Each problem here is worth 3pts. Nice presentation worth 3pts.

Bonus 1 Let $J = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and define $W = \{M \in \mathbb{R}^{2\times 2} \mid MJ = JM\}$.

(a.) show W is a subspace of $\mathbb{R}^{2\times 2}$

(b.) Find a basis β for W and calculate $\left[\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right]_\beta$ in terms of a and b.

Bonus 2 We define **hyperbolic numbers** $\mathcal{H} = \{x + jy \mid x, y \in \mathbb{R}\}$ where $j^2 = 1$ and we suppose $\{1, j\}$ are linearly independent ($\mathcal{H} = 1\mathbb{R} \oplus j\mathbb{R}$) . In particular, we add and multiply in the natural fashion:

$$(a + bj) + (x + jy) = a + x + j(b + y)$$

$$(a + bj)(x + jy) = ax + by + j(ay + bx)$$

for all $a + jb, x + jy \in \mathcal{H}$. Scalar multiplication is a special case of the vector multiplication in $\mathcal{H}$; $c(a + bj) = ca + j(cb)$. It is straightforward to verify $\mathcal{H}$ forms a vector space over $\mathbb{R}$. Prove the following:

(a.) $\Psi(x + jy) = \begin{bmatrix} x & y \\ y & x \end{bmatrix}$ is a vector space isomorphism from $\mathcal{H}$ to W from **Bonus 1**.

(b.) $\Psi(1) = I$ and $\Psi(zw) = \Psi(z)\Psi(w)$ for all $z, w \in \mathcal{H}$

(c.) If $z \in \mathcal{H}$ has $z^{-1} \in \mathcal{H}$ such that $zz^{-1} = 1$ then $\Psi(z^{-1}) = (\Psi(z))^{-1}$

(d.) Find all $z \in \mathcal{H}$ for which z^{-1} does not exist. Show that each such element z has nonzero $w \in \mathcal{H}$ for which $zw = 0$.

Remark: a vector space V paired with a multiplication is called an **algebra**. For example, $\mathbb{C}$ is an algebra since $\mathbb{C}$ is a vector space where we also have a natural method to multiply vectors. Likewise, $\mathbb{R}^{n\times n}$ or $\mathbb{C}^{n\times n}$ form algebras with respect to the usual matrix multiplication. When given two algebras it is interesting to ask if they are isomorphic as algebras. This requires they have the same linear structure (which is the sense of isomorphism this course focuses primarily upon) **and** the multiplication is preserved in the natural fashion. To be precise, if $\mathcal{A}$ has multiplication $\star$ and $\mathcal{B}$ has multiplication $\circ$ then $\Psi : \mathcal{A} \to \mathcal{B}$ is an algebra isomorphism if

$$\Psi(x + cy) = \Psi(x) + c\Psi(y), \qquad \Psi(x \star y) = \Psi(x) \circ \Psi(y)$$

for all $x, y \in \mathcal{A}$ and $c \in \mathbb{F}$. We also insist that when $\mathcal{A}$ has a multiplicative identity $1_\mathcal{A}$ and likewise $1_\mathcal{B}$ for $\mathcal{B}$ then $\Psi(1_\mathcal{A}) = 1_\mathcal{B}$. In the above problem, we see $\Psi(1) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ hence Ψ is an algebra isomorphism of hyperbolic numbers and the subspace of 2×2 matrices of the special form $\begin{bmatrix} x & y \\ y & x \end{bmatrix}$.

Bonus 3 The direct product algebra on $\mathcal{A} = \mathbb{R}^n$ is defined by

$$(x_1, \dots, x_n) \star (y_1, \dots, y_n) = (x_1 y_1, \dots, x_n y_n)$$

for all $x, y \in \mathbb{R}^n$.

(a.) show the direct product algebra is a unital, associative algebra

(b.) which elements of $\mathcal{A}$ are invertible with respect to the given multiplication ?

