The solutions should be written neatly with the work clearly labeled. Do not omit scratch work. I need to see all steps. Skipping details will result in a loss of credit. Also, don't be afraid of the long-winded problem statements, often more words does not mean proportionally more work in my problems. I have estimated a length for each problem and given appropriate white-space. If you find it it not enough then feel free to add paper. Otherwise, I think this will help the grader. **Feel free to use a calculator to do the row reductions for Problems 43,44,45,47,48,49,50,53 and 64. It is not enough to write the answer to the matrix calculation, you should write the both A and $rref(A)$ and explain what you are doing, reference the Example in my notes or the book you are mimicking or what theorem you are using etc...** Finally, be warned that I will expect you can do the row-reductions w/o a calculator on the test so if you need practice still then by all means practice! There are 500 points to earn in this Problem Set. Thanks and enjoy.

Problem 35 [15pts] Let $A, B \in \mathbb{R}^{m\times n}$. **Prove** that if $AE_{ij} = BE_{ij}$ for all i, j then $A = B$.

Problem 36 [15pts] Let A, B be invertible matrices. **Prove** that if $A = B$ then $A^{-1} = B^{-1}$. (your proof should use little more than the definition of invertible matrices)

Problem 37 [15pts] Let $A \in \mathbb{R}^{n\times n}$ be similar to $B \in \mathbb{R}^{n\times n}$. **Prove** that A^k is similar to B^k for every $k \in \mathbb{N}$.

Problem 38 [15pts] **Prove** that $V = \mathbb{R}^{m \times n}$ forms a vector space. You may reference theorems in the notes (excluding Example 4.1.4 obviously).

Problem 39 [15pts] Consider Example 4.2.11, if I instead defined $W = \{f \in \mathcal{F}(\mathbb{R}) \mid \int_{-1}^{1} f(x)\, dx = 1\}$ would W form a vector space? Explain why or why not.

Problem 40 [15pts] Let $W = \{a\sin(x) + b\cos(x) + ce^x \ |a, b, c \in \mathbb{R}\}$. **Prove** that $W \leq \mathcal{F}(\mathbb{R})$.

Problem 41 [15pts] We define P_4 to be the set of polynomials up to order 4.

Let $W = \{g \in P_4 \mid g(2) = 0 \text{ and } g(17) = 0\}$. Show that $W \leq P_4$.

Problem 42 [20pts] Let $W = \{\ [w, x, y, z]^T \in \mathbb{R}^{\,4\times 1} \mid x = 2y, \text{ and } w + z = a\}$. For what value of $a \in \mathbb{R}$ is $W \leq \mathbb{R}^{\,4\times 1}$? Set a to that value and prove W is a subspace of $\mathbb{R}^{\,4\times 1}$ in that case.

Problem 43 [20pts] Consider the following sets of column vectors:

$$S = \{\ [1, 0, 2, 3]^T,\ [2, 4, 5, 6]^T,\ [0, 0, 3, 2]^T\ \}$$

$$T = \{\ [2, 0, 10, 10]^T,\ [1, 0, 0, 0]^T,\ [0, 3, 0, 0]^T\ \}$$

Which of the vectors in T are in the $span(S)$? For those vector(s) in T which are in the span, give the linear combinations of vectors in S which produce those vector(s).

Problem 44 [20pts] Let $S = \{x, x+1, x^3\}$ and let $T = \{1, x^2+x, x^3+2x+1\}$. Which of the polynomials in T in the span of S? Give the linear combinations which prove your assertions.

Problem 45 [20pts] Let $A = \begin{bmatrix} 2 & 5 \\ -3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 4 & 0 \\ -3 & 2 \end{bmatrix}$. Is E_{12} in $span\{A, B, C\}$? **Explain** why or why not in terms of explicit calculations.

Problem 46 [20pts] Let $A_3 = \{M \in \mathbb{R}^{3\times 3} | M^T = -M\}$. **Prove** that $A_3 \leq \mathbb{R}^{3\times 3}$. Also, find a basis for A_3 and state the dimension of A_3.

Problem 47 [20pts] Is the set of vectors S (defined below) linearly independent?

$$S = \{\ [1,0,2]^T,\ [1,1,0]^T,\ [0,0,3]^T, [1,1,1]^T\ \}$$

If not then pick a subset $T \subset S$ which is linearly independent and explain how the vectors in $S - T$ are linearly dependent on the vectors in T. What is a basis for the subspace $span(S)$?

