Closed book, no matrix operations calculator. Remember, you must justify your answers. $\mathbb{F}$ denotes a field and V, W denote finite-dimensional vector spaces over $\mathbb{F}$ unless otherwise specified. Also, $L(V,W)$ denotes the set of linear transformations from V to W.

Problem 1 Solve $x + y = 3, y + z = 5, x + z = 4$.

Problem 2 Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix}$ and $v = [1, 2]$. Calculate $A^2 + vv^T$

Problem 3 Let $S = \{(1, 2, 3, 4), (1, 2, 4, 0)\}$. Find a basis for $S^{\perp}$.

Problem 4 Notice I define V_1, V_2, V_3, V_4, V_5 in the calculations below.

$$\text{rref}\underbrace{\left[\begin{array}{ccc|cc} 1 & 2 & 3 & 6 & 3 \\ -2 & 4 & 2 & 4 & 2 \\ -3 & 1 & 7 & 5 & -2 \end{array}\right]}_{V_1,\ V_2,\ V_3,\ V_4,\ V_5} = \left[\begin{array}{ccc|cc} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array}\right] \quad \text{rref}\underbrace{\left[\begin{array}{cc|ccc} 3 & 6 & 3 & 2 & 1 \\ 2 & 4 & 2 & 4 & -2 \\ -2 & 5 & 7 & 1 & -3 \end{array}\right]}_{V_5,\ V_4,\ V_3,\ V_2,\ V_1} = \left[\begin{array}{cc|ccc} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{array}\right]$$

(a.) is $V_4 \in \text{span}\{V_1, V_2, V_3\}$?

(b.) is $V_1 \in \text{span}\{V_4, V_5\}$?

(c.) let $A = [V_5|V_4|V_3]$, find a basis for $\text{Col}(A)$ and also find a basis for $\text{Null}(A)$.

Problem 5 Let $A = \begin{bmatrix} 1 & -2 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$

(a.) calculate $A^T A$.

(b.) find A^{-1} (this you should be able to do without a single row-reduction)

Problem 6 Consider $\left\{ \begin{array}{c} x - 2y = a, \\ x + y - z = b \\ x + y + z = c \end{array} \right\}$ for given $a, b, c \in \mathbb{R}$. Solve this system. Hint: use what you already learned in the previous problem.

Problem 7 If $\beta = \{(1,1,1), (-2,1,1), (0,-1,1)\}$ then what kind of basis is β? Find $[(1,2,3)]_\beta$

Problem 8 If $T(x,y,z) = (x+2y, y+z, x+3z)$ defines a linear transformation on T then find $[T]$ and $[T]_{\beta,\beta}$ where β is the basis given in the previous problem. In view of your calculation, what are the eigenvalues of T ? (thou shall **not** calculate/factor $\det(T - \lambda I)$)

Problem 9 Let A, B be multipliable matrices over $\mathbb{F}$. Prove $(AB)^T = B^T A^T$

Problem 10 Let $T : V \to V$ be a linear transformation for which $T^2 + 3T + 2 = 0$. Is T diagonalizable ? What are the eigenvalues of T ?

Problem 11 Denote the standard bases $e_i \in \mathbb{F}^n$ for $1 \leq i \leq n$ whereas $\bar{e}_j \in \mathbb{F}^m$ for $1 \leq j \leq m$. Recall, we define for $A \in \mathbb{F}^{m \times n}$ and $x \in \mathbb{F}^n$ the column-vector product by $(Ax)_i = \sum_{j=1}^{n} A_{ij} x_j$ which means $Ax = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} x_j \bar{e}_i$. **Prove the following:**
$T \in L(\mathbb{F}^n, \mathbb{F}^m)$ **if and only if** there exists $A \in \mathbb{F}^{m \times n}$ for which $T(x) = Ax$ for all $x \in \mathbb{F}^n$.

Problem 12 Let $Q(x, y, z) = x^2 + 2yz$. Find the matrix of Q and its eigenvalues. Find the formula for Q in eigencoordinates $\bar{x}, \bar{y}, \bar{z}$

Problem 13 [10pts] Suppose A, B are square. Also A is nilpotent order 3 and B is nilpotent order 2. Prove that $M = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right]$ is nilpotent. What is the order of M? Is M invertible ?

Problem 14 Let $W = \{(x, y, z) \mid x \geq 0\}$ is W a subspace of $\mathbb{R}^3$, **prove or disprove**.

