MATH 321 QUIZ 1 (TAKEHOME, DUE START OF CLASS MONDAY)

Copying answers and steps is strictly forbidden. Evidence of copying results in zero for copied and copier. Working together is encouraged, share ideas not calculations. Explain your steps. The calculations and answers should be written neatly on one-side of paper which is attached and neatly stapled in the upper left corner. Box your answers where appropriate. Please do not fold. Thanks!

Problem 1 If possible, calculate $\begin{bmatrix} 1 & 2 & 0 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} 1 & 6 \\ 0 & 0 \\ 7 & 8 \end{bmatrix} = \left[\begin{array}{c|c} 1+0+0 & 6+0+0 \\ \hline 3+0+35 & 18+0+40 \end{array}\right] = \boxed{\begin{bmatrix} 1 & 6 \\ 38 & 58 \end{bmatrix}}$

Problem 2 If possible, calculate $\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}^2}_{(2\times 3)(2\times 3)}$

$(2\times 3)(2\times 3) \Rightarrow$ NOT POSSIBLE, This matrix cannot be multiplied by itself.

Problem 3 If possible, calculate $\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}^{-1}$ NOT POSSIBLE. There are many ways to explain. For example, $\text{rref}\begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \therefore A^{-1}$ d.n.e. Or $A = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}$ has $A\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \therefore A^{-1}$ d.n.e. Or the formula $\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ does not hold in the case $ad-bc=0$.

Problem 4 If possible, calculate $\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}^8$

Let $A = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}$. Notice $A^2 = \begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}$

Thus $A^8 = (A^2)^4 = \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}^4 = \begin{bmatrix} 3^4 & 0 \\ 0 & 3^4 \end{bmatrix} = \boxed{\begin{bmatrix} 81 & 0 \\ 0 & 81 \end{bmatrix}}$

Problem 5 If possible, calculate $e_1 \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$(2\times1)(2\times2) \Rightarrow$ not possible.

forget it.

Problem 6 If possible, calculate $\begin{bmatrix} a & b \\ c & d \end{bmatrix} e_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \boxed{\begin{bmatrix} a \\ c \end{bmatrix}}$

(I suppose to be completely clear I should have said $e_1 \in \mathbb{R}^2$ for Problems 6 and 5 to be unambiguous)

Problem 7 Let $Q(x,y) = x^2 + 2xy - y^2$. Find $A = A^T$ for which $Q(v) = v^T Av$ where $v^T = [x,y]$.

Notice $A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$ has $A_{12} = A_{21}$ as $A = A^T$.

Furthermore, $Q(x,y) = [x,y]\begin{bmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = [x,y]\begin{bmatrix} A_{11}x + A_{12}y \\ A_{12}x + A_{22}y \end{bmatrix}$

$= A_{11}x^2 + A_{12}xy + A_{12}yx + A_{22}y^2$

Compare $x^2 + 2xy - y^2 = A_{11}x^2 + 2A_{12}xy - A_{22}y^2$ to read-off $A_{11} = 1$, $A_{12} = 1$, $A_{22} = -1$ $\therefore$ $\boxed{A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}$

Problem 8 Let A, B, C be square matrices. Define $[A,B] = AB - BA$. Show

$$[A+B,C] = [A,C] + [B,C].$$

Consider,
$$\begin{aligned} [A+B,C] &= (A+B)C - C(A+B) &&: \text{def}^n \text{ of } [\,,\,] \\ &= AC + BC - CA - CB &&: \text{dist. prop of matrix mult.} \\ &= (AC - CA) + (BC - CB) &&: \text{algebra of matrices} \\ &= [A,C] + [B,C]. \end{aligned}$$

Problem 9 Let A be a nonzero square matrix such that $A^2 \neq 0$ yet $A^3 = 0$. Is A invertible? Is $I + A$ invertible?

