Please work the problems in the white space provided and clearly box your solutions. You are allowed one 3″ × 5″ notecard. Enjoy! Each problem is worth 10pts.

Problem 1 Convert $f(x,y,z) = \sin(z^3) + \dfrac{x^4}{x^2+y^2}$ to cylindrical coordinates and calculate ∇f in terms of the cylindrical coordinate frame. (leave answer in terms of $\hat{r}, \hat{\theta}, \hat{z}$ and the cylindrical coordinates r, θ, z)

$$f(r,\theta,z) = \sin(z^3) + \frac{1}{r^2}(r\cos\theta)^4 = \underline{\sin(z^3) + r^2\cos^4\theta}.$$

$$\nabla f = \hat{r}\underbrace{\frac{\partial f}{\partial r}}_{2r\cos^4\theta} + \frac{1}{r}\hat{\theta}\underbrace{\frac{\partial f}{\partial \theta}}_{-4r^2\cos^3\theta\sin\theta} + \hat{z}\underbrace{\frac{\partial f}{\partial z}}_{3z^2\cos(z^3)} = \boxed{(2r\cos^4\theta)\hat{r} - (4r\cos^3\theta\sin\theta)\hat{\theta} + 3z^2\cos(z^3)\,\hat{z}}$$

Problem 2 Find the multivariate power series expansion of $f(x,y) = \dfrac{1}{1-x^2-y^2}$ centered about $(0,0)$ to fourth order. Classify the critical point $(0,0)$ as a local min., max. or saddle:

$$f(x,y) = \frac{1}{1-(x^2+y^2)} = 1 + x^2+y^2 + (x^2+y^2)^2 + \cdots$$ geometric series!

$$\boxed{f(x,y) = 1 + x^2 + y^2 + x^4 + 2x^2y^2 + y^4 + \cdots}$$

We see $f_{xx}(0,0) = f_{yy}(0,0) = 2$ and $f_{xy}(0,0) = 0$ hence $D = (2)(2) - 0 = 4 > 0$ and $f_{xx}(0,0) > 0$ $\therefore$ $f(0,0) = 1$ is local minimum.

Problem 3 Find all critical points of $f(x,y,z) = x^2y - xy + y^2$ and classify each by the second derivative test either as a local minimum, maximum or saddle:

$\nabla f = \langle 2xy - y,\ x^2 - x + 2y\rangle = \langle 0, 0\rangle$ for critical pts.

$(2x-1)y = 0 \implies x = 1/2$ or $y = 0$

$x^2 - x + 2y = 0$

If $\underline{x = 1/2}$ then $\frac{1}{4} - \frac{1}{2} + 2y = 0 \hookrightarrow 2y = \frac{1}{2} - \frac{1}{4} = \frac{1}{4} \Rightarrow \underline{y = \frac{1}{8}}$.

If $\underline{y = 0}$ then $x^2 - x = x(x-1) = 0$ $\therefore$ $\underline{x = 0}$ or $\underline{x = 1}$.

We find three critical pts. $(1/2, 1/8)$, $(0,0)$, $(1,0)$.

$f_{xx} = 2y$, $f_{xy} = 2x - 1$, $f_{yy} = 2$ thus $D = 4y - (2x-1)^2$

	D	fxx	conclusion
(1/2, 1/8)	1/2	1/4	local min.
(0, 0)	−1		Saddle pt.
(1, 0)	−1		Saddle pt.

$$\overbrace{9x^2 + 4y^2}^{g(x,y)} = 36$$

Problem 4 A ninja hound is caged in an region with boundary $9x^2 + 4y^2 = 36$. Once his energy reaches a level of 5 he can change form into an eagle and escape by flying over the ellipse-shaped fence. Given that his energy level is $f(x, y) = 3 + xy$ is there any place(s) in the cage which will allow him the energy to escape?
(please justify your answer by the method of Lagrange Multipliers and multivariate calculus, guessing the answer is only worth 20% credit)

$\nabla f = \langle y, x \rangle = \langle 0, 0 \rangle \Rightarrow x = 0, y = 0$ for local critical pt.

$\underline{f(0,0) = 3}$.

