Closed book, no matrix operations calculator. Remember, you must justify your answers. There are 30 extra points, this means you are free to skip 30pts of problems. Use your time wisely.

Problem 1 [20pts] Let $A = \begin{bmatrix} 1 & 0 & 2 \\ -2 & 1 & 4 \\ -3 & 1 & 3 \end{bmatrix}$ find A^{-1}

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ -2 & 1 & 4 & 0 & 1 & 0 \\ -3 & 1 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[r_3+3r_1]{r_2+2r_1} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 8 & 2 & 1 & 0 \\ 0 & 1 & 9 & 3 & 0 & 1 \end{array}\right] \xrightarrow{r_3-r_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 1 & 0 & 0 \\ 0 & 1 & 8 & 2 & 1 & 0 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right]$$

$$\xrightarrow[r_2-8r_3]{r_1-2r_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 2 & -2 \\ 0 & 1 & 0 & -6 & 9 & -8 \\ 0 & 0 & 1 & 1 & -1 & 1 \end{array}\right] \quad \therefore \quad A^{-1} = \begin{bmatrix} -1 & 2 & -2 \\ -6 & 9 & -8 \\ 1 & -1 & 1 \end{bmatrix}.$$

We can check, $AA^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. ✓

Problem 2 [10pts] Let $\beta = \{(1,-2,-3), (0,1,1), (2,4,3)\}$. Also, let $v = (1,6,6)$.Find the coordinate vector of v with respect to the β basis; that is, find $[v]_\beta$.

$$[v]_\beta = [\beta]^{-1} v = \begin{bmatrix} -1 & 2 & -2 \\ -6 & 9 & -8 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 6 \\ 6 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Btw, if you didn't know why:

$$\beta = \{v_1, v_2, v_3\} \rightarrow v = x_1 v_1 + x_2 v_2 + x_3 v_3 = \underbrace{\left[v_1 \middle| v_2 \middle| v_3 \right]}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$[\beta][v]_\beta = v$$

$$\therefore [v]_\beta = [\beta]^{-1} v.$$

Problem 3 [10pts] Let $W = \text{span}\{(1,3,2,3), (0,1,2,2)\}$. Find a basis for $W^\perp$.

$$W^\perp = \{x \in \mathbb{R}^4 \mid x \cdot w = 0 \ \forall w \in W\}$$

$$= \text{Null}\begin{bmatrix} 1 & 3 & 2 & 3 \\ 0 & 1 & 2 & 2 \end{bmatrix}$$

$$= \text{Null}\begin{bmatrix} 1 & 0 & -4 & -3 \\ 0 & 1 & 2 & 2 \end{bmatrix} \Rightarrow x \in W^\perp \text{ has } \begin{array}{l} x_1 = 4x_3 + 3x_4 \\ x_2 = -2x_3 - 2x_4 \end{array}$$

thus $x = (4x_3 + 3x_4, -2x_3 - 2x_4, x_3, x_4)$

$= x_3(4, -2, 1, 0) + x_4(3, -2, 0, 1)$

$$\therefore \boxed{W^\perp = \text{span}\{(4,-2,1,0), (3,-2,0,1)\}}$$

Problem 4 [10pts] Find the volume of a parallel piped with edges $(1,2,3),(0,2,2)$ and $(0,0,3)$.

$$\text{Vol} = \left| \det\begin{bmatrix} 1 & 0 & 0 \\ 2 & 2 & 0 \\ 3 & 2 & 3 \end{bmatrix}\right| = |1\cdot 2\cdot 3| = \boxed{6}$$

Problem 5 [10pts] Find the solution set of the system of equations $x+4y+z=0$ and $y-3z=0$.

$y = 3z$ & $x = -4y - z = -4(3z) - z = -13z$

Thus $\underline{\{(-13z, 3z, z) \mid z \in \mathbb{R}\} = \text{span}\{(-13,3,1)\}}$.

Sol$^{\underline{n}}$ set.

Or, $\text{Null}\begin{bmatrix} 1 & 4 & 1 \\ 0 & 1 & -3 \end{bmatrix} = \text{span}\{(-13, 3, 1)\}$.