(non zero)

① If $A^2 \neq 0$ then $\exists x_0 \in \mathbb{R}^n$ such that $A^2x_0 \neq 0$. Consider

Let $v = A^2x_0 \neq 0$ and notice $Av = A(A^2x_0) = A^3x_0 = 0x_0 = 0$
hence $Av = 0 \not\Rightarrow v = 0$ $\therefore$ A^{-1} does not exist.

② Consider $(I+A)(I-A+A^2) = I - \cancel{A} + \cancel{A^2} + \cancel{A} - \cancel{A^2} + \underbrace{A^3}_{0} = I$

$\therefore$ $(I+A)^{-1} = I - A + A^2$

Comment: I usually make guess like $I + aA + bA^2$ then work out a, b to make it work. see ↻

$$(I+A)(I+aA+bA^2) =$$

$$= I + \underbrace{aA + bA^2 + A + aA^2}_{} + \cancelto{0}{bA^{3}} = I$$

$$a = -1, \quad b = 1$$

(for the LHS to reduce to I we must choose $a=-1$ and $b=1$)

Problem 10 Suppose A is a symmetric matrix. Prove A^n is symmetric for all $n \in \mathbb{N}$.

$n=1$ / $A^T = A$ is given. Suppose $(A^k)^T = A^k$ for some $k \in \mathbb{N}$. Consider,

$$\begin{aligned} (A^{k+1})^T &= (A^k A)^T && \text{: def}^{\underline{n}} \text{ of } A^{k+1} \\ &= A^T (A^k)^T && \text{: socks-shoes prop.} \\ &= A \cdot A^k && \text{: } A^T = A \text{ and induct. hypo.} \\ &= A^{k+1} && \text{: def}^{\underline{n}} \text{ of } A^{k+1} \end{aligned}$$

Thus $(A^n)^T = A^n$ $\forall n \in \mathbb{N}$, by proof by mathematical induction. //

Problem 11 Suppose there exist matrices X which are multipliable with A, B such that $AX = BX$. Is it true, possible, or false that $A = B$?

<u>POSSIBLE</u>. If X is invertible then certainly

$$AX = BX \Rightarrow AXX^{-1} = BXX^{-1} \Rightarrow AI = BI \Rightarrow A = B.$$

However, $\exists X$ for which $\not\Rightarrow A = B$. For example, $X = 0$, we have $A(0) = B(0)$ for any A, B, there is no need for $A = B$. To be concrete,

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

yet $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \neq \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$.

Problem 12 Solve (freestyle, I judge your answer not your work here, but, keep in mind, this might be a test question so wolfram alpha etc. is really not the style which helps you prepare)

$$x - y + z + t = 14, \qquad x - z = 1, \qquad z + t = 12, \qquad x - t = 3.$$

$$+\begin{pmatrix} z + t = 12 \\ x - t = 3 \end{pmatrix} \quad \Rightarrow \quad z + x = 15$$

$$+\begin{pmatrix} x - z = 1 \\ z + x = 15 \end{pmatrix}$$

$$\Rightarrow 2x = 16$$

$$\therefore x = 8$$

$$t = x - 3 = 8 - 3 = 5 = t$$

$$z = 12 - t = 12 - 5 = 7 = z$$

$$y = x + z + t - 14 = 8 + 7 + 5 - 14 = 6 = y$$

$$\boxed{\begin{aligned} x &= 8 \\ y &= 6 \\ z &= 7 \\ t &= 5 \end{aligned}}$$

Problem 13 Suppose that the matrix below is the augmented coefficient matrix for a linear system of equations $Av = b$ where $v = (x_1, x_2, x_3, x_4)$,

$$[A|b] = \left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 9 \\ 1 & 2 & 3 & 0 & 14 \\ 1 & -1 & 0 & 0 & -1 \end{array}\right] \quad \text{and you're given:} \quad rref[A|b] = \left[\begin{array}{cccc|c} 1 & 0 & 1 & 1 & 4 \\ 0 & 1 & 1 & 1 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Perform the following two tasks:

