The solutions should be written neatly on lined or unlined paper with the work clearly labeled. Do not omit scratch work. I need to see all steps. Skipping details will result in a loss of credit. Thanks and enjoy.

Problem 1 [10pts] (like § 1.3 # 82) Notice that there are only a few possible forms for the rref of a 2×3 matrix:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & * & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & * & * \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & * \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

where $*$ denotes an arbitrary value. Use this notation to list out the possible forms of the rref of an arbitrary 4×2 matrix.

Problem 2 [10pts] Show that

$$A = \begin{bmatrix} 3 & 1 & -5 & 11 \\ 2 & 1 & -4 & 8 \\ 3 & 0 & -3 & 10 \end{bmatrix} \quad \text{has } rref(A) = \begin{bmatrix} 1 & 0 & -1 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Use the calculation above to

(i.) find the general solution of $Au = 0$.

(ii.) argue that $3x + y - 5z = 11, 2x + y - 4z = 8, 3x - 3z = 10$ has no solution.

Problem 3 [5pts] (§ 1.1 # 83) A system of linear equations is called **underdetermined** if there are less equations than variables. What can you say about the solution of an undetermined sytem of linear equations?

Problem 4 [15pts] (§ 1.1 # 84) An system of linear equations is called **overdetermined** if there are more equations than variables. Give examples of overdetermined systems of linear equations with

(i.) a unique solution,

(ii.) infinitely many solutions,

(iii.) no solutions.

Problem 5 [5pts] (§ 1.3 # 79) Prove that if $A \in \mathbb{R}^{m \times n}$ then $Ax = 0$ is a consistent system of equations.

Problem 6 [20pts] (§ 1.4 # 87,88) Let $c \in \mathbb{R}$. Claim: If $A \in \mathbb{R}^{m \times n}$ and $u, v \in \mathbb{R}^{n \times 1}$ are solutions to $Ax = 0$ then $u + v$ and cu are solutions of $Ax = 0$. Prove or disprove by giving counter-examples.

Problem 7 [20pts] (§ 1.4 # 87,88) Let $c \in \mathbb{R}$. Claim: If $A \in \mathbb{R}^{m \times n}$, $b \in \mathbb{R}^{n \times 1}$ and $u, v \in \mathbb{R}^{n \times 1}$ are solutions to $Ax = b$ then $u + v$ and cu are solutions of $Ax = b$. Prove or disprove by giving counter-examples.

Problem 8 [10pts] (§ 1.4 # 48) Find a quadratic polynomial whose graph contains the points $(-2, 33)$, $(2, -1)$ and $(3, -8)$.

Problem 9 [10pts] Find a cubic polynomial whose graph contains the points $(1, 2)$, $(2, 2)$, $(3, 2)$ and $(4, 2)$.

Problem 10 [15pts] (Anton §1.2 #24) Solve the following nonlinear system for x, y, z,

$$\frac{1}{x} + \frac{2}{y} - \frac{4}{z} = 1$$
$$\frac{2}{x} + \frac{3}{y} + \frac{8}{z} = 0$$
$$-\frac{1}{x} + \frac{9}{y} + \frac{10}{z} = 5.$$

Problem 11 [15pts] (Anton §1.2 #20) Let $\alpha, \beta, \gamma \in \mathbb{R}$ such that $0 \leq \alpha \leq 2\pi$, $0 \leq \beta \leq 2\pi$ and $0 \leq \gamma \leq \pi$. Solve

$$2\sin\alpha - \cos\beta + 3\tan\gamma = 3$$
$$4\sin\alpha + 2\cos\beta - 2\tan\gamma = 2$$
$$-\sin\alpha - 5\cos\beta + 5\tan\gamma = 9.$$

Problem 12 [10pts] Let $Z = \begin{bmatrix} a & c \\ b & d \end{bmatrix}$. Recall that there are no real solutions of the equation $x^2 + 1 = 0$. The same is not true for matrices. Show that $J = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ solves $Z^2 + I = 0$. Let $a, b, c, d \in \mathbb{R}$ and define $Z = aI + bJ$ and $W = cI + dJ$. Calculate ZW and interpret what this calculation represents.

Problem 13 [10pts] (§ 2.1 # 9 of Lay) Let $A = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & -5 \\ 3 & k \end{bmatrix}$. Which value of k makes $AB = BA$? Show your work.

Problem 14 [20pts] (Lay Chapter 2 supp. # 11) Suppose that $AB = \begin{bmatrix} 5 & 4 \\ -2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 3 \\ 2 & 1 \end{bmatrix}$. Calculate A.