Problem 6 [10pts] Let $T(x,y,z) = (x+4y+z, y-3z)$. Find the standard matrix $[T]$ for T and find a basis for Ker(T). (hint: the previous problem is related)

$$T(x,y,z) = \begin{bmatrix} x+4y+z \\ y-3z \end{bmatrix} = \begin{bmatrix} 1 & 4 & 1 \\ 0 & 1 & -3 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} \therefore \boxed{[T] = \begin{bmatrix} 1 & 4 & 1 \\ 0 & 1 & -3 \end{bmatrix}}$$

$T: \mathbb{R}^3 \longrightarrow \mathbb{R}^2 \hookrightarrow [T] \in \mathbb{R}^{2\times 3}$

the $\text{Ker}(T) = \text{Null}\begin{bmatrix} 1 & 4 & 1 \\ 0 & 1 & -3 \end{bmatrix}$ has basis $\boxed{\{(-13, 3, 1)\}}$.

Problem 7 [10pts] Fun facts:

$$\text{rref}\underbrace{\left[\begin{array}{ccc|cc} 1 & 2 & 3 & 6 & 3 \\ -2 & 4 & 2 & 4 & 2 \\ -3 & 1 & 7 & 5 & -2 \end{array}\right]}_{V_1,\ V_2,\ V_3,\ V_4,\ V_5} = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \quad \text{rref}\underbrace{\left[\begin{array}{cc|ccc} 3 & 6 & 3 & 2 & 1 \\ 2 & 4 & 2 & 4 & -2 \\ -2 & 5 & 7 & 1 & -3 \end{array}\right]}_{V_5,\ V_4,\ V_3,\ V_2,\ V_1} = \left[\begin{array}{cc|ccc} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

Notice I define V_1, V_2, V_3, V_4, V_5 in the calculations above. Given the data above, answer the following: (when I say answer, I mean "yes" or "no" followed by a brief sentence to explain why. Little if any additional calculation is needed to answer these if you understand the CCP)

(a.) is $V_4 \in \text{span}\{V_1, V_2, V_3\}$?

Yes, by CCP and given fun facts, $V_4 = V_1 + V_2 + V_3$.

(b.) is $V_2 \in \text{span}\{V_4, V_5\}$?

No, by CCP and right matrix, see $c_1 V_4 + c_2 V_5 = V_2$ has no sol$^{\underline{ns}}$.

Problem 8 [10pts] Suppose $A \in \mathbb{R}^{m \times p}$ and $B \in \mathbb{R}^{p \times n}$. Prove that $(AB)^T = B^T A^T$.

$$\begin{aligned}
((AB)^T)_{ji} = (AB)_{ij} &= \sum_{k=1}^{p} A_{ik} B_{kj} \\
&= \sum_{k=1}^{p} B_{kj} A_{ik} \\
&= \sum_{k=1}^{p} (B^T)_{jk} (A^T)_{ki} \\
&= \underbrace{(B^T A^T)_{ji}}_{\text{holds } \forall j, i} \quad \therefore \quad \underline{(AB)^T = B^T A^T}.
\end{aligned}$$

Problem 9 [10pts] Suppose A is symmetric and B is antisymmetric. Define $M = [A, B] = AB - BA$. Prove M is symmetric.

Suppose $A^T = A$ and $B^T = -B$. Consider,

$$\begin{aligned}
M^T = (AB - BA)^T = (AB)^T - (BA)^T &= B^T A^T - A^T B^T \\
&= -BA + AB \\
&= [A, B] = M. //
\end{aligned}$$

Problem 10 [10pts] Let $B \in \mathbb{R}^{n \times n}$ and define $T(A) = AB + BA$ for all $A \in \mathbb{R}^{n \times n}$. Show that T is a linear transformation.

$$\begin{aligned}
T(A_1 + cA_2) &= (A_1 + cA_2)B + B(A_1 + cA_2) \\
&= A_1 B + BA_1 + c(A_2 B + BA_2) \\
&= T(A_1) + cT(A_2)
\end{aligned}$$

Thus, by $c = 1$, T is additive, and by $A_1 = 0$, T is homogeneous.