Now search the boundary for extrema by Lagrange's Method,

$$\nabla f = \lambda \nabla g$$

$$\langle y, x \rangle = \lambda \langle 18x, 8y \rangle$$

$y = 18\lambda x \longrightarrow \lambda = \dfrac{y}{18x} = \dfrac{x}{8y}$

$x = 8\lambda y$

$\therefore 8y^2 = 18x^2$

or $4y^2 = 9x^2$

Hence, $9x^2 + 4y^2 = 9x^2 + 9x^2 = 36 \quad \therefore 18x^2 = 36$

$x^2 = 2$

$\underline{x = \pm\sqrt{2}}$.

Hence, $y^2 = \frac{9}{4}x^2 = \frac{9}{4}(\pm\sqrt{2})^2 = \frac{9}{2}$

$\therefore y = \frac{\pm 3}{\sqrt{2}}$

We find 4 possible extremal points

$(\sqrt{2}, 3/\sqrt{2}),\ (\sqrt{2}, -3/\sqrt{2}),\ (-\sqrt{2}, 3/\sqrt{2}),\ (-\sqrt{2}, -3/\sqrt{2})$

Observe,

$$f\left(\pm\sqrt{2}, \frac{\pm 3}{\sqrt{2}}\right) = 3 + (\pm\cancel{\sqrt{2}})\left(\frac{\pm 3}{\cancel{\sqrt{2}}}\right) = \underline{6}.$$

$$f\left(\pm\sqrt{2}, \frac{\mp 3}{\sqrt{2}}\right) = 3 + (\pm\sqrt{2})\left(\frac{\mp 3}{\sqrt{2}}\right) = 3 - 3 = \underline{0}.$$

YES, near either $(\sqrt{2}, 3/\sqrt{2})$ or $(-\sqrt{2}, -3/\sqrt{2})$ the hound may transform to an eagle and escape.

Problem 5 Suppose the mass per unit area is given by $\sigma(x, y) = x^2$. Find the center of mass for the rectangular region $R = [0, 1] \times [0, 1]$.

$$M = \int_0^1\int_0^1 x^2\,dx\,dy = \int_0^1 x^2 dx \int_0^1 dy = \left(\tfrac{1}{3}\right)(1) = \underline{\tfrac{1}{3}}.$$

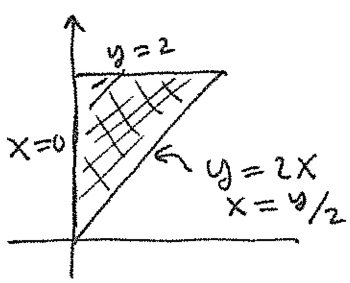
$$x_{cm} = \frac{1}{M}\iint_R x\sigma\,dA = 3\int_0^1\int_0^1 x^3\,dx\,dy = 3\int_0^1 \frac{dy}{4} = \underline{\frac{3}{4}}.$$

$$y_{cm} = \frac{1}{M}\iint_R y\sigma\,dA = 3\int_0^1\int_0^1 x^2 y\,dx\,dy = 3\int_0^1 x^2 dx \int_0^1 y\,dy$$
$$= 3\left(\tfrac{1}{3}\right)\left(\tfrac{1}{2}\right)$$
$$= \tfrac{1}{2}.$$ (not surprising given the symmetry here.)

Hence, c.o.m. is at $\boxed{(3/4, 1/2)}$

Problem 6 Calculate $\int_0^1 \int_{2x}^2 e^{y^2}\,dy\,dx$

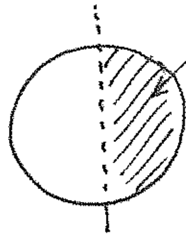
<u>cannot</u> integrate $\int e^{y^2} dy$ directly
we have to <u>swap</u> bounds.



note $0 \le y \le 2$
$0 \le x \le y/2$

$$\int_0^1\int_{2x}^2 e^{y^2}\,dy\,dx = \int_0^2\int_0^{y/2} e^{y^2}\,dx\,dy = \int_0^2 \left(xe^{y^2}\Big|_0^{y/2}\right)dy$$
$$= \int_0^2 \frac{y}{2} e^{y^2}\,dy$$
$$= \frac{1}{4} e^{y^2}\Big|_0^2$$
$$= \boxed{\frac{1}{4}(e^4 - 1)}$$

Problem 7 Suppose $R = \{(x,y) \mid 0 \leq x^2+y^2 \leq 9, \text{ and } x \geq 0\}$. Calculate $\iint_R \underbrace{\sin(x^2+y^2)\,dA}_{\sin(r^2)\, r\, dr\, d\theta}$