(i.) write the **system of equations** $Av = b$ in "scalar" form.

$$\boxed{\begin{aligned} x_1 + x_2 + 2x_3 &= 9 \\ x_1 + 2x_2 + 3x_3 &= 14 \\ x_1 - x_2 &= -1 \end{aligned}}$$

(ii.) give the **general solution** with pivot variables as the dependent variables.

$$\boxed{\left.\begin{aligned} x_1 &= 4 - x_3 - x_4 \\ x_2 &= 5 - x_3 - x_4 \end{aligned}\right\} \text{ for } x_3, x_4 \in \mathbb{R}}$$

(could set $x_3 = s$, $x_4 = t$ and write $\boxed{\begin{aligned} x_1 &= 4 - s - t \\ x_2 &= 5 - s - t \\ x_3 &= s \\ x_4 &= t \end{aligned} \text{ for } s, t \in \mathbb{R}}$

(iii.) give the **solution set** parametrized by the non-pivot variables.

$$\boxed{\text{Sol}^{\text{n}} \text{ Set} = \{(4 - x_3 - x_4,\ 5 - x_3 - x_4,\ x_3,\ x_4) \mid x_3, x_4 \in \mathbb{R}\}}$$

$$= \{(4 - \alpha - \beta,\ 5 - \alpha - \beta,\ \alpha,\ \beta) \mid \alpha, \beta \in \mathbb{R}\}$$

α, β just parameters here, other notations possible...

Problem 14 Find the solution set of the system of equations below using row-reduction.

$$x + y + z = 2, \qquad 2x + 2y + 2z = 4, \quad y - z = 2$$

Use the non-pivot variables to parametrize the solution set.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 2 & 2 & 2 & 4 \\ 0 & 1 & -1 & 2 \end{array}\right] \xrightarrow{r_2 - 2r_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 \end{array}\right] \xrightarrow{r_1 - r_3} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 2 \end{array}\right]$$

thus $x + 2z = 0$ and $y - z = 2$ hence,

$x = -2z$, $y = 2 + z$, $z = z$

$$\boxed{\text{Sol}^{\text{n}} \text{ set} = \{(-2z,\ 2 + z,\ z) \mid z \in \mathbb{R}\}}$$

Problem 15 Is $(1,2,0,0) \in \text{span}\{(1,0,2,2),(0,1,2,2),(1,1,0,0)\}$? Explain.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{array}\right] \xrightarrow[r_4-2r_1]{r_3-2r_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & -2 & -2 \\ 0 & 2 & -2 & -2 \end{array}\right] \xrightarrow[r_4-2r_2]{r_3-2r_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -4 & -6 \\ 0 & 0 & -4 & -6 \end{array}\right]$$

$$\xrightarrow{r_4-r_3} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & -4 & -6 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{r_3/-4} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow[r_2-r_3]{r_1-r_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & -1/2 \\ 0 & 1 & 0 & 1/2 \\ 0 & 0 & 1 & 3/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Thus, by CCP, we find, $-\frac{1}{2}(1,0,2,2)+\frac{1}{2}(0,1,2,2)+\frac{3}{2}(1,1,0,0)=(1,2,0,0)$ that is to show $(1,2,0,0) \in \text{span}\, S$. (where S is defined)

Problem 16 Let $S = \{(1,0,2,2),(0,1,2,2),(1,1,0,0)\}$. Is S a LI set?

Indeed, by the calculation of PROBLEM 15 we have $\text{rref}\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 2 & 2 & 0 \\ 2 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$. It is clear that $\nexists$ any linear dependence among $\{e_1, e_2, e_3\}$ thus $\nexists$ any linear dependence among the vectors of S by the CCP. (or, can argue they correspond to pivot columns hence form LI set, basically same argument)

Problem 17 Let $n \geq 5$. Suppose $\{v_1, v_2, v_3, v_4\} \subset \mathbb{R}^n$ such that $\text{rref}[v_1|v_2|v_3|v_4|w] = [e_1|2e_1|e_2|e_1+e_2|e_5]$. Use the given information to answer the following:

pivot columns.

oops, that last column must read e_3.