Problem 11 [10pts] Let $T : V \to W$ and $S : W \to W$ be linear mappings of finite dimensional vector spaces V, W over $\mathbb{R}$. Suppose S is injective. Prove $T \circ S$ is injective.

CURSES. $T = 0 \Rightarrow T \circ S = 0$ certainly <u>NOT</u> <u>INJECTIVE</u>.

Suppose T was injective. If $T(S(x)) = 0 = T(0)$

then T injective $\Rightarrow S(x) = 0 = S(0) \Rightarrow \underline{x = 0}$.

So, we need injectivity of <u>both</u> T & S. //

Problem 12 [10pts] Is $W = \{(s+t, 3s-t, t, s) \mid s, t \in \mathbb{R}\}$ a subspace? Prove or disprove.

$$W = \text{span}\{(1,3,0,1), (1,-1,1,0)\} \leq \mathbb{R}^4.$$

YES, this is a subspace as it is a span.

Problem 13 [10pts] Is $W = \{(s+t, 3s-t, 2, 1) \mid s, t \in \mathbb{R}\}$ a subspace? Prove or disprove.

Notice $(0,0,0,0) \notin W$ $\therefore$ $\underline{W \text{ not a subspace.}}$

Problem 14 [10pts] Suppose $f(x) = (2x-3)^2$. Let $\beta = \{x^2, x, 1\}$ and find $[f(x)]_\beta$.

$$f(x) = 4x^2 - 12x + 9$$

$$\therefore \underline{[f(x)]_\beta = (4, -12, 9)}.$$

Can show lin dep.
$(x^2+4x-1) - (x^2+x) = \underline{3x-1}$.

Problem 15 [10pts] Is $\{x^2+x,\ 3x-1,\ x^2+4x-1\}$ a basis for $P_2 = \text{span}\{1, x, x^2\}$? Prove or disprove.

$$c_1(x^2+x) + c_2(3x-1) + c_3(x^2+4x-1) = 0$$

$$(c_1+c_3)x^2 + (c_1+3c_2+4c_3)x + (-c_2-c_3) = 0$$

$c_2 = -c_3$
$c_1 = -c_3$
$c_1 - 3c_3 + 4c_3 = c_1 + c_3 \neq 0$
∞ly many sol'n,

~~Thus $c_2 = -c_3$, $c_1 = -c_3$ and $c_1 - c_2 - c_3 = 0$~~

$c_1 + c_3 - c_3 = 0 \therefore \underline{c_1 = 0}$.

~~Moreover, $c_3 = -c_1 = 0$ & $c_2 = 0$~~

~~Consequently, $\beta = \{x^2+x, 3x-1, x^2+4x-1\}$ is LI.~~

~~But, $\dim(P_2) = 3$ $\therefore$ $\{x^2+x, 3x-1, x^2+4x-1\}$ is Basis.~~

~~Alternatively: Could show span $\beta \neq P_2$ as $\#\beta = 3$.~~

Problem 16 [10pts] Suppose A, B are square. Let $Ax = 0$ have only the $x = 0$ solution. Also, suppose $By = 0$ has only the $y = 0$ solution. Prove that $M = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right]$ is invertible.

$Ax = 0$ iff $x = 0$ $\Rightarrow$ A^{-1} exists.
$By = 0$ iff $y = 0$ $\Rightarrow$ B^{-1} exists.

$$\boxed{M^{-1} = \left[\begin{array}{c|c} A^{-1} & 0 \\ \hline 0 & B^{-1} \end{array}\right]} \text{ as } \underbrace{\left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right]\left[\begin{array}{c|c} A^{-1} & 0 \\ \hline 0 & B^{-1} \end{array}\right] = \left[\begin{array}{c|c} AA^{-1} & 0 \\ \hline 0 & BB^{-1} \end{array}\right] = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right]}_{\text{proves } \left[\begin{array}{c|c} A^{-1} & 0 \\ \hline 0 & B^{-1} \end{array}\right] = M^{-1}.}$$