$R: \quad 0 \leq r \leq 3$

$\quad\;\; -\pi/2 \leq \theta \leq \pi/2$

$$\iint_R \sin(x^2+y^2)\,dA = \int_0^3 \int_{-\pi/2}^{\pi/2} r\sin(r^2)\,d\theta\,dr$$

$$= \pi \int_0^3 r\sin(r^2)\,dr$$

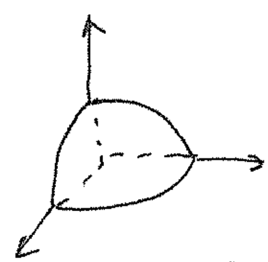
$$= -\frac{\pi}{2}\cos(r^2)\Big|_0^3$$

$$= -\frac{\pi}{2}\big(\cos(9) - \cos(0)\big) = \boxed{\frac{\pi}{2}\big(1-\cos(9)\big)}$$

Problem 8 Let B be part the ball of radius R centered at the origin $(x^2+y^2+z^2 \leq R^2)$ such that $x \geq 0$ and $y \geq 0$ and $z \geq 0$. Suppose $\underbrace{\delta(x,y,z) = \frac{1}{1+x^2+y^2+z^2}}_{\delta = \frac{1}{1+\rho^2}}$ is the mass-density $\frac{dm}{dV}$ of B. Calculate the mass of B :

note $\frac{dm}{dV} = \delta \Rightarrow dm = \delta\, dV$ hence

$$M = \iiint_B \delta\, dV = \iiint_B \frac{\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta}{1+\rho^2}$$



$0 \leq \theta, \phi \leq \pi/2$

$0 \leq \rho \leq R.$

$$= \int_0^R \int_0^{\pi/2} \int_0^{\pi/2} \left(\frac{\rho^2}{1+\rho^2}\right)\sin\phi\, d\phi\, d\theta\, d\rho$$

$$= \int_0^R \left(\frac{1+\rho^2-1}{1+\rho^2}\right)d\rho \int_0^{\pi/2} \sin\phi\, d\phi \int_0^{\pi/2} d\theta$$

$$= \left(\int_0^R \left(1 - \frac{1}{1+\rho^2}\right)d\rho\right)\left(-\cos\phi\Big|_0^{\pi/2}\right)\left(\frac{\pi}{2}\right)$$

$$= \left(R - \tan^{-1}(R)\right)\left(-\cos\left(\frac{\pi}{2}\right) + \cos(0)\right)\left(\frac{\pi}{2}\right)$$

$$= \boxed{\frac{\pi}{2}\left(R - \tan^{-1}(R)\right)}$$

Problem 9 Calculate the volume of the ellipsoid $x^2/a^2 + y^2/b^2 + z^2/c^2 \le 1$ where $a, b, c > 0$.

Let $x = au$, $y = bv$, $z = cw$ thus

$$x^2/a^2 + y^2/b^2 + z^2/c^2 = u^2+v^2+w^2 \le 1$$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \det\begin{bmatrix} \partial_u x & \partial_v x & \partial_w x \\ \partial_u y & \partial_v y & \partial_w y \\ \partial_u z & \partial_v z & \partial_w z \end{bmatrix} = \det\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = abc.$$

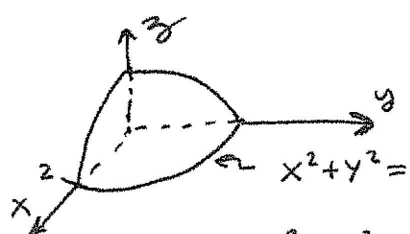
Hence, by change of variables theorem,

$$\iiint_{x^2/a^2+y^2/b^2+z^2/c^2\le 1} dx\,dy\,dz = \iiint_{u^2+v^2+w^2\le 1} abc\,du\,dv\,dw = abc\overbrace{\iiint_{u^2+v^2+w^2\le 1} du\,dv\,dw}^{\text{volume of unit-sphere}} = abc\left(\frac{4\pi}{3}\right)$$