Problem 15 [30pts] Let $A \in \mathbb{R}^{n\times n}$ and $B, C \in \mathbb{R}^{n\times p}$. Prove that

(i.) If $AB = AC$ for all $A \in \mathbb{R}^{n\times n}$ then $B = C$,

(ii.) If A is invertible and $AB = AC$ then $B = C$,

(iii.) There exists $A \neq 0$ such that $AB = AC$ yet $B \neq C$.

Problem 16 [20pts] (Anton §1.5 #9b, c) Suppose $k_1, k_2, k_3, k_4, k \neq 0$. Find the inverse matrix of

$$A = \begin{bmatrix} 0 & 0 & 0 & k_1 \\ 0 & 0 & k_2 & 0 \\ 0 & k_3 & 0 & 0 \\ k_4 & 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} k & 0 & 0 & 0 \\ 1 & k & 0 & 0 \\ 0 & 1 & k & 0 \\ 0 & 0 & 1 & k \end{bmatrix}.$$

Problem 17 [5pts](§ 1.1 # 75) If B is a square matrix prove that $B + B^T$ is symmetric.

Problem 18 [20pts](§ 1.1 # 83) A **probability vector** $\vec{p}$ is a vector with non-negative components that have a sum of one, in other words $\vec{p}$ has the form

$$\vec{p} = < p_1, p_2, \ldots, p_n > \quad \text{such that} \quad p_1 + p_2 + \cdots + p_n = 1 \quad \text{and} \quad p_j \geq 0 \quad \text{for all } j.$$

Prove the following: If $\vec{p}$ and $\vec{q}$ are probability vectors and $a, b \in \mathbb{R}$ are non-negative scalars such that $a + b = 1$ then $a\vec{p} + b\vec{q}$ is a probability vector.

Problem 19 [20pts] (§ 1.2 # 69) A **stochastic matrix** is a square matrix whose columns are probability vectors. Let $A = \begin{bmatrix} .85 & .03 \\ .15 & .97 \end{bmatrix}$ be a stochastic matrix which models the migration of people to and from the city to the suburbs. In particular, if c_k denotes the number of people (in thousands) in the city in year k and s_k denotes the number of people (in thousands) living in the suburbs in year k then letting $X_k = [c_k, s_k]^T$ we can model the migration of people by the following matrix product:

$$X_{k+1} = AX_k.$$

This model assumes the population stays constant and that there are only two places to live, the city or the suburbs. In addition, the model says that 15% of city people move to suburbs while only 3% of the suburb people move to the city. Suppose that $X_0 = [400, 300]^T$. Calculate the number of people living in the city and the number of people in suburbs after 1 year and then after 2 years.

Problem 20 [5pts] (§ 2.2 # 19 of Lay) Suppose $A, B, C \in \mathbb{R}^{n \times n}$ are invertible matrices. Solve $C^{-1}(A + X)B^{-1} = I$ for X.

Problem 21 [20pts] (§ 2.4 # 64) Let $A = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}$.

(a.) show that $A^2 - 3A + I = 0$,

(b.) let $B = 3I - A$ and prove $B = A^{-1}$,

Problem 22 [20pts](§ 1.1 # 82 and § 2.1 # 65) The **trace** of a square matrix is defined as follows: let $A \in \mathbb{R}^{n \times n}$ then

$$trace(A) = \sum_{i=1}^{n} A_{ii} = A_{11} + A_{22} + \cdots A_{nn}.$$

Prove the following statements are true for all $A, B \in \mathbb{R}^{n \times n}$, $C \in \mathbb{R}^{m \times n}$, $D \in \mathbb{R}^{n \times m}$ and $c \in \mathbb{R}$,

$$\begin{array}{ll} (i.) & trace(A + B) = trace(A) + trace(B) \\ (ii.) & trace(cA) = c\, trace(A) \\ (iii.) & trace(A^T) = trace(A) \\ (iv.) & trace(CD) = trace(DC) \end{array}$$

Problem 23 [10pts] (§ 2.1 #s 9 of Lay) We define the **commutator** of A with B by $[A, B] = AB - BA$. Let $A, B \in \mathbb{R}^{n \times n}$ show that it is not possible for $[A, B] = I$.

Problem 24 [10pts] (§ 2.3 # 66) Suppose we are given a block-matrix $M = \begin{bmatrix} A & 0_{p\times q} \\ 0_{q\times p} & B \end{bmatrix}$ which is a $(p+q)\times(p+q)$ matrix with square blocks $A \in \mathbb{R}^{\,p\times p}$ and $B \in \mathbb{R}^{\,q\times q}$ and $0_{p\times q}, 0_{q\times p}$ denote zero matrices. Prove that M is invertible iff A and B are invertible.

Problem 25 [10pts] (§ 3.2 #s 69) Suppose $A \in \mathbb{R}^{\,n\times n}$ is invertible, prove that $det(A^{-1}) = 1/det(A)$.