Problem 17 [20pts] Let $T : P_2 \to P_2$ be defined by $T(f(x)) = \dfrac{df}{dx} + f(x)$. Let $\beta = \{1, x, x^2\}$ form the basis for P_2. Calculate $[T]_{\beta,\beta}$ and find the dimension of $\text{Ker}(T)$.

$$T(\underbrace{a+bx+cx^2}_{f(x)}) = b + 2cx + a + bx + cx^2$$

$$T(f(x)) = (b+a) + (2c+b)x + cx^2$$

$$[T(f(x))]_\beta = (b+a,\ 2c+b,\ c) = \begin{bmatrix} a+b \\ b+2c \\ c \end{bmatrix}$$

$$\Rightarrow \boxed{[T]_{\beta\beta} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}}$$

has rank $[T]_{\beta\beta} = 3$

also $\det [T]_{\beta\beta} = 1 \neq 0$

$\Rightarrow \text{Ker}[T]_{\beta\beta} = \{0\}$

$\therefore \boxed{\dim(\text{Ker}[T]_{\beta\beta}) = 0}$

Problem 18 [20pts] Let $W = \text{span}\{\underbrace{(1,0,-1)}_{v_1}, \underbrace{(2,-1,-1)}_{v_2}\}$. Find an orthonormal basis for W and calculate $\text{Proj}_W(a,b,c)$.

$$\underline{u_1 = \tfrac{1}{\sqrt{2}}(1,0,-1)}.$$

$$\tilde{u}_2 = v_2 - (v_2 \cdot u_1)u_1 = (2,-1,-1) - \tfrac{1}{2}(3)(1,0,-1)$$

$$\tilde{u}_2 = (2 - 3/2,\ -1,\ -1 + 3/2) = (1/2,\ -2/2,\ 1/2)$$

$$\tilde{u}_2 = \tfrac{1}{2}(1,-2,1) \hookrightarrow \underline{u_2 = \tfrac{1}{\sqrt{6}}(1,-2,1)}.$$

$$\begin{aligned}\text{Proj}_W(a,b,c) &= ((a,b,c)\cdot u_1)u_1 + ((a,b,c)\cdot u_2)u_2 \\ &= \tfrac{1}{2}(a-c)(1,0,-1) + \tfrac{1}{6}(a-2b+c)(1,-2,1) \\ &= \left(\tfrac{1}{2}(a-c) + \tfrac{1}{6}(a-2b+c),\ \tfrac{-1}{3}(a-2b+c),\ \tfrac{1}{2}(c-a) + \tfrac{1}{6}(a-2b+c)\right) \\ &= \left(\tfrac{2}{3}a - \tfrac{1}{3}b - \tfrac{1}{3}c,\ \tfrac{-1}{3}a + \tfrac{2}{3}b - \tfrac{1}{3}c,\ -\tfrac{1}{3}a - \tfrac{1}{3}b - \tfrac{1}{3}c\right)\end{aligned}$$

Problem 19 [10pts] You are given the following data about matrices A, B and C

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 3 & 3 & 3 \end{bmatrix}, \quad \text{rref}(B) = \begin{bmatrix} 1 & 6 & 8 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 3 & 0 \end{bmatrix}$$

Which pair of the above matrices can represent the same linear transformation T with respect to a different choice of basis ?

$\text{rank}(A) = 1$, $\text{rank}(B) = 1$ $\text{rank}(C) = 2$

$\therefore$ A & B can represent $T : V \longrightarrow V$ for different choices of basis.

$\text{rank}(T) = \text{rank}([T]_{\beta\gamma}) = \text{rank}(A) = \text{rank}(B)$.

Problem 20 [10pts] Let $W = \{A \in \mathbb{R}^{3\times 3} \mid \text{trace}(A) = 0,\ \&\ A^T = A\}$. Find an isomorphism from W to $P_n = \text{span}\{1, x, \ldots, x^n\}$ for appropriate n.

$$A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \quad \text{with } \text{trace}(A) = a + e + i = 0$$
$$\& \quad b = d,\ c = g,\ f = h$$

$$A = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & -a-d \end{pmatrix} \in W$$

$$\underline{\psi\begin{pmatrix} a & b & c \\ b & d & e \\ c & e & -a-d \end{pmatrix} = ax^4 + bx^3 + cx^2 + dx + e}.$$

Problem 21 [10pts] Let $A \in \mathbb{R}^{n\times n}$. Suppose A has eigenvalue 3. Prove A^2 has eigenvalue 9.

