Closed book, no matrix operations calculator. Remember, you must justify your answers.

Problem 1 [25pts] Let $A = \begin{bmatrix} 1 & 0 & 2 \\ -2 & 1 & 4 \\ -3 & 1 & 3 \end{bmatrix}$ find A^{-1}

Problem 2 [20pts] Let a, b, c be constants. Solve the following system of equations **by using the result of the previous problem.**

$$\begin{aligned} x + 2z &= a \\ -2x + y + 4z &= b \\ -3x + y + 3z &= c \end{aligned}$$

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Problem 3 [10pts] Suppose A, B are square. Let $Ax = 0$ have only the $x = 0$ solution. Also, suppose $By = 0$ has only the $y = 0$ solution. Prove that $M = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right]$ is invertible.

Problem 4 [15pts] Find the solution set of the system of equations $x + 4y + z = 0$ and $y - 3z = 0$.

Problem 5 [15pts] Let $T(x, y, z) = (x + 4y + z, y - 3z)$. Find the standard matrix $[T]$ for T and find a basis for $\text{Ker}(T)$.

Problem 6 [20pts] Fun facts:

$$\text{rref}\underbrace{\left[\begin{array}{ccc|cc} 1 & 2 & 3 & 6 & 3 \\ -2 & 4 & 2 & 4 & 2 \\ -3 & 1 & 7 & 5 & -2 \end{array}\right]}_{V_1,\ V_2,\ V_3,\ V_4,\ V_5} = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \quad \text{rref}\underbrace{\left[\begin{array}{cc|ccc} 3 & 6 & 3 & 2 & 1 \\ 2 & 4 & 2 & 4 & -2 \\ -2 & 5 & 7 & 1 & -3 \end{array}\right]}_{V_5,\ V_4,\ V_3,\ V_2,\ V_1} = \left[\begin{array}{cc|ccc} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

Notice I define V_1, V_2, V_3, V_4, V_5 in the calculations above. Given the data above, answer the following: (when I say answer, I mean "yes" or "no" followed by a brief sentence to explain why. Little if any additional calculation is needed to answer these if you understand the CCP)

(a.) is $V_4 \in \text{span}\{V_1, V_2, V_3\}$?

(b.) is $V_3 \in \text{span}\{V_4, V_5\}$?

(c.) is $V_1 \in \text{span}\{V_4, V_5\}$?

(d.) let $A = [V_5|V_4|V_3]$, find a basis for $\text{Col}(A)$ and also find a basis for $\text{Null}(A)$.

Problem 7 [20pts] Let $B \in \mathbb{R}^{n\times n}$ and define $T(A) = AB + BA$ for all $A \in \mathbb{R}^{n\times n}$. Show that T is a linear transformation.

Problem 8 [10pts] Suppose $f(x) = (2x-3)^2$. Let $\beta = \{x^2, x, 1\}$ and find $[f(x)]_\beta$.

Problem 9 [25pts] Suppose $W = \{(f(x), g(x)) \mid f(x), g(x) \in P_2,\ f(1) = 0,\ g(0) = 0\}$. Show that $W \leq P_2 \times P_2$ and find a basis for W. Find $\dim(W)$.

Problem 10 [10pts] Let $v = (a, b)$ and find $[v]_\beta$ given basis $\beta = \{(2, 1), (1, 1)\}$.

Problem 11 [10pts] Suppose A in not invertible. Why is zero an eigenvalue of A?

Problem 12 [10pts] Suppose $\det[u|v|w] = 2$. Furthermore, suppose A is a 3×3 matrix for which $Au = u$, $Av = 3v$ and $Aw = 5w$. Calculate $\det(A)$ and $\text{trace}(A)$.

Problem 13 [20pts] Suppose $S = \{v, w\}$ is a set of nonzero orthogonal vectors.
Prove $T = \{v + 2w, 3v - w\}$ is LI.

Problem 14 [10pts] Suppose $T : V \to V$ is a linear transformation on a two-dimensional vector space V with basis $\beta = \{v_1, v_2\}$. You are given that the complexification of T has:

$$T_{\mathbb{C}}(v_1 + iv_2) = 6v_1 + 2v_2 + i(8v_1 + 4v_2)$$

Find the matrix of T in the β basis; that is, find $[T]_{\beta,\beta}$.

Problem 15 [15pts] Prove: if T is injective then $\mathrm{Ker}(T) = \{0\}$.

Problem 16 [15pts] Let U and W be subspaces of a vector space V. Prove $U \cap W \leq V$.

Problem 17 [10pts] Suppose that If $A = \mathrm{diag}\left(\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} \right)$ where this notation indicates that A is block-diagonal with the diagonal blocks as given. Find the eigenvalues of A and state the algebraic and geometric multiplicity of each eigenvalue. We use the notation λ_j has algebraic multiplicity a_j and geometric multiplicity g_j

Problem 18 [20pts] Let $W = \mathrm{span}\{(1,3,2,3),(0,1,2,2)\}$. Find a basis for $W^{\perp}$.

Problem 19 [20pts] Let $T : P_2 \to P_2$ be defined by $T(f(x)) = \dfrac{df}{dx} + f(x)$. Let $\beta = \{1, x, x^2\}$ form the basis for P_2. Calculate $[T]_{\beta,\beta}$ and find the dimension of Ker(T).

Problem 20 [15pts] Consider $T : V \to W$ where $V = \text{span}\{v_1, v_2\}$ and $W = \text{span}\{w_1, w_2, w_3\}$. Assume T is a linear transformation. List the possible dimensions for Ker(T) and Range(T). Write your answer in tabular form as indicated below.

dim(V)	dim($Ker(T)$)	dim($Range(T)$)

Problem 21 [15pts] Suppose $A \in \mathbb{R}^{m\times p}$ and $B \in \mathbb{R}^{p\times n}$. Prove that $(AB)^T = B^T A^T$.