(a) is $\{v_1, v_2\}$ LI ? No. By CCP, $2V_1 = V_2$.

(b) is $\{v_1, v_3\}$ LI ? YES, $\{V_1, V_3\}$ are in pivot columns hence form LI set.

(c) is $\{v_2, v_4\}$ LI ? YES. The reason is that $\{2e_1, e_1+e_2\}$ clearly forms a LI set as $2e_1 \neq k(e_1+e_2)$. Thus by CCP $\{V_2, V_4\}$ are also LI.

(d) is $w \in \text{span}\{v_1, v_2, v_3, v_4\}$?

No. $W \notin \text{span}\{V_1, V_2, V_3, V_4\}$ as the eq$\underline{\text{n}}$ $c_1V_1 + c_2V_2 + c_3V_3 + c_4V_4 = W$ is inconsistent by the given rref. Equivalently, it is impossible to build e_3 with e_1 & e_2 hence W is likewise not in span of V_1, V_2, V_3, V_4 by CCP.

Problem 18 Let $k \in \mathbb{R}$ and define $S = \{(1,1,0),(0,1,1),(1,k,1)\}$. For which values of k is it true that $\mathbb{R}^3 = \text{span}(S)$?

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 1 & 1 & k & b \\ 0 & 1 & 1 & c \end{array}\right] \xrightarrow{r_2 - r_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & k-1 & b-a \\ 0 & 1 & 1 & c \end{array}\right] \xrightarrow{r_3 - r_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & k-1 & b-a \\ 0 & 0 & 2-k & c-b+a \end{array}\right]$$

It follows, if $k \neq 2$ then we can obtain $\text{rref}\underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & k \\ 0 & 1 & 1 \end{bmatrix}}_{A} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

hence A^{-1} exists and $A\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \rightarrow \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = A^{-1}\begin{bmatrix} a \\ b \\ c \end{bmatrix}$

for all $(a,b,c) \in \mathbb{R}^3$ $\therefore$ $\text{span}(S) = \mathbb{R}^3$. However, if $k = 2$

then

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 1 & 1 & 2 & b \\ 0 & 1 & 1 & c \end{array}\right] \rightarrow \left[\begin{array}{ccc|c} 1 & 0 & 1 & a \\ 0 & 1 & 1 & b-a \\ 0 & 0 & 0 & c-b+a \end{array}\right]$$

Hence, any $(a,b,c) \in \mathbb{R}^3$ for which $c - b + a \neq 0$ will fall out side of span S. **In closing, only for $k \neq 2$ do we find span $S = \mathbb{R}^3$.**

Problem 19 Consider $S = \{(1,2,3),(1,1,0),(1,1,1),(3,4,4)\}$. Find all LI subsets $T \subseteq S$ such that $\text{span}(T) = \mathbb{R}^3$.

$$[S] = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 1 & 1 & 4 \\ 3 & 0 & 1 & 4 \end{bmatrix} \xrightarrow[r_3 - 3r_1]{r_2 - 2r_1} \left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 0 & -1 & -1 & -2 \\ 0 & -3 & -2 & -5 \end{array}\right] \xrightarrow[r_3 - 3r_2]{r_1 + r_2} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & -1 & -2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \xrightarrow{r_2 + r_3} \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\Big\downarrow -r_2$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

$$\therefore \text{rref}\begin{bmatrix} 1 & 1 & 1 & 3 \\ 2 & 1 & 1 & 4 \\ 3 & 0 & 1 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}$$

We need 3 vectors to span $\mathbb{R}^3$ and no more as we know k-vectors for $k > 3$ form Linearly Dependent set. So, the question is to select all LI triples from S"

$T = \{(1,2,3),(1,1,0),(1,1,1)\}$
$T = \{(1,2,3),(1,1,0),(3,4,4)\}$
$T = \{(1,2,3),(1,1,1),(3,4,4)\}$
$T = \{(1,1,0),(1,1,1),(3,4,4)\}$

$\leftarrow$ all clearly LI by CCP in view of the rref $[S]$ calculated above.