$\exists\, v \neq 0$ s.t. $Av = 3v$

$$\Rightarrow \quad AAv = A(3v) \quad \Rightarrow \quad A^2 v = 9v$$

$\therefore\ \lambda = 9$ is e-value for A^2.

Problem 22 [10pts] Let $\beta = \{u, v, w\}$ be an orthonormal basis for $\mathbb{R}^3$. Define:

$$A = uu^T + 2vv^T + 3ww^T.$$

$u^T u = 1$
$v^T v = 1$
$w^T w = 1$
others zero.

Show β is an eigenbasis for A. Also, calculate the determinant and trace of A.

$$Au = \cancel{A}\, uu^Tu + 2vv^Tu + 3ww^Tu = u. \Rightarrow \underline{\lambda_1 = 1}.$$
$$Av = 2vv^Tv = 2v \quad \therefore \quad \underline{\lambda_2 = 2}.$$
$$Aw = 3ww^Tw = 3w \quad \therefore \quad \underline{\lambda_3 = 3}.$$

$$\therefore \quad \det(A) = \lambda_1\lambda_2\lambda_3 = \boxed{6.}$$
$$\text{trace}(A) = \lambda_1 + \lambda_2 + \lambda_3 = 1 + 2 + 3 = \boxed{6.}$$

Problem 23 [10pts] Let $Q(x,y) = 4xy$. Find a formula for Q based on eigencoordinates. Describe $Q(x,y) = 1$ qualitatively, is this an ellipse, a parabola, a hyperbola, a plane? What is it? (you do not need to explicitly graph to solve this)

$$[Q] = \begin{bmatrix} 0 & 2 \\ 2 & 0 \end{bmatrix} \quad \det([Q]-\lambda I) = \det\begin{bmatrix} -\lambda & 2 \\ 2 & -\lambda \end{bmatrix} = \lambda^2 - 4 = (\lambda-2)(\lambda+2).$$

$$\therefore \quad Q(\bar{x}\bar{v}_1 + \bar{y}\bar{v}_2) = \underbrace{2\bar{x}^2 - 2\bar{y}^2 = 1}_{\text{hyperbola.}}$$

$\{\bar{v}_1, \bar{v}_2\}$ orthonormal e-basis with $\lambda_1 = 2$, $\lambda_2 = -2$

Problem 24 [10pts] Suppose $T : V \to V$ is a linear transformation on a two-dimensional vector space V with basis $\beta = \{v_1, v_2\}$. You are given that the complexification of T has:

$$T_{\mathbb{C}}(v_1 + iv_2) = 6v_1 + 2v_2 + i(8v_1 + 4v_2) \quad = T(v_1) + iT(v_2)$$

Find the matrix of T in the β basis; that is, find $[T]_{\beta,\beta}$. We have:

$$T(v_1) = 6v_1 + 2v_2 \quad \therefore \quad [T(v_1)]_\beta = (6, 2)$$
$$T(v_2) = 8v_1 + 4v_2 \quad \therefore \quad [T(v_2)]_\beta = (8, 4)$$

$$\therefore \quad \underline{[T]_{\beta\beta} = \begin{bmatrix} 6 & 8 \\ 2 & 4 \end{bmatrix}}.$$

Problem 25 [20pts] Suppose that A is a 9×9 block-diagonal square matrix:

$$A = \text{diag}\left(\begin{bmatrix} 3 & 1 & 0 & 0 \\ 0 & 3 & 1 & 0 \\ 0 & 0 & 3 & 1 \\ 0 & 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix} \right)$$

(a.) provide the eigenvalues and characteristic polynomial of A.

$\lambda_1 = 3$, $\lambda_2 = 4$ $\quad \det(A - \lambda I) = \underline{(3-\lambda)^4(4-\lambda)^5 = P(\lambda)}$.

