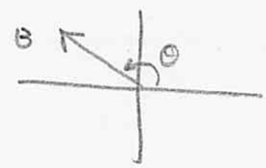
Please work the problems in the white space provided and clearly box your solutions. You are allowed one 3″ × 5″ notecard. Enjoy!

Problem 1 (5pts) Find a nonzero vector which is orthogonal to both $\widehat{x}$ and $\widehat{y} + \widehat{z}$.

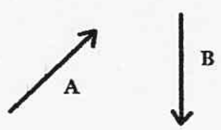
$$\hat{x} \times (\hat{y} + \hat{z}) = \hat{x} \times \hat{y} + \hat{x} \times \hat{z}$$
$$= \boxed{\hat{z} - \hat{y} = \langle 0, -1, 1 \rangle}$$

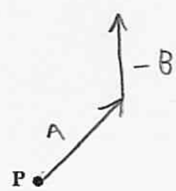
Problem 2 (9pts) Find the magnitude, standard-angle and unit-vector for $\vec{B} = \langle -2, \sqrt{3} \rangle$.

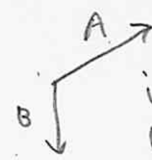
$$B = \sqrt{4+3} = \boxed{\sqrt{7} = B}$$
$$\boxed{\hat{B} = \frac{1}{\sqrt{7}} \langle -2, \sqrt{3} \rangle}$$
$$\tan^{-1}\left(\frac{\sqrt{3}}{-2}\right) = -40.89^\circ \longrightarrow \boxed{\theta = 139.11^\circ}$$



Problem 3 (6pts) For the vectors pictured below. Draw $\vec{A} - \vec{B}$ and explain which direction $\vec{A} \times \vec{B}$ points. Please draw $\vec{A} - \vec{B}$ starting at the point P.







$\vec{A} \times \vec{B}$ into page by RHR.

Problem 4 (15pts) Let $\vec{A} = \langle 1, 1, 1 \rangle$ and $\vec{B} = \langle 0, 3, 4 \rangle$. Find the following:

(a) a vector of length 4 in the direction of $\vec{B}$

$$4\hat{B} = 4 \frac{1}{\sqrt{9+16}} \vec{B} = \frac{4}{5} \langle 0, 3, 4 \rangle = \boxed{\left\langle 0, \frac{12}{5}, \frac{16}{5} \right\rangle}$$

(b) the projection of $\vec{A}$ in the direction of $\vec{B}$

$$\text{Proj}_{\vec{B}}(\vec{A}) = (\vec{A} \cdot \hat{B})\hat{B} = \left(\frac{\langle 1,1,1 \rangle \cdot \langle 0,3,4 \rangle}{\| \langle 0,3,4 \rangle \|^2} \right) \langle 0, 3, 4 \rangle$$
$$= \frac{7}{25} \langle 0, 3, 4 \rangle = \boxed{\left\langle 0, \frac{21}{25}, \frac{28}{25} \right\rangle}$$

(c) the angle between $\vec{A}$ and $\vec{B}$

$$\theta = \cos^{-1}\left(\frac{\vec{A} \cdot \vec{B}}{AB} \right) = \cos^{-1}\left(\frac{7}{\sqrt{3} \cdot 5} \right) = \boxed{36.07^\circ}$$

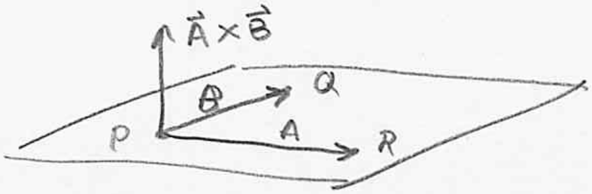
Problem 5 (7pts) A plane contains the point $(1,1,1)$. Also, the line $\vec{r}(t) = \langle 2+t,\, 3t,\, 6\rangle$ is normal to the plane. Find the Cartesian equation of the plane.