* What if $\text{rref}[S] = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$, how would the answer be different?

Problem 20 Find the set of all possibly degenerate cubic polynomials whose graphs contain the points $(1,1)$, $(2,1)$ and $(-1,0)$.

$g(x) = Ax^3 + Bx^2 + Cx + D$

(from * with t set to β)

$$\boxed{\left\{ \left(-\tfrac{1}{3} + \tfrac{\beta}{2}\right)x^3 + \left(\tfrac{1}{2} - \beta\right)x^2 + \left(\tfrac{5-3\beta}{6}\right)x + \beta \;\middle|\; \beta \in \mathbb{R} \right\}}$$

$g(1) = A + B + C + D = 1$

$g(2) = 8A + 4B + 2C + D = 1$

$g(-1) = -A + B - C + D = 0$

$$\left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 8 & 4 & 2 & 1 & 1 \\ -1 & 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow[r_3 + r_1]{r_2 - 8r_1} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & -4 & -6 & -7 & -7 \\ 0 & 2 & 0 & 2 & 1 \end{array}\right] \xrightarrow{r_2 + 2r_3} \left[\begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & -6 & -3 & -5 \\ 0 & 2 & 0 & 2 & 1 \end{array}\right]$$

$$\xrightarrow{r_1 - r_3/2} \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1/2 \\ 0 & 0 & -6 & -3 & -5 \\ 0 & 2 & 0 & 2 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{cccc|c} 1 & 0 & 1 & 0 & 1/2 \\ 0 & 1 & 0 & 1 & 1/2 \\ 0 & 0 & -6 & -3 & -5 \end{array}\right]$$

clear $D = t$ serves as parameter of sol$^{=}$

$-6C = -5 + 3t$

$B = 1/2 - t$

$A = 1/2 - C$

$= \frac{1}{2} + \frac{1}{6}(-5 + 3t)$

$= \frac{-2}{6} + \frac{1}{2}t$

$$\therefore \quad \boxed{g(x) = \left(\frac{-1}{3} + \frac{t}{2}\right)x^3 + \left(\frac{1}{2} - t\right)x^2 + \left(\frac{5-3t}{6}\right)x + t} \; *$$

Problem 21 Suppose $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$. Furthermore, suppose

$$EA = \begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ 6 & 7 & 8 \end{bmatrix}$$

Which elementary matrix E makes the equation above true?

Observe

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ 6 & 7 & 8 \end{bmatrix}$$

$$\boxed{E_{r_1 \leftrightarrow r_2} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E}$$

Check it:

$$\begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 3 & 5 \\ 7 & 6 & 8 \end{bmatrix}$$

oops, this was AE which is a column swap.

$$EA = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix} = \begin{bmatrix} 3 & 4 & 5 \\ 1 & 2 & 3 \\ 6 & 7 & 8 \end{bmatrix} \checkmark$$

Problem 22 Consider the the system of equations $Av = 0$ where

$$A = \begin{bmatrix} 1 & 4 & 6 \\ -1 & -3 & -6 \\ 2 & 8 & 13 \end{bmatrix}$$

and v is notation for $v = [x, y, z]^T$. Solve this system by explicit row reduction.

$$\begin{bmatrix} 1 & 4 & 6 \\ -1 & -3 & -6 \\ 2 & 8 & 13 \end{bmatrix} \xrightarrow[r_3 - 2r_1]{r_2 + r_1} \begin{bmatrix} 1 & 4 & 6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 - 4r_2} \begin{bmatrix} 1 & 0 & 6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{r_1 - 6r_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Hence $\text{rref}[A|0] = [I|0]$ $\therefore$ $Av = 0 \Rightarrow \boxed{v = 0}$