(b.) give the dimension of each eigenspace of A

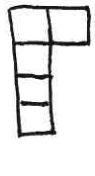
$\dim E_1 = 1$, $\dim E_2 = 4$

(c.) write down the minimal polynomial of A

$\underline{g(t) = (t-3)^4(t-4)^2}$. (see biggest blocks)

(d.) Let $K_2 = \text{Ker}(T - 4I)^5$ and define $T(v) = Av$ for all $v \in \mathbb{R}$. Let $N_2 = (T - 4I)|_{K_2}$. Give the Tableau which corresponds to N_2.

$\lambda_2 = 4$ has 4-evectors LI in E_2, and, one 2-cycle



$\star \Rightarrow \underline{\dim V = 3}$

Problem 26 [10pts] Suppose $\overbrace{\{v_1, v_2, v_3\}}^{*}$ and $\{w_1, w_2\}$ are LI. Consider $T : V \to W$ where $V = \text{span}\{v_1, v_2, v_3\}$ and $W = \text{span}\{w_1, w_2\}$. Assume T is a linear transformation. Fill in the last column with an appropriate number or comment.

$\dim V = \dim(\text{Ker}\, T) + \dim(\text{Range}(T))$

$\underline{\text{Note}}$: $\dim(\text{Range}(T)) \leq 2$ as $\text{Range}(T) \leq W$ and $\dim W \leq 2$.

Also, NOT POSSIBLE $\dim V = 3$

	dim(V)	dim($Ker(T)$)	dim($Range(T)$)
	1	1	0
	1	0	1
	2	0	2
	2	1	1
	2	2	0
*	3	0	~~3~~ BUT, THIS is NOT POSSIBLE!
	3	1	2
	3	2	1
	3	3	0

Problem 27 [10pts] Let S be the set of symmetric $n \times n$ real matrices and A be the set of real antisymmetric $n \times n$ matrices. Show that $\mathbb{R}^{n\times n}/S \approx A$. Challenge: find dim($S$) and dim($A$).

Let $\varphi(X) = \frac{1}{2}(X - X^T)$.

$\varphi(X) = 0 \Rightarrow (X - X^T)/2 = 0 \Rightarrow X = X^T \therefore \text{Ker}\,\varphi = S$.

Also, $\varphi(x)^T = \frac{1}{2}(x - x^T)^T = \frac{1}{2}(x^T - x) = -\varphi(x) \therefore \varphi(x) \in A$

Indeed, $y \in A \Rightarrow \varphi(y) = \frac{1}{2}(y - y^T) = \frac{1}{2}(y + y) = y \therefore \varphi(\mathbb{R}^{n\times n}) = A$.

By 1st isomorphism thm,

$$\mathbb{R}^{n\times n}/\text{Ker}\,\varphi \approx \varphi(\mathbb{R}^{n\times n}) \longrightarrow \underline{\mathbb{R}^{n\times n}/S \approx A}.$$

Problem 28 [10pts] Let $A \in \mathbb{R}^{n\times n}$ be symmetric. Prove A has real eigenvalues. (you are free to use results we proved about the complexification of A including the Hermitian inner product for $\mathbb{C}^n$)

Let $\lambda \in \mathbb{C}$ be e-value for A acting on $\mathbb{C}^n$; $Ax = \lambda x$ for $x \neq 0$. Consider, $\langle cv, w\rangle = c\langle v, w\rangle$ & $\langle v, cw\rangle = \bar{c}\langle v, w\rangle$.

Also, $\langle v, A^{\dagger} w\rangle = \langle Av, w\rangle$ where $A^{\dagger} = (\bar{A})^T$. Consider,

$$\langle Ax, x\rangle = \langle x, (\bar{A})^T x\rangle = \langle x, Ax\rangle \quad \text{as } \underline{\bar{A} = A \text{ \& } A^T = A}.$$

$$\Rightarrow \langle \lambda x, x\rangle = \langle x, \lambda x\rangle$$

$$\Rightarrow \lambda\langle x, x\rangle = \bar{\lambda}\langle x, x\rangle$$

$$\Rightarrow \boxed{\lambda = \bar{\lambda}} \quad \text{as } \langle x, x\rangle \neq 0 \text{ for } x \neq 0.$$