{M}dx - \underbrace{dy}_{N=-1} = 0} \text{---} (*)$$

Observe $\frac{\partial M}{\partial y} = \frac{\partial}{\partial y}(Q) - \frac{\partial}{\partial y}(Py) = 0 - P\frac{\partial y}{\partial y} = -P$

Yet $\frac{\partial N}{\partial x} = \frac{\partial}{\partial x}(-1) = 0$ thus $(*)$ not exact.

Multiply by $I = I(x)$

$$(IQ - IPy)dx - I\,dy = 0$$

For exactness we require the form above be closed,

$$\frac{\partial}{\partial y}[IQ - IPy] = -IP = \frac{\partial}{\partial x}[-I] \Rightarrow \frac{dI}{dx} = IP$$

But, $\frac{dI}{I} = P\,dx \Rightarrow \ln|I| = \int P\,dx \Rightarrow \boxed{I = \exp\left(\int P\,dx\right)}$

"this is the so-called integrating factor" for $(*)$