For the next problem, it'd be easier to do Problem 23 first as we can then answer this one, well, if I had not put that pesky phrase "by explicit row reduction"

Observe $E_{r_1 - 6r_3} E_{r_1 - 4r_2} E_{r_2 + r_1} E_{r_3 - 2r_1} A = I$

Problem 23 Is the matrix below invertible? If so, find the inverse.

$$A = \begin{bmatrix} 1 & 4 & 6 \\ -1 & -3 & -6 \\ 2 & 8 & 13 \end{bmatrix}$$

By the row reduction above, identify that:

$$A^{-1} = \begin{bmatrix} 1 & 0 & -6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -4 & -6 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 9 & -4 & -6 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}}$$

check it: $\begin{bmatrix} 1 & 4 & 6 \\ -1 & -3 & -6 \\ 2 & 8 & 13 \end{bmatrix} \begin{bmatrix} 9 & -4 & -6 \\ 1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Problem 24 Consider $T(x, y, z) = (x + 4y + 6z,\ -x - 3y - 6z,\ 2x + 8y + 13z)$. Show that T is a linear bijection on $\mathbb{R}^3$. Notice, the previous problem saves you calculation here.

$$T(x,y,z) = \begin{bmatrix} x + 4y + 6z \\ -x - 3y - 6z \\ 2x + 8y + 13z \end{bmatrix} = \begin{bmatrix} 1 & 4 & 6 \\ -1 & -3 & -6 \\ 2 & 8 & 13 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$\therefore$ T is linear mapping from $\mathbb{R}^3 \to \mathbb{R}^3$ by FTLA.

Moreover, $\rightarrow$

$\boxed{\text{PROBLEM 24}}$ Continued, note $A = [T]$ where we proved A^{-1} exists in PROBLEM 23. Observe, if $b \in \mathbb{R}^3$ then $T(A^{-1}b) = [T]A^{-1}b = AA^{-1}b = Ib = b$ thus T is a surjection. Furthermore, if

$$T(x) = T(y) \Rightarrow Ax = Ay \Rightarrow A^{-1}Ax = A^{-1}Ay \Rightarrow x = y$$

thus T is an injective mapping on $\mathbb{R}^3$.

Therefore, T is both injective and surjective on $\mathbb{R}^3$ hence T is a <u>linear</u> <u>bijection</u> on $\mathbb{R}^3$.

Problem 25 A kind robot tells you that $rref\left[\begin{array}{ccc|c} 1 & 0 & 1 & 16 \\ 2 & 2 & 0 & 20 \\ 3 & 1 & 0 & 24 \end{array}\right] = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 9 \end{array}\right]$. Define vectors as follows:

$$v_1 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix} \quad v_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad w = \begin{bmatrix} 16 \\ 20 \\ 24 \end{bmatrix}$$

Answer the following questions: (you do not have to explain, just this once)

(a) Is $w \in span(\{v_1, v_2, v_3\})$? If it is then provide the linear combination of the vectors v_1, v_2, v_3 which yields w.

Yes, by CCP we see : $\underline{W = 7V_1 + 3V_2 + 9V_3}$.

(b) Is $\{v_1, v_2, v_3\}$ a linearly independent set of vectors? (Yes ~~No~~ Maybe)

These correspond to pivot columns hence are LI.

(c) Is $\{v_1, v_2, v_3, w\}$ a linearly independent set of vectors? (Yes No Maybe)

we saw $W = 7V_1 + 3V_2 + 9V_3$ hence they possess a linear dependence

Remark : also 4-vectors in $\mathbb{R}^3$ are automatically Linearly Dependent

Problem 26 Define a function $T : \mathbb{R}^3 \to \mathbb{R}^3$ by the formula below:

$$T(x, y, z) = (x + z,\ 2x + 2y, 3x + y).$$

(a) find the standard matrix for T

$$\left[T(1,0,0) \middle| T(0,1,0) \middle| T(0,0,1)\right] = \boxed{\begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 0 \\ 3 & 1 & 0 \end{bmatrix}}$$

(b) is T linear? Why?