(c.) find the matrix representation of $\mathcal{A}$ with respect to the standard basis

Bonus 4 **challenge:** show $\mathcal{H}$ is isomorphic to $\mathbb{R}^2$ with the direct product algebra and use this isomorphism to solve the quadratic equation in $\mathcal{H}$: for $b, c \in \mathcal{H}$

$$z^2 + bz + c = 0.$$

Bonus 5 The direct product algebra on $\mathcal{A} = \mathbb{C}^n$ is defined by

$$(z_1, \dots, z_n) \star (w_1, \dots, w_n) = (z_1 w_1, \dots, z_n w_n)$$

for all $z, w \in \mathbb{C}^n$. Find the matrix representation of $\mathcal{A}$ with respect to the basis β

$$\beta = \{e_1, ie_1, e_2, ie_2, \dots, e_n, ie_n\}.$$

Bonus 6 The n-hyperbolic numbers $\mathcal{H}_n = 1\mathbb{R} \oplus j\mathbb{R} \oplus \cdots \oplus j^{n-1}\mathbb{R}$ are numbers of the form $z = x_1 + x_2 j + x_3 j^2 + \cdots + x_n j^{n-1}$ where $j^n = 1$. If $\beta = \{1, j, j^2, \dots, j^{n-1}\}$ serves as the basis for $\mathcal{H}_n$ then find the matrix representation of z.

Bonus 7 If we visualize $\mathcal{H}_3$ as $\mathbb{R}^3$ then explain geometrically where the zero-divisors are found. Note, η is a zero divisor if there is $\alpha \neq 0$ for which $\eta\alpha = 0$.

Bonus 8 The n-complicated numbers $\mathcal{C}_n = 1\mathbb{R} \oplus k\mathbb{R} \oplus \cdots \oplus k^{n-1}\mathbb{R}$ are numbers of the form $\eta = x_1 + x_2 k + x_3 k^2 + \cdots + x_n k^{n-1}$ where $k^n = -1$. If $\beta = \{1, k, k^2, \dots, k^{n-1}\}$ serves as the basis for $\mathcal{C}_n$ then find the matrix representation of η.

Bonus 9 Find an algebra isomorphism of $\mathcal{C}_3$ and $\mathcal{H}_3$. In addition, show $\mathcal{H}_3$ is isomorphic to $\mathcal{B} = \mathbb{C} \times \mathbb{R}$ where we define $(z, x) \star (w, y) = (zw, xy)$ for each $(z, x), (w, y) \in \mathbb{C} \times \mathbb{R}$.

Bonus 10 **challenge:** solve the quadratic equation in $\mathcal{H}_3$; given $b, c \in \mathcal{H}_3$, find all $z \in \mathcal{H}_3$ for which

$$z^2 + bz + c = 0.$$

Bonus 11 Let $\mathcal{A}$ be an algebra of dimension n with basis $\{e_1, \dots, e_n\}$. Suppose we define

$$e_i \star e_j = \sum_{k=1}^{n} C_{ij}^k e_k$$

for appropriate **structure constants** $C_{ij}^k \in \mathbb{R}$. If $x = \sum_{i=1}^n x_i e_i$ then the **length** of x is defined by $\|x\| = \sqrt{\sum_{i=1}^n (x_i)^2}$. Show that there exists $M > 0$ for which $\|v \star w\| \leq M\|v\|\|w\|$ for all $v, w \in \mathcal{A}$. Also, find the smallest value possible for M which holds for the hyperbolic numbers.

Bonus 12 Consider $\beta = \{1, e_1, e_2, e_1 \wedge e_2\}$ serve to generate $V = \text{span}(\beta)$ as a real vector space of dimension 4. I'll arrange the $\wedge$ products in a table:

$\wedge$	1	e_1	e_2	$e_1 \wedge e_2$
1	1	e_1	e_2	$e_1 \wedge e_2$
e_1	e_1	0	$e_1 \wedge e_2$	0
e_2	e_2	$-e_1 \wedge e_2$	0	0
$e_1 \wedge e_2$	$e_1 \wedge e_2$	0	0	0

Extend the table linearly as to define $\wedge : V \times V \to V$ as a unital associative multiplication. Define the left multiplication with respect to the wedge product: $\ell_x(y) = x \wedge y$. Then,

$$\begin{aligned} [\ell_x]_{\beta,\beta} &= [[\ell_x(1)]_\beta | [\ell_x(e_1)]_\beta | [\ell_x(e_2)]_\beta | [\ell_x(e_1 \wedge e_2)]_\beta] \\ &= [[x]_\beta | [x \wedge e_1]_\beta | [x \wedge e_2]_\beta | [x \wedge e_1 \wedge e_2]_\beta] \end{aligned}$$

