Please show your work. Enjoy! There are at least 150pts to earn here. Copying answers and steps is strictly forbidden. Evidence of copying results in zero for copied and copier. Working together is encouraged, share ideas not calculations. Explain your steps. Thanks!

Problem 1 Give the correct solution for Problem 99. (fix my solution which is so very wrong currently)

Problem 2 find a solution of $x^2 - 65y^2 = 1$ by solving $x^2 - 65y^2 = -1$. Note the latter equation is easily solved by setting $y = 1$. Follow method of Problem 74.

Problem 3 Suppose $d \in \mathbb{Z}$ is square-free and $a, b, x, y \in \mathbb{Z}$. Prove the following:
If $a + b\sqrt{d} = x + y\sqrt{d}$ then $a = x$ and $b = y$.

Problem 4 Find 4 distinct units in $\mathbb{Z}[\sqrt{-5}]$. How many units are there in $\mathbb{Z}[\sqrt{-5}]$?

Problem 5 Consider $p = 13$.

(a) is p a prime in $\mathbb{Z}$?
(b) is p a prime in $\mathbb{Q}$?
(c) is p a prime in $\mathbb{Z}[i]$?
(d) is p a prime in $\mathbb{Z}[i, j, k]$ (this is just half the Hurwitz integers)

Problem 6 Consider $\mathbb{Z}[i]$. Use the Euclidean algorithm to calculate the $\gcd(10 + 7i, 2 - 3i)$ and to find $a, b \in \mathbb{Z}[i]$ for which: $a(10 + 7i) + b(2 - 3i) = \gcd(10 + 7i, 2 - 3i)$.

Problem 7 Suppose $a, b \in \mathbb{Z}$ and $a + ib$ is a Gaussian prime. Prove $a - ib$ is a Gaussian prime.

Problem 8 Find a Gaussian prime factorization of 102 in $\mathbb{Z}[i]$

Problem 9 Give an example of how the prime divisor property fails for non-real primes in the Hurwitz integers. That is, find a Hurwitz prime p and Hurwitz integers α, β for which $p|(\alpha\beta)$ yet p fails to divide α or β. (assume all divisors need to be right divisors as defined on page 6 of Lecture 15)

Problem 10 Defining new number systems by matrix representations is fun. Consider the following:

$$z = a + bj$$

where $a, b \in \mathbb{R}$ and we define the formal symbol j by what follows:

$$M(a + bj) = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$$

and the multiplication of $z = a + bj$ and $w = c + jd$ is implicitly defined by $M(zw) = M(z)M(w)$ where the multiplication of $M(z)$ and $M(w)$ is simply that of 2×2 matrices. Given this set-up, answer the following:

(a) express $(a + bj)(c + jd) = c_1 + c_2 j$ for appropriate $c_1, c_2 \in \mathbb{R}$
(b) what simple rule does j follow ?
(c) If $N = \{x + yj \mid x, y \in \mathbb{R}\}$ then find all solutions in N to the quadratic equation $z^2 + \alpha z = 0$ where $\alpha = a + jb \in N$. It may be helpful to set $z = x + jy$ for $x, y \in \mathbb{R}$.