YES, we have $T(x,y,z) = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 0 \\ 3 & 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ which shows that T is linear.

(c) is T one-one ?

YES, the columns of $[T]$ form LI set by Problem 25b, thus T is 1-1.

(d) is $w = (16, 20, 24) \in T(\mathbb{R}^3)$? (again Problem ~~25~~ 25 helps)

YES,

$$T(7,3,9) = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & 0 \\ 3 & 1 & 0 \end{bmatrix}\begin{bmatrix} 7 \\ 3 \\ 9 \end{bmatrix} = \begin{bmatrix} 16 \\ 20 \\ 24 \end{bmatrix} \therefore \underline{\begin{bmatrix} 16 \\ 20 \\ 24 \end{bmatrix} \in T(\mathbb{R}^3)}.$$

Problem 27 Prove that $(c_1+c_2)v = c_1v + c_2v$ for all $v \in \mathbb{R}^n$ and $c_1, c_2 \in \mathbb{R}$. Let $1 \le i \le n$,

$$\begin{aligned}[(c_1+c_2)v]_i &= (c_1+c_2)v_i &&: \text{def}^{\underline{n}} \text{ of scalar multiplication}\\ &= c_1v_i + c_2v_i &&: \text{prop. of } \mathbb{R}\#\\ &= (c_1v)_i + (c_2v)_i &&: \text{def}^{\underline{n}} \text{ of scalar mult. again}\\ &= [c_1v + c_2v]_i &&: \text{def}^{\underline{n}} \text{ of } + \text{ in } \mathbb{R}^n \Rightarrow (c_1+c_2)v = c_1v + c_2v. //\end{aligned}$$

Problem 28 Let $T(x,y,z) = (x+y-z, y+z)$ and $S(a,b) = (2a+b, a-b, 3a+b)$. If possible, calculate the standard matrix for $S \circ T$ and give the formula for $S \circ T$. If possible, calculate the standard matrix for $T \circ S$ and give the formula for $S \circ T$.

$T: \mathbb{R}^3 \to \mathbb{R}^2$ and $S: \mathbb{R}^2 \to \mathbb{R}^3 \Rightarrow T \circ S: \mathbb{R}^2 \xrightarrow{S} \mathbb{R}^3 \xrightarrow{T} \mathbb{R}^2$

AND ~~But~~, $S \circ T$ is also defined $S \circ T: \mathbb{R}^3 \xrightarrow{T} \mathbb{R}^2 \xrightarrow{S} \mathbb{R}^3$

Observe $[T] = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$ and $[S] = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 3 & 1 \end{bmatrix}$ thus,

$[T \circ S] = [T][S] = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 3 & 1 \end{bmatrix} = \boxed{\underbrace{\begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix}}_{[T \circ S]}}$ $\therefore \boxed{(T \circ S)(a,b) = (-b, 4a)}$

$\begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} -b \\ 4a \end{bmatrix}$

$[S \circ T] = [S][T] = \begin{bmatrix} 2 & 1 \\ 1 & -1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 3 & -1 \\ 1 & 0 & -2 \\ 3 & 4 & -2 \end{bmatrix} = [S \circ T]}$ hence $(S \circ T)(x,y,z) =$

Problem 29 Find the set of all $n \times n$ matrices which commute with all $n \times n$ matrices. That is, find all matrices $X \in \mathbb{R}^{n \times n}$ such that $AX = XA$ for all $A \in \mathbb{R}^{n \times n}$.

$\boxed{\underline{\text{Answer}}: \{\lambda I \mid \lambda \in \mathbb{R}\}.}$

How to see this?

Consider $A = E_{ij}$ or $A = [e_i|0|\cdots|0]$ and

carefully compare AX vs. XA to obtain

X must be diagonal with matching diag. entries.