Problem 26 [10pts] (§ 3.2 #s 72) A square matrix $A \in \mathbb{R}^{\,n\times n}$ is **nilpotent of order** k if for some smallest positive integer k the product $A^k = 0$. Prove that if $A \in \mathbb{R}^{\,n\times n}$ is nilpotent then $det(A) = 0$

Problem 27 [20pts] (§ 3.2 #s 74) A square matrix $A \in \mathbb{R}^{\,n\times n}$ is **skew-symmetric** if $A^T = -A$. Prove that if $n \in 2\mathbb{Z}+1$ then $det(A) = 0$. What if $n \in 2\mathbb{Z}$? Explain.

Problem 28 [20pts] (§ 3.2 #s 75) The matrix $A = \begin{bmatrix} 1 & a & a^2 \\ 1 & b & b^2 \\ 1 & c & c^2 \end{bmatrix}$ is an example of a **Vandermonde matrix**. Calculate the determinant by performing elementary row operations to show that

$$det(A) = (b-a)(c-a)(c-b).$$

Problem 29 [20pts] Define a **Vandermonde matrix** $V(t)$ as follows:

$$V(t) = \begin{bmatrix} 1 & t & t^2 & t^3 \\ 1 & x_1 & x_1^2 & x_1^3 \\ 1 & x_2 & x_2^2 & x_2^3 \\ 1 & x_3 & x_3^2 & x_3^3 \end{bmatrix}$$

This matrix provides us a convenient way of creating a cubic polynomial that passes through distinct zeros $(x_1,0)$, $(x_2,0)$, $(x_3,0)$. Define $f(x) = det(V(x))$ and explain why $f(x_1) = f(x_2) = f(x_3) = 0$. Based on an analogy to the preceding problem state an explicit formula for $f(t)$.

Problem 30 [40pts] (§ 2.4 #s 84,86,87) If $A, B \in \mathbb{R}^{\,n\times n}$ then we say A, B are **similar matrices** iff there exists an invertible matrix $P \in \mathbb{R}^{\,n\times n}$ such that $B = P^{-1}AP$, in such a case we say B is the similarity transform of A by P. Assume A, B are square matrices and prove the following claims:

(a.) Similarity transformation is an equivalence relation,

(b.) if A, B are invertible and A is similar to B then A^{-1} is similar to B^{-1},

(c.) if A is similar to B then A^T is similar to B^T.

(d.) if A is similar to B then $det(A) = det(B)$

Proofs to complete the lecture notes

Problem 31 [20pts] Prove the concatenation proposition 2.3.11:

Let $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times p}$ then

$$AB = A[col_1(B)|col_2(B)|\cdots|col_p(B)] = [Acol_1(B)|Acol_2(B)|\cdots|Acol_p(B)].$$

Problem 32 [25pts]Prove items 2, 5, 9, 10 and 11 from proposition 2.3.13.

If $A, B, C \in \mathbb{R}^{m\times n}$, $X, Y \in \mathbb{R}^{n\times p}$, $Z \in \mathbb{R}^{p\times q}$ and $c_1, c_2 \in \mathbb{R}$ then

1. $(A + B) + C = A + (B + C)$,
2. $(AX)Z = A(XZ)$,
3. $A + B = B + A$,
4. $c_1(A + B) = c_1A + c_2B$,
5. $(c_1 + c_2)A = c_1A + c_2A$,
6. $(c_1c_2)A = c_1(c_2A)$,
7. $(c_1A)X = c_1(AX) = A(c_1X) = (AX)c_1$,
8. $1A = A$,
9. $I_mA = A = AI_n$,
10. $A(X + Y) = AX + AY$,
11. $A(c_1X + c_2Y) = c_1AX + c_2AY$,
12. $(A + B)X = AX + BX$,

Problem 33 [20pts] Prove 1, 2 , 4, and 5 of Proposition 2.9.3.

Let $A, B \in \mathbb{R}^{n\times n}$ and $c \in \mathbb{R}$ then

1. $(A^T)^T = A$
2. $(AB)^T = B^TA^T$
3. $(cA)^T = cA^T$
4. $(A + B)^T = A^T + B^T$
5. $(A^T)^{-1} = (A^{-1})^T$.

Problem 34 [25pts] Prove the following proposition holds for all $k \in \mathbb{N}$.
Your proof should include a careful induction argument.

If $A_1, A_2, \ldots, A_k \in \mathbb{R}^{n\times n}$ are invertible then

$$(A_1A_2\cdots A_k)^{-1} = A_k^{-1}A_{k-1}^{-1}\cdots A_2^{-1}A_1^{-1}$$