need point and normal. $\langle 1,3,0\rangle$ normal to plane.
We have both. Use $(1,1,1)$ as base point,

$$1(x-1) + 3(y-1) + 0(z-1) = 0$$

$$x - 1 + 3y - 3 = 0 \hookrightarrow \boxed{x + 3y = 4}$$

Problem 6 (10pts) Find the Cartesian equation of the plane containing the points $P = (1,1,1)$, $Q = (0,1,0)$ and $R = (0,0,1)$.



$$\vec{A} = R - P = (0,0,1) - (1,1,1) = \langle -1,-1,0\rangle$$

$$\vec{B} = Q - P = (0,1,0) - (1,1,1) = \langle -1, 0, -1\rangle$$

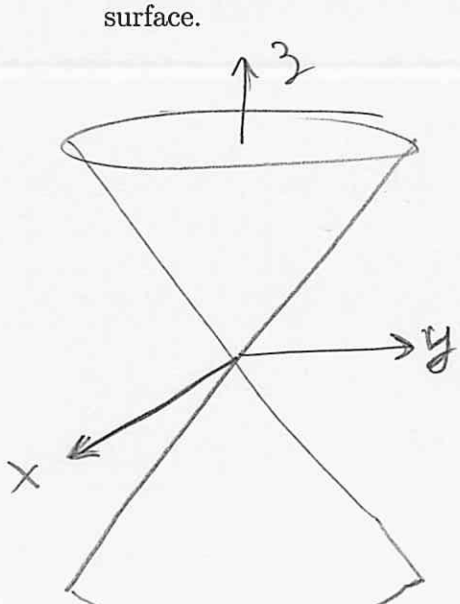
$$\vec{A}\times\vec{B} = \det\begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ -1 & -1 & 0 \\ -1 & 0 & -1 \end{bmatrix}$$

$$\vec{A}\times\vec{B} = \langle 1, -1, -1\rangle \quad \underline{\text{normal to plane}}$$

Checked $\vec{A}\cdot(\vec{A}\times\vec{B})$ and $\vec{B}\cdot(\vec{A}\times\vec{B}) = 0$. We find,

$$1\cdot(x-1) - 1(y-1) - 1(z-1) = 0 \Rightarrow \boxed{x - y - z = -1}$$ ← you can easily check P, Q, R all solve this.

Problem 7 (5pts) Suppose S is a surface is the collection of all $(x,y,z) \in \mathbb{R}^3$ such that $z^2 = x^2 + y^2$. Find a parametrization of S which does not use square roots and sketch and name the surface.



$z^2 = x^2 + y^2$ is a $\boxed{\text{cone}}$

Let $x = z\cos\theta$
$y = z\sin\theta$

$$x^2 + y^2 = z^2(\cos^2\theta + \sin^2\theta) \neq z^2$$

$$\boxed{\vec{r}(z,\theta) = \langle z\cos\theta,\, z\sin\theta,\, z\rangle}$$

Problem 8 (14pts) Let $\vec{\gamma}(t) = \langle 5\sin t, 4\cos t, 3\cos t\rangle$ for $t \geq 0$. Find the arclength function of this curve based at $t = 0$. Also, find the T, N and B vectorfields at time t (or arclength s if you prefer). Finally, find the curvature and torsion of the curve.

$$\frac{d\vec{\gamma}}{dt} = \langle 5\cos t, -4\sin t, -3\sin t\rangle = \vec{V}$$

$$V = \|\vec{V}\| = \sqrt{25\cos^2 t + 16\sin^2 t + 9\sin^2 t} = \sqrt{25} = 5.$$

Hence $s(t) = \int_0^t V\,d\bar{t} = \int_0^t 5\,d\bar{t} = 5\bar{t}\Big|_0^t = \boxed{5t = s(t)}$

$$T(t) = \frac{1}{V}\vec{V} = \frac{1}{5}\langle 5\cos t, -4\sin t, -3\sin t\rangle$$

$$\frac{dT}{dt} = \frac{1}{5}\langle -5\sin t, -4\cos t, -3\cos t\rangle$$

$$\left\|\frac{dT}{dt}\right\| = \frac{1}{5}\sqrt{25\sin^2 t + 16\cos^2 t + 9\cos^2 t} = \frac{\sqrt{25}}{5} = 1.$$