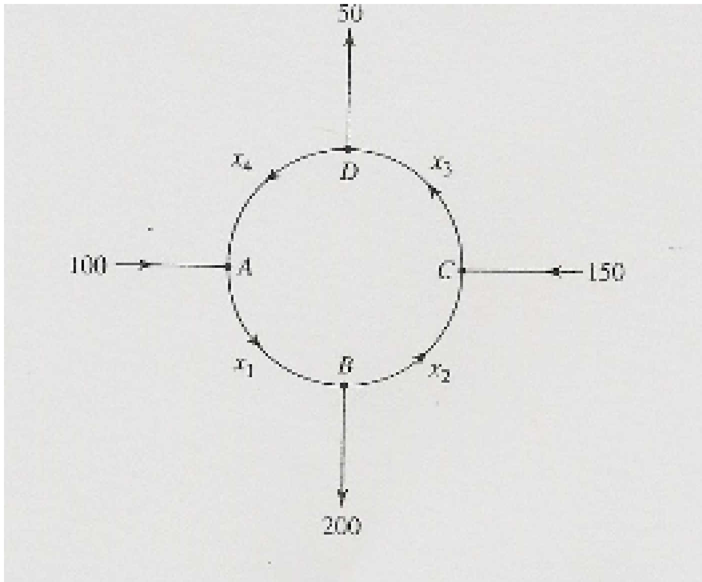
Problem 48 [15pts] Given A (defined below) find a basis for $Col(A)$, $Row(A)$ and $Null(A)$.

$$A = \begin{bmatrix} 2 & 5 & 3 & 1 \\ -1 & 1 & 0 & 1 \\ 1 & 2 & 0 & 2 \end{bmatrix}$$

Problem 49 [15pts] Given A (defined below) find a basis for $Col(A)$, $Row(A)$ and $Null(A)$. This time make sure the basis for $Row(A)$ and $Col(A)$ are chosen from the actual rows and columns of A.

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

Problem 50 [15pts] Find the general solution for the traffic flow problem pictured below:



Notice traffic only flows in the direction of the arrows. What is the minimum flow possible through x_2?

Problem 51 [20pts] Find the standard matrix for the linear operator $T : \mathbb{R}^{3\times 1} \to \mathbb{R}^{3\times 1}$ defined below:

$$T([x, y, z]^T) = [x + y, z - x, x + y + z]^T.$$

Also find a basis for the $Null(T)$ and $Range(T)$. What are the rank and nullity of this linear transformation? Is this linear transformation injective, surjective or both?

Problem 52 [20pts] Suppose the linear operator $S : \mathbb{R}^{3\times 1} \to \mathbb{R}^{2\times 1}$ is defined by

$$S([x, y, z]^T) = [x - y, z - y]^T.$$

Perform the following tasks, you should take T to be the operator defined in Problem 51,

(i.) Find the standard matrix of S,

(ii.) Calculate $(S \circ T)([x, y, z]^T)$ explicitly from the definition of the composite of two operators; namely $(S \circ T)([x, y, z]^T) = S(T([x, y, z]^T))$.

(iii.) Determine the standard matrix of $S \circ T$ from the formula you derived in part (ii.).

(iv.) Multiply the standard matrices of S and T to check your answer from part (iii.)

Problem 53 [25pts] Let $\beta = \{e_1 + e_2, e_2 + e_3, e_3 + e_1\}$ be a basis for $\mathbb{R}^{3\times 1}$. Furthermore, let $\beta' = \{2e_1, 4e_2 - e_1\}$ be a basis for $\mathbb{R}^{2\times 1}$. Find the matrix of the linear transformation S from problem 52 with respect to this new set of nonstandard bases. In other words, find $[S]_{\beta,\beta'}$. Also, find the matrix of T from problem 51 with respect to the new basis; $[T]_{\beta,\beta}$. Last, find $[T]_{\beta,\beta_e}$ where I mean β_e to denote the standard basis for $\mathbb{R}^{3\times 1}$.