Problem 30 Find all 2×2 matrices which commute with $B = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$\boxed{\underline{\text{Answer}}: X = \begin{bmatrix} 2c+d & c \\ c & d \end{bmatrix} \text{ with } c, d \in \mathbb{R}.}$

How to find? Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ set $BX = XB$ and reduce the system of 4-eq$^{\underline{n}}$s and 4-unknowns to just 2 indep. eq$^{\underline{n}}$s.

Given $A X = X A \quad \forall A \in \mathbb{R}^{n\times n}$. (scratch work)

Let $X = \sum_{i,j} X_{ij} E_{ij}$ and consider $A = [0|\cdots|e_j|\cdots 0] = E_{ij}$
e_i

$$X[0|\cdots|e_i|\cdots|0] = [0|\cdots|Xe_i|\cdots 0] = [0|\cdots|col_i(X)|\cdots 0]$$

$$X[e_i|0|\cdots|0] = [col_i(X)|\cdots]$$

$$[e_i|0|\cdots|0]X = \begin{bmatrix} 0 \\ \hline e_i^T \\ \hline \vdots \\ 0 \end{bmatrix} X = \begin{bmatrix} 0 \\ \hline \vdots \\ \hline row_1(X) \\ \hline \vdots \\ 0 \end{bmatrix} \Leftarrow i^{th} \text{ slot}$$

$$\therefore \quad [col_i(X)|0|\cdots|0] = \begin{bmatrix} 0 \\ \hline \vdots \\ 0 \\ \hline row_1(X) \\ \hline 0 \\ \vdots \\ 0 \end{bmatrix}$$

$$\Rightarrow (col_i(X))_j = \begin{cases} 0 & \text{if } j \neq i \\ row_1(X) & \text{if } j = i \end{cases}$$

a b c d

$$\begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & -2 & 0 & -1 \\ 1 & 0 & -2 & -1 \\ \cancel{0} & \cancel{1} & \cancel{-1} & \cancel{0} \end{bmatrix} \sim \begin{bmatrix} 0 & 2 & -2 & 0 \\ 1 & -2 & 0 & -1 \\ \cancel{0} & \cancel{2} & \cancel{-2} & \cancel{0} \\ 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \text{~~use~~ only diag. non zero}$$

$$\sim \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & -2 & 0 & -1 \end{bmatrix} \qquad X = \lambda I.$$

$$\sim \begin{bmatrix} 0 & 1 & -1 & 0 \\ 1 & 0 & -2 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & -1 \\ 0 & 1 & -1 & 0 \end{bmatrix}$$

$\cancel{a}+b-c$
$a-2b+d=0$
$a+2c-d=0$

$a = 2c+d$
$b = c$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\left[\begin{array}{c|c} a+c & b+d \\ \hline a-c & b-d \end{array}\right] = \left[\begin{array}{c|c} a+b & a-b \\ \hline c+d & c-d \end{array}\right]$$

$\cancel{a}+c = \cancel{a}+b \quad \therefore \underline{b=c}.$
$b+d = a-b$
$a-c = c+d$
$b-\cancel{d} = c-\cancel{d} \quad \therefore \underline{b=c}.$

$$\left[\begin{array}{c|c} 2c+d & c \\ \hline c & d \end{array}\right]$$

$$\begin{array}{l} (b+d = a-b) \\ + (a-c = c+d) \\ \hline b+c+d+a = c-b+a+d. \end{array}$$

$2b = a-d$
$2c = a-d$

(scratch work)

$$\left[\begin{array}{c|c} 2c+d & c \\ \hline c & d \end{array}\right]\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \neq \left[\begin{array}{c|c} 3c+d & c+d \\ \hline c+d & c-d \end{array}\right]$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\left[\begin{array}{c|c} 2c+d & c \\ \hline c & d \end{array}\right] \neq \left[\begin{array}{c|c} 3c+d & c+d \\ \hline c+d & c-d \end{array}\right]$$