Thus $\underline{N = T'}$ and $\kappa(t) = \frac{1}{V}\left\|\frac{dT}{dt}\right\| = \boxed{\frac{1}{5} = \kappa}$

$$B = T \times N$$

$$= \det\begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ \cos t & -\frac{4}{5}\sin t & -\frac{3}{5}\sin t \\ -\sin t & -\frac{4}{5}\cos t & -\frac{3}{5}\cos t \end{bmatrix}$$

$$= \left\langle 0, +\frac{3}{5}(\cos^2 t + \sin^2 t), -\frac{4}{5}(\cos^2 t + \sin^2 t)\right\rangle$$

$$= \underbrace{\boxed{\langle 0, +3/5, -4/5\rangle}}_{B}$$

Hence $\frac{dB}{dt} = 0 \Rightarrow \frac{dB}{dt}\cdot N = 0$

$\Rightarrow \boxed{\tau = 0}$

ANSWERS: please record here ↓

arclength: $5t$

T: $\langle \cos t, \frac{-4}{5}\sin t, \frac{-3}{5}\sin t\rangle$

N: $\langle -\sin t, \frac{-4}{5}\cos t, \frac{-3}{5}\cos t\rangle$

B: $\langle 0, +3/5, -4/5\rangle$

κ: $1/5$

τ: 0

Problem 9 (10pts) Suppose $\vec{a} = t\hat{x}$ is the acceleration at time t for a ninja. Given that the ninja is initially springing into action with velocity $\langle 1,2,3\rangle$ at the position $(0,0,0)$ find both the velocity and position of the ninja at time t.

$$\vec{a} = \frac{d\vec{V}}{dt} = t\hat{x} \Rightarrow \vec{V}(t) = \tfrac{1}{2}t^2\hat{x} + \vec{C}$$

$$\vec{V}(0) = \langle 1,2,3\rangle = \tfrac{1}{2}(0)^2\hat{x} + \vec{C}_1$$

$$\therefore \vec{C}_1 = \langle 1,2,3\rangle$$

We find $\boxed{\vec{V}(t) = \langle 1+\tfrac{1}{2}t^2, 2, 3\rangle}$

$$\vec{V}(t) = \frac{d\vec{r}}{dt} \Rightarrow \vec{r}(t) = \langle t + \tfrac{1}{6}t^3, 2t, 3t\rangle + \vec{C}_2$$

But, $\vec{r}(0) = (0,0,0) = \langle 0,0,0\rangle + \vec{C}_2 \quad \therefore \underline{\vec{C}_2 = 0}$.

Hence, $\boxed{\vec{r}(t) = \langle t + t^3/6, 2t, 3t\rangle}$

Problem 10 (5pts) Show $\frac{d}{dt}[\vec{A}\bullet\vec{B}] = \frac{d\vec{A}}{dt}\bullet\vec{B} + \vec{A}\bullet\frac{d\vec{B}}{dt}$. You may assume $\vec{A},\vec{B}$ are vectors in two-dimensions.

$$\frac{d}{dt}(\vec{A}\cdot\vec{B}) = \frac{d}{dt}\left(\sum_{i=1}^{n} A_i B_i\right) \quad : \text{def}^{\underline{n}} \text{ of dot-product}$$