$\therefore \boxed{\frac{4\pi}{3}abc}$

Problem 10 Calculate $\displaystyle\int_0^2\int_0^{\sqrt{4-x^2}}\int_0^{\sqrt{4-x^2-y^2}} (zx^2+zy^2+z^3)\,dz\,dy\,dx. = \mathcal{I}$

$$\begin{aligned} 0 &\le z \le \sqrt{4-x^2-y^2} \\ 0 &\le y \le \sqrt{4-x^2} \\ 0 &\le x \le 2 \end{aligned} \longrightarrow \quad x^2+y^2+z^2 = 4 \text{ for } 0 \le x \le 2.$$



$x^2+y^2=4,\ z=0$

$0 \le \theta, \phi, \ \pi/2$

$0 \le \rho \le 2.$

Note, $zx^2+zy^2+z^3 = z(x^2+y^2+z^2) = z\rho^2 = \rho^3\cos\phi$

$dV = \rho^2 \sin\phi\, d\phi\, d\theta\, d\rho$

$$\begin{aligned}
\mathcal{I} &= \int_0^2\int_0^{\pi/2}\int_0^{\pi/2} \rho^5\cos\phi\,\sin\phi\,d\phi\,d\theta\,d\rho \\
&= \left(\int_0^2 \rho^5\,d\rho\right)\left(\int_0^{\pi/2} d\theta\right)\left(\int_0^{\pi/2}\cos\phi\sin\phi\,d\phi\right) \\
&= \left(\frac{2^6}{6}\right)\left(\frac{\pi}{2}\right)\left(\frac{1}{2}\sin^2\phi\right)\Big|_0^{\pi/2} \\
&= \frac{2^4}{6}\cdot\pi\cdot 1 \\
&= \boxed{\frac{8\pi}{3}}
\end{aligned}$$

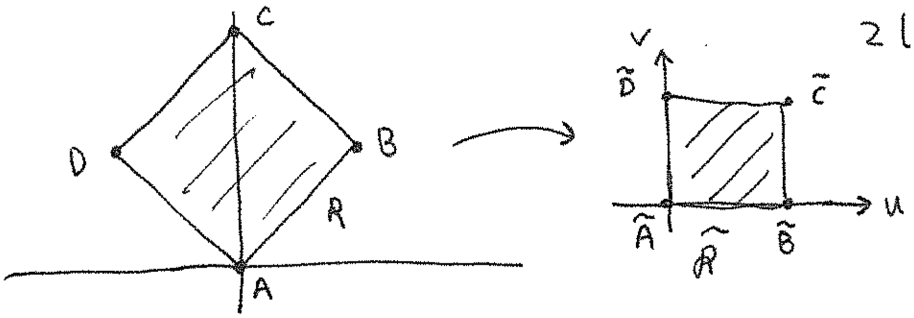
Problem 11 Let R be the diamond with vertices $(0,0),(1,1),(0,2)$ and $(-1,1)$. Change coordinates to calculate the integral:
note: $\int \sinh(u)\,du = \cosh(u) + c$.

$$\iint_R \sinh\underbrace{\left(\frac{x+y}{2}\right)}_{u} \sin\underbrace{\left(\frac{y-x}{2}\right)}_{v} dA$$

$$2u = x+y$$
$$2v = y-x$$

$2(u+v) = 2y \quad \therefore \quad y = u+v$

$2(u-v) = 2x \quad \therefore \quad x = u-v$



(x,y)	(u,v)
$(0,0)$	$(0,0)$
$(1,1)$	$(1,0)$
$(0,2)$	$(1,1)$
$(-1,1)$	$(0,1)$

$$\iint_R \sinh\left(\frac{x+y}{2}\right) \sin\left(\frac{y-x}{2}\right) dA = \int_0^1\int_0^1 \sinh(u)\sin(v)\left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

$$= \int_0^1\int_0^1 \sinh(u)\sin(v)\left|\det\begin{bmatrix} x_u & x_v \\ y_u & y_v\end{bmatrix}\right| du\,dv$$

$$= \int_0^1\int_0^1 \sinh(u)\sin(v)\underbrace{\det\begin{bmatrix} 1 & -1 \\ 1 & 1\end{bmatrix}}_{2} du\,dv$$

$$= 2\int_0^1 \sinh(u)\,du \int_0^1 \sin(v)\,dv$$

$$= 2\left(\cosh(u)\Big|_0^1\right)\left(-\cos(v)\Big|_0^1\right)$$

$$= 2(\cosh(1)-1)(-\cos(1)+\cos(0))$$

$$= \boxed{2(\cosh(1)-1)(1-\cos(1))}$$