Problem 54 [15pts] Let the set of polynomials up to order 3 have basis $\beta = \{1, x, x^2, x^3\}$. Define $T : P_3 \to P_3$ by $T[f](x) = f'(x)$ for all $f \in P_3$. **Prove** that T is a linear operator and find its matrix with respect to β. (note this is exceedingly similar to Ex. 5.5.3)

Problem 55 [15pts] In Problem 40 you proved that $W = \{a\sin(x)+b\cos(x)+ce^x \mid a, b, c \in \mathbb{R}\} \leq \mathcal{F}(\mathbb{R})$. Show that $T : W \to W$ defined by $T[f](x) = f''(x) + f(x)$ for all $f \in W$ is a linear operator. Also, find the matrix of T with respect to the basis $\{\sin(x), \cos(x), e^x\}$. Calculate $Ker(T)$ and $Range(T)$.

Problem 56 [15pts] Integral transforms provide a sophisticated indirect approach to tackle difficult calculus problems. One example is the **Laplace Transform**. For appropriate functions we define the Laplace Tranform by

$$\mathcal{L}[f](s) = \int_0^\infty e^{-st} f(t)dt$$

Common notation for this is $\mathcal{L}[y] = Y$. We transform $y(t)$ to $Y(s)$, this takes us from the "time domain" to the "frequency domain" in EE-lingo. **Prove** that $\mathcal{L}$ is a linear operator from $S = \{f \in \mathcal{F}(\mathbb{R}) \mid \mathcal{L}[f] \in \mathcal{F}(\mathbb{R})\}$ to $\mathcal{F}(\mathbb{R})$. *I'm just trying to say you should show linearity under the assumption that the Laplace transform of the functions exists, in general not all functions have a well-defined transform... I just want you to see this is yet another example of an interesting linear transformation*

Problem 57 [15pts] Suppose that $T_A : \mathbb{R}^{\ m\times n} \to \mathbb{R}^{\ n\times p}$ and $T_B : \mathbb{R}^{\ m\times n} \to \mathbb{R}^{\ n\times p}$ are linear operators with standard matrices A and B respective. **Prove** that $T_A + T_B$ is a linear operator and the standard matrix of $T_A + T_B$ is $A + B$.

Problem 58 [15pts] Suppose $T : \mathbb{R}^{\,2\times 1} \to \mathbb{R}^{\,2\times 1}$ is an invertible linear operator. If

$$P = \{\vec{r}_o + u\vec{a} + v\vec{b} \mid 0 \leq u, v \leq 1\} \subset \mathbb{R}^{\,2\times 1}$$

describes a paralellogram based at $\vec{r}_o$ with sides $\vec{a}, \vec{b}$ then what is $T(P)$ geometrically speaking? How are the area of P and the area of $T(P)$ related ?

Problem 59 [15pts] Define $\mathbb{R}^3 = \{(x,y,z) \mid x,y,z \in \mathbb{R}\}$ and define vector addition by $(x,y,z) + (a,b,c) = (x+a, y+b, z+c)$ and scalar multiplication by $\lambda(x,y,z) = (\lambda x, \lambda y, \lambda z)$ for all $\lambda \in \mathbb{R}$ and $(x,y,z),(a,b,c) \in \mathbb{R}^3$. **Find an isomorphism** to A_3 from Problem 46. Then **prove** your construction is an isomorphism. Is there any particular geometric significance to the determinant or trace of a matrix in A_3 given the correspondence you just discovered?

Problem 60 [15pts] It's straightforward exercise to prove that the complex numbers $\mathbb{C} = \{a+ib \mid a,b \in \mathbb{R}\}$ form a vector space with vector addition $(a+ib)+(c+id) = a+c+i(b+d)$ and real scalar multiplication $r(a+ib) = ra+irb$ for all $a+ib, c+id \in \mathbb{C}$ and $r \in \mathbb{R}$. **Find a vector space isomorphism** $\Phi : \mathbb{C} \to \mathbb{R}^2$ where I'm denoting the usual $\mathbb{R}^2 = \{(x,y) \mid x,y \in \mathbb{R}\}$ and I assume $\mathbb{R}^2$ has the standard vector addition and scalar multiplication (see Problem 59 for the $n=3$ case). Assume complex numbers are multiplied by the usual rule; $(a+ib)(x+iy) = ax+ibx+iay+i^2by = ax-by+i(bx+ay)$. **Find a multiplication** on $\mathbb{R}^2$ called $\star$ by assuming that $\Phi(zw) = \Phi(z)\star\Phi(w)$. This construction is originally due to Gauss in the early 19-th century. This bridge between real 2-vectors and complex numbers is very useful in the construction of n-dimensional complex theory which is from this view point just a $2n$-dimensional real theory with a particular structure. This is one of the first things you'll do in the complex variables course (although, you may not emphasize it's a vector space isomorphism).