$: \text{linearity of } \frac{d}{dt}$

$$= \sum_{i=1}^{n} \frac{d}{dt}(A_i B_i) \quad \text{usual product rule}$$

$$= \sum_{i=1}^{n}\left(\frac{dA_i}{dt}B_i + A_i\frac{dB_i}{dt}\right) = \sum \frac{dA_i}{dt}B_i + \sum A_i\frac{dB_i}{dt}$$

$$= \frac{d\vec{A}}{dt}\cdot\vec{B} + \vec{A}\cdot\frac{d\vec{B}}{dt}$$

Problem 11 (5pts) Calculate, and simplify as much as possible, the following derivative:

$$\frac{d}{dt}[\vec{r}\bullet(\vec{r}\,'\times\vec{r}\,'')] = \frac{d\vec{r}}{dt}\cdot(\vec{r}\,'\times\vec{r}\,'') + \vec{r}\cdot\frac{d}{dt}(\vec{r}\,'\times\vec{r}\,'')$$

$$= \vec{r}\,'\cdot(\underbrace{\vec{r}\,'\times\vec{r}\,''}_{\substack{\text{perpendicular}\\ \text{to } \vec{r}\,' \text{ hence}\\ \text{dots to zero.}}}) + \vec{r}\cdot(\underbrace{\vec{r}\,''\times\vec{r}\,''}_{\text{zero}} + \underbrace{\vec{r}\,'\times\vec{r}\,'''}_{\text{remains}})$$

$$\therefore \boxed{\frac{d}{dt}\left[\vec{r}\cdot(\vec{r}\,'\times\vec{r}\,'')\right] = \vec{r}\cdot(\vec{r}\,'\times\vec{r}\,''') = \vec{r}\cdot\left(\frac{d\vec{r}}{dt}\times\frac{d^3\vec{r}}{dt^3}\right)}$$

Problem 12 (4pts) Parametrize the ellipse $x^2/a^2 + y^2/b^2 = 1$ found in the $z = 3$ plane.

$$x = a\cos t, \quad y = b\sin t, \quad z = 3 \qquad \text{or} \qquad \boxed{\vec{r}(t) = \langle a\cos t,\ b\sin t,\ 3\rangle}$$

Problem 13 (4pts) Find the equation of the sphere $x^2 + y^2 + z^2 = R^2$ in cylindrical coordinates.

$x^2+y^2=r^2$ and z is a cylindrical coord.

thus $\boxed{r^2+z^2=R^2}$

Problem 14 (6pts) Suppose $\langle 4, 7\rangle = s\vec{A} + t\vec{B}$ where $\vec{A} = \langle 1, 1\rangle$ and $\vec{B} = \langle 1, -1\rangle$. Find s and t.

$$\underbrace{\langle 4,7\rangle \cdot \vec{A}}_{11} = (s\vec{A}+t\vec{B})\cdot\vec{A} = s\vec{A}\cdot\vec{A} + \cancel{t\vec{B}\cdot\vec{A}}^{\,0} = s(2)$$

$$\Rightarrow 11 = 2s \quad \therefore \quad \boxed{s = 11/2}$$

$$\underbrace{\langle 4,7\rangle \cdot \vec{B}}_{-3} = (s\vec{A}+t\vec{B})\cdot\vec{B} = \cancel{s\vec{A}\cdot\vec{B}}^{\,0} + t\vec{B}\cdot\vec{B} = t(2)$$

$$\Rightarrow -3 = 2t \quad \therefore \quad \boxed{t = -3/2}$$

Problem 15 (5pts) Let $\vec{A}, \vec{C}$ be nonzero, non-colinear vectors. Let γ be a curve parametrized by:

$$\vec{\gamma}(t) = \vec{r}_o + f(t)\vec{A} + g(t)\vec{C}$$

for $t \in \mathbb{R}$ where $f, g : \mathbb{R} \to \mathbb{R}$ are smooth functions and $\vec{r}_o$ is a constant vector. Find the torsion of γ.

$$\frac{d\vec{r}}{dt} = \frac{df}{dt}\vec{A} + \frac{dg}{dt}\vec{C}$$

$$\vec{a} = \frac{d^2\vec{r}}{dt^2} = f''\vec{A} + g''\vec{C} = a_T T + a_N N$$

we can argue γ lies in plane parametrized by $\vec{r}(u,v) = \vec{r}_o + u\vec{A} + v\vec{B}$.

I'll be interested to see if anyone saw how...