Problem 15 Let $W = \{ax^2 + bx^9 \mid a, b \in \mathbb{C}\}$. Is $W \leq P(\mathbb{C})$? **Prove or disprove**.

Problem 16 Let $S, T \in L(V, W)$ and define $U = \{x \in V \mid S(x) = T(x)\}$. Is $U \leq V$? **Prove or disprove**.

Problem 17 **Prove:** if $T : V \to V$ is injective then $\text{Ker}(T) = \{0\}$

Problem 18 Let $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and define $T(A) = BA$ for all $A \in \mathbb{R}^{2\times 2}$. Calculate the matrix of T with respect to the basis $\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ for $\mathbb{R}^{2\times 2}$.

Bonus: *Reformulate your answer for* $[T]_{\beta,\beta}$ *in terms of the* $\otimes$*-product of* 2×2 *matrices.*

Problem 19 Let $T(f(x)) = f''(x)$ for $T : P_4(\mathbb{R}) \to P_2(\mathbb{R})$. Prove T is a surjection.

pick one of the pair below to solve:

Problem 20 Suppose $f(x) = (2x - 3)^2$. Let $\beta = \{x^2, x, 1\}$ and find $[f(x)]_\beta$.

Problem 21 Prove $\{x^2, x + 1\}$ is a LI subset of $P_2(\mathbb{R})$.

pick one of the pair below to solve:

Problem 22 Let $T : V \to V$ be a linear transformation with eigenvectors v, w with eigenvalues α, β respective such that $\alpha \neq \beta$. Prove $\{v, w\}$ is a LI subset of V.

Problem 23 Suppose $S = \{v, w\}$ is a set of nonzero g-orthogonal vectors in a real geometry (V, g). Prove $T = \{v + 2w, 3v - w\}$ is LI.

pick one of the pair below to solve:

Problem 24 Suppose $T : V \to V$ is a linear transformation on a two-dimensional vector space V with basis $\beta = \{v_1, v_2\}$. You are given that the complexification of T has:

$$T_{\mathbb{C}}(v_1 + iv_2) = 6v_1 + 2v_2 + i(8v_1 + 4v_2)$$

Find the matrix of T in the β basis; that is, find $[T]_{\beta,\beta}$.

Problem 25 Let (V, g) be a real geometry. Show that the composition of g-orthogonal transformations on V is once more a g-orthgonal transformation on V.

Problem 26 Let U and W be subspaces of a vector space V. Prove $U \cap W \leq V$.

Problem 27 Suppose $\det[u|v|w] = 2$. Furthermore, suppose A is a 3×3 matrix for which $Au = u$, $Av = 3v$ and $Aw = 5w$. Calculate $\det(A)$ and $\text{trace}(A)$.

Problem 28 Let R be a 3×3 rotation matrix. Is it possible for $\text{tr}(R) = a$. What condition is needed for $a \in \mathbb{R}$? By what angle does R rotate most vectors ?

Problem 29 Suppose that If $A = \text{diag}\left(\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} \right)$ where this notation indicates that A is block-diagonal with the diagonal blocks as given. Find the eigenvalues of A and state the algebraic and geometric multiplicity of each eigenvalue. We use the notation λ_j has algebraic multiplicity a_j and geometric multiplicity g_j (we consider $A \in \mathbb{C}^{8\times 8}$). Also, find the characteristic and minimal polynomials of A (no need to multiply out!, please leave in factored form)

Problem 30 Define a metric on $V = P_1(\mathbb{R}) = \text{span}\{1, x\}$ by the rule $g(f(x), g(x)) = \int_0^1 f(x)g(x)\,dx$.

- **(a.)** show g is a metric.
- **(b.)** find G with respect to $\{1, x\}$
- **(c.)** Let $\alpha(a + bx) = a - b$ define $\alpha \in V^*$. Find $\sharp\alpha$ (aka find the Riesz' vector of α)

Problem 31 Suppose $T \in L(V, V)$ has $Y \leq V$ as an invariant subspace. Let $S : V/Y \to V/Y$ be defined by $S(x + Y) = T(x) + Y$. Show that S is well-defined.

Problem 32 Prove the rank nullity theorem for $T : V \to V$ using an explicit basis extension argument (aka in the style of the solution to this question on Test 1)

Problem 33 Suppose U and W are finite-dimensional vector spaces over $\mathbb{F}$. If $T : U \to V$ and $S : V \to W$ are linear transformations and $S \circ T$ is an isomorphism then what can we say about the dimension of V ? (prove your assertion)