For example,

$$[L_{e_1}]_{\beta,\beta} = [[e_1]_\beta | [e_1 \wedge e_1]_\beta | [e_1 \wedge e_2]_\beta | [e_2 \wedge e_1 \wedge e_2]_\beta] = [[e_1]_\beta | 0 | [e_1 \wedge e_2]_\beta | 0] = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(a.) Complete my work by finding $[L_{e_2}]_{\beta,\beta}$, $[L_{e_1 \wedge e_2}]_{\beta,\beta}$ and the far more relaxing $[L_1]_{\beta,\beta}$.

(b.) Show $[L_{e_1}]_{\beta,\beta}[L_{e_2}]_{\beta,\beta} = [L_{e_1 \wedge e_2}]_{\beta,\beta}$

Remark: The problem above shows how we can represent the wedge product in terms of matrix multiplication. In particular, the wedge product on $\mathbb{R}^2$ is easily implemented with 4×4 matrices. Generally, we can follow much the same construction to build the wedge product algebra on $\mathbb{R}^n$ by representing it on $2^n \times 2^n$ matrices. That said, that's not really how I think about the wedge product.

Bonus 13 The exterior algebra of $\mathbb{R}^3$ is formed by the direct sum of k-forms for $k = 0, 1, 2, 3$;

$$\Omega\mathbb{R}^3 = \mathbb{R} \oplus \Lambda_1\mathbb{R}^3 \oplus \Lambda_2\mathbb{R}^3 \oplus \Lambda_3\mathbb{R}^3.$$

In the previous problem you explored some properties of the isomorphisms $\omega : \mathbb{R}^3 \to \Lambda_1\mathbb{R}^3$ and $\Phi : \mathbb{R}^3 \to \Lambda_2\mathbb{R}^3$. **Hodge Duality** exchanges a p-form for an $(n-p)$-form. We define Hodge duality on $\mathbb{R}^3$ as a linear map $* \in \mathcal{L}(\Omega\mathbb{R}^3)$ by the rules

$$*\omega_v = \Phi_v, \qquad *\Phi_v = \omega_v, \qquad *dx \wedge dy \wedge dz = 1, \qquad *1 = dx \wedge dy \wedge dz.$$

In this creative problem I want you to try to define Hodge duality for Euclidean four dimensional space with coordinates t, x, y, z. We'll define

$$W_{(\alpha,a,b,c)} = \alpha dt + \omega_{(a,b,c)} = \alpha dt + adx + bdy + cdz$$

for each $(\alpha, a, b, c) \in \mathbb{R}^4$; that is, $W_{(\alpha,v)} = \alpha dt + \omega_v$ for all $\alpha \in \mathbb{R}$ and $v \in \mathbb{R}^3$. Note $W : \mathbb{R}^4 \to \Lambda_1\mathbb{R}^4$ defines an isomorphism.

(a.) Find an isomorphism $\Psi : \mathbb{R}^4 \to \Lambda_3\mathbb{R}^4$ denoted by $\Psi((\alpha, v)) = \Psi_{(\alpha,v)}$

(b.) Give rules for $* : \Lambda_p\mathbb{R}^4 \to \Lambda_{4-p}$ for $p = 0, 1, 2, 3, 4$ such that $*W_{(\alpha,v)} = \Psi_{(\alpha,v)}$ and $*\Psi_{(\alpha,v)} = W_{(\alpha,v)}$. It suffices to define $*$ on the basis for p-forms as we may linear extend from such rules. I'll do two whole cases for you: $*1 = dt \wedge dx \wedge dy \wedge dz$ and $*dt \wedge dx \wedge dy \wedge dz = 1$. Now you just need to explain how it works for $p = 1, 2, 3$.

(c.) It is a quirk of three dimensions that $dim(\Lambda_1\mathbb{R}^3) = 3 = dim(\Lambda_2\mathbb{R}^3)$. Once more, in four dimensions W gives a natural isomorphism of vectors and one-forms. Discuss two-forms verses three-forms in regard to isomorphism with four dimensional vectors.