Problem 61 [15pts] A *derivation* on $\mathcal{F}(\mathbb{R}, \mathbb{C})$ is a linear operator $T : \mathcal{F}(\mathbb{R}, \mathbb{C}) \rightarrow \mathcal{F}(\mathbb{R}, \mathbb{C})$ which satisfies the product rule $T[fg](x) = T[f](x)g(x) + f(x)T[g](x)$. Remember the product of operators is defined to be the composite of the operators; for example if T is an operator then $T^2[f] = (T \circ T)[f] = T[T[f]]$. Prove the following claims using theorems from calculus:

(i.) If $D[f](x) = \frac{df}{dx}$ then D is derivation,

(ii.) Let $a \in \mathbb{C}$ then $a[f](x) = af(x)$ defines the *multiplication by a operator.* If $T[f] = af$ then show that $T \circ D = D \circ T$. (*note you should check that* $(T \circ D)[f] = (D \circ T)[f]$ *for arbitrary* $f \in \mathcal{F}(\mathbb{R}, \mathbb{C})$, *this is what is needed to prove* $T \circ D = D \circ T$)

(iii.) The differential equation $y'' + 5y' + 6y = 0$ can be written as $L[y] = 0$ for $L = D^2 + 5D + 6$ using the notation introduced in part (ii.). Show that $L = (D + 2)(D + 3)$
(*to do this you should use what you just proved in (ii.).*

(iv.) Show that the set of solutions to $y'' + 5y' + 6y = 0$ forms a subspace. *This proves that* L *is a linear operator. You might also notice that the set of solutions is the kernel of* L.

Generally speaking, given an n-th order homogeneous constant coefficient ordinary linear differential equation we can write it as an operator equation $L[y] = 0$. The operator then factors into n-linear factors which correspond to n-linearly independent solutions in a precise way which we describe in our math 334 course. This set of solutions forms an n-dimensional subspace of $\mathcal{F}(\mathbb{R}, \mathbb{R})$. This is not suprising if you remember that solving an n-th order differential equation is an indirect way of doing an n-fold integration. Each integration yields an integration constant which corresponds to a parameter tracing out one of the n-dimensions of the solution set. All of this happens inside the infinite dimensional function space.

Problem 62 [15pts] Prove that if a linear transformation $L : V \to W$ is 1-1 and $dim(V) = dim(W)$ then $L(V) = W$. In other words, **prove that injective linear transformations from an n-dimensional vector space to another n-dimensional vector space are necessarily surjective. Give examples** of how this fails to be true if $dim(V) \neq dim(W)$. Also, **give an example** of a function from $\mathbb{N}$ to $\mathbb{N}$ which is 1-1 but not onto. The fact that 1-1 implies onto for linear mappings from a *finite dimensional* vector space V to itself is rather special. You might recall that the same is true for finite sets. If $f : S \to S$ is an onto function then it's also 1-1 if we know that S is a finite set. In the case of the finite dimensional vector space, we don't have a finite set, however we do have a finite basis.

Problem 63 [15pts] Let V be a vector space and suppose W_1 and W_2 are subspaces which only intersect at the zero vector ($W_1 \cap W_2 = \{0\}$). Furthermore suppose every vector in V can be written as $w_1 + w_2$ for some choice of $w_1 \in W_1$ and $w_2 \in W_2$. Given these conditions we say that V is the **direct sum** of W_1 and W_2 and this concept is denoted by stating $V = W_1 \oplus W_2$. Suppose β_1 is a basis for W_1 and β_2 is a basis for W_2 prove that $\beta_1 \cup \beta_2$ is a basis for V.

Problem 64 [15pts] Consider the following sets of vectors

$$S = \{\ [1,0,2]^T,\ [1,3,4]^T,\ [0,3,2]^T,\ [3,3,8]^T\ \}$$

$$T = \{\ [2,3,6]^T,\ [-2,3,-2]^T, [0,-4,-2]^T\ \}$$

Answer the following calculations by utilzing the matrix techniques concerning spanning and linear independence of column vectors we found in Chapter 4.

(i.) is $T \subset span(S)$? If not find a subset of T which is a subset of $span(S)$. Denote this subset by T_S.

(ii.) Is T_S a linearly independent set?

(iii.) Find a basis for $span(S)$ that consists of vectors from T_S and vectors in S.