Each A-Rank problem here is worth 3pts. The Hokage-Rank is worth 20pts.

Bonus 14 (A-Rank) Insel, Spence and Friedberg §5.2#18 page 281

Bonus 15 (A-Rank) Insel, Spence and Friedberg §5.2#19 page 281

Bonus 16 (A-Rank) Insel, Spence and Friedberg §5.2#20 page 281

Bonus 17 (A-Rank) Insel, Spence and Friedberg §5.2#21 page 281

Bonus 18 (A-Rank) Insel, Spence and Friedberg §5.4#25 page 323

Bonus 19 (A-Rank) Insel, Spence and Friedberg §5.4#38 page 325

Bonus 20 (A-Rank) Insel, Spence and Friedberg §6.4#14 page 373

Bonus 21 (A-Rank) Insel, Spence and Friedberg §6.4#15 page 374

Bonus 22 (A-Rank) Insel, Spence and Friedberg §6.6#10 page 402

Bonus 23 (Hokage) Let $\pi_j : \mathbb{R}^n \to \mathbb{R}^n$ be the projection defined by $\pi(x) = x - (x \bullet e_j)e_j$ for each $x \in \mathbb{R}^n$ for $j = 1, \dots, n$. Suppose $\mathcal{P}$ is an $(n-1)$-dimensional paralell-piped which is formed by the convex-hull of $v_1, \dots, v_{n-1} \in \mathbb{R}^n$ suspended at base-point $p \in (0, \infty)^n$;

$$\mathcal{P} = \left\{ p + \sum_{j=1}^{n-1} \alpha_j v_j \;\middle|\; \alpha_j \in [0,1] \ \& \ \sum_{j=1}^{n-1} \alpha_j \leq 1 \right\}$$

Let $n \in \mathbb{R}^n$ be a unit-vector in $\{v_1, \cdots, v_{n-1}\}^\perp$. The $(n-1)$-**area** of $\mathcal{P}$ is given by $\mathbf{area}(\mathcal{P}) = |\det[v_1|\cdots|v_{n-1}|n]|$. We can study the area of the **shadows** formed by $\mathcal{P}$ on the coordinate hyperplanes. Let $\mathcal{P}_j = \pi_j(\mathcal{P})$ define the shadow of $\mathcal{P}$ on the $x_j = 0$ coordinate plane. Notice,

$$\mathcal{P}_j = \left\{ \pi_j(p) + \sum_{i=1}^{n-1} \alpha_i \pi_j(v_i) \;\middle|\; \alpha_j \in [0,1] \ \& \ \sum_{j=1}^{n-1} \alpha_j \leq 1 \right\}$$

which shows $\mathcal{P}_j$ is formed by the convex-hull $\pi_j(v_1), \dots, \pi_j(v_n)$ of attached at basepoint $\pi_j(p)$. It follows that the $(n-1)$-area of the $\mathcal{P}_j$ can be calculated as follows:

$$\mathbf{area}(\mathcal{P}_j) = |\det[\pi_j(v_1)|\cdots|\pi_j(v_{n-1})|e_j]|.$$

since e_j is perpendicular to $\mathcal{P}_j$. In the case $n = 2$ the 1-dimensional paralell-piped is just a line-segment. For example, if $v_1 = (1,1)$ then $(1/\sqrt{2}, -1/\sqrt{2})$ is perpendicular to v_1 and

$$\det \begin{bmatrix} 1 & 1/\sqrt{2} \\ 1 & -1/\sqrt{2} \end{bmatrix} = -2/\sqrt{2} = -\sqrt{2} \quad \Rightarrow \quad \mathbf{area}(\mathcal{P}) = \sqrt{2}.$$

Of course, this is actually the length of the line-segment. Also, notice

$$\mathbf{area}(\mathcal{P}_1)^2 + \mathbf{area}(\mathcal{P}_2)^2 = 1^2 + 1^2 = \sqrt{2}^2 = \mathbf{area}(\mathcal{P})^2.$$

This is not suprising. However, perhaps the fact this generalizes to n-dimensions in the following sense is not already known to you:

$$\mathbf{area}(\mathcal{P}_1)^2 + \mathbf{area}(\mathcal{P}_2)^2 + \cdots \mathbf{area}(\mathcal{P}_n)^2 = \mathbf{area}(\mathcal{P})^2$$

Prove it. You might call this the generalized Pythagorean identity, I'm not sure its history or formal name. That said, the formula I give for generalized area could just as well be termed generalized volume. Also, you could **define**

$$v_1 \times v_2 \times \cdots \times v_{n-1} = \det \left[\begin{array}{c|c|c|c|c} & & & & e_1 \\ & & & & e_2 \\ v_1 & v_2 & \cdots & v_{n-1} & \vdots \\ & & & & e_n \end{array} \right] \in \mathbb{R}^n$$

where we insist the determinant is calculated via the Laplace expansion by minors along the last column. You can show $v_1 \times v_2 \times \cdots \times v_{n-1} \in \{v_1, \ldots, v_{n-1}\}^{\perp}$. But, if n is a unit-vector which spans $\{v_1, \ldots, v_{n-1}\}^{\perp}$ then the $(n-1)$-ry cross-product must be a vector parallel to n and thus:

$$v_1 \times v_2 \times \cdots \times v_{n-1} = \left[(v_1 \times v_2 \times \cdots \times v_{n-1}) \bullet n\right] n$$

Note, $n \bullet n = 1$ as we assumed n is unit-vector and we can show

$$(v_1 \times v_2 \times \cdots \times v_{n-1}) \bullet n = \det[v_1|v_2|\ldots|v_{n-1}|n]$$

Notice this generalized cross-product is just an extension of the heurstic determinant commonly used in multivariate calculus to define the standard cross-product. In particular, the following is equivalent to the column-based definition

$$v_1 \times v_2 \times \cdots \times v_{n-1} = \det \begin{bmatrix} e_1 \quad e_2 \quad \cdots \quad e_n \\ v_1^T \\ v_2^T \\ \vdots \\ v_{n-1}^T \end{bmatrix}$$

where we insist the determinant is calculated via the Laplace expansion by minors along the first row. In any event, my point in this discussion is merely that we can calculate higher-dimensional volumes with determinants and these go hand-in-hand with generalized tertiary cross-products. In particular,

$$||v_1 \times v_2 \times \cdots \times v_{n-1}|| = \mathbf{vol}(\mathcal{P})$$

where $\mathcal{P}$ is formed by the convex hull of $v_1, \ldots, v_{n-1}$. When $n = 2$ this gives vector length, when $n = 3$ this is the familar result that the area of the parallelogram with sides $\vec{A}, \vec{B}$ is just $||\vec{A} \times \vec{B}||$.