Closed book, no matrix operations calculator. Remember, you must justify your answers. \mathbb{F} denotes a field and V, W denote finite-dimensional vector spaces over \mathbb{F} unless otherwise specified. Also, L(V, W) denotes the set of linear transformations from V to W.

Problem 1 Solve x + y = 3, y + z = 5, x + z = 4.

Problem 2 Let
$$A = \begin{bmatrix} 1 & 2 \\ 3 & 3 \end{bmatrix}$$
 and $v = [1, 2]$. Calculate $A^2 + vv^T$

Problem 3 Let $S = \{(1, 2, 3, 4), (1, 2, 4, 0)\}$. Find a basis for S^{\perp} .

Problem 4 Notice I define V_1, V_2, V_3, V_4, V_5 in the calculations below.

$$\operatorname{rref} \underbrace{\left[\begin{array}{ccc|c} 1 & 2 & 3 & 6 & 3 \\ -2 & 4 & 2 & 4 & 2 \\ -3 & 1 & 7 & 5 & -2 \end{array} \right]}_{V_{1}, \ V_{2}, \ V_{3}, \ V_{4}, \ V_{5}} = \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{array} \right] \ \operatorname{rref} \underbrace{\left[\begin{array}{ccc|c} 3 & 6 & 3 & 2 & 1 \\ 2 & 4 & 2 & 4 & -2 \\ -2 & 5 & 7 & 1 & -3 \end{array} \right]}_{V_{5}, \ V_{4}, \ V_{3}, \ V_{2}, \ V_{1}} = \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{array} \right]$$

- (a.) is $V_4 \in \text{span}\{V_1, V_2, V_3\}$?
- **(b.)** is $V_1 \in \text{span}\{V_4, V_5\}$?
- (c.) let $A = [V_5|V_4|V_3]$, find a basis for Col(A) and also find a basis for Null(A).

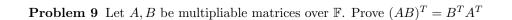
Problem 5 Let
$$A = \begin{bmatrix} 1 & -2 & 0 \\ 1 & 1 & -1 \\ 1 & 1 & 1 \end{bmatrix}$$

- (a.) calculate $A^T A$.
- (b.) find A^{-1} (this you should be able to do without a single row-reduction)

Problem 6 Consider $\begin{cases} x-2y=a, \\ x+y-z=b \\ x+y+z=c \end{cases}$ for given $a,b,c\in\mathbb{R}$. Solve this system. Hint: use what you already learned in the previous problem.

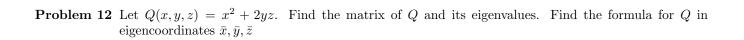
Problem 7 If $\beta = \{(1,1,1), (-2,1,1), (0,-1,1)\}$ then what kind of basis is β ? Find $[(1,2,3)]_{\beta}$

Problem 8 If T(x,y,z) = (x+2y,y+z,x+3z) defines a linear transformation on T then find [T] and $[T]_{\beta,\beta}$ where β is the basis given in the previous problem. In view of your calculation, what are the eigenvalues of T? (thou shall **not** calculate/factor $\det(T - \lambda I)$)



Problem 10 Let $T: V \to V$ be a linear transformation for which $T^2 + 3T + 2 = 0$. Is T diagonalizable? What are the eigenvalues of T?

Problem 11 Denote the standard bases $e_i \in \mathbb{F}^n$ for $1 \leq i \leq n$ whereas $\bar{e}_j \in \mathbb{F}^m$ for $1 \leq j \leq m$. Recall, we define for $A \in \mathbb{F}^{m \times n}$ and $x \in \mathbb{F}^n$ the column-vector product by $(Ax)_i = \sum_{j=1}^n A_{ij}x_j$ which means $Ax = \sum_{i=1}^m \sum_{j=1}^n A_{ij}x_j\bar{e}_i$. **Prove the following:** $T \in L(\mathbb{F}^n, \mathbb{F}^m)$ if and only if there exists $A \in \mathbb{F}^{m \times n}$ for which T(x) = Ax for all $x \in \mathbb{F}^n$.



Problem 13 [10pts] Suppose A, B are square. Also A is nilpotent order 3 and B is nilpotent order 2. Prove that $M = \begin{bmatrix} A & 0 \\ \hline 0 & B \end{bmatrix}$ is nilpotent. What is the order of M? Is M invertible?

Problem 14 Let $W = \{(x, y, z) \mid x \ge 0\}$ is W a subspace of \mathbb{R}^3 , **prove or disprove**.

Problem 15 Let $W = \{ax^2 + bx^9 \mid a, b \in \mathbb{C}\}$. Is $W \leq P(\mathbb{C})$? **Prove or disprove**.

Problem 16 Let $S, T \in L(V, W)$ and define $U = \{x \in V \mid S(x) = T(x)\}$. Is $U \leq V$? **Prove or disprove.**

Problem 17 Prove: if $T: V \to V$ is injective then $Ker(T) = \{0\}$

Problem 18 Let $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and define T(A) = BA for all $A \in \mathbb{R}^{2 \times 2}$. Calculate the matrix of T with respect to the basis $\beta = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$ for $\mathbb{R}^{2 \times 2}$. **Bonus:** Reformulate your answer for $[T]_{\beta,\beta}$ in terms of the \otimes -product of 2×2 matrices.

Problem 19 Let T(f(x)) = f''(x) for $T: P_4(\mathbb{R}) \to P_2(\mathbb{R})$. Prove T is a surjection.

pick one of the pair below to solve:

- **Problem 20** Suppose $f(x) = (2x 3)^2$. Let $\beta = \{x^2, x, 1\}$ and find $[f(x)]_{\beta}$.
- **Problem 21** Prove $\{x^2, x+1\}$ is a LI subset of $P_2(\mathbb{R})$.

pick one of the pair below to solve:

- **Problem 22** Let $T: V \to V$ be a linear transformation with eigenvectors v, w with eigenvalues α, β respective such that $\alpha \neq \beta$. Prove $\{v, w\}$ is a LI subset of V.
- **Problem 23** Suppose $S = \{v, w\}$ is a set of nonzero g-orthogonal vectors in a real geometry (V, g). Prove $T = \{v + 2w, 3v - w\}$ is LI.

pick one of the pair below to solve:

Problem 24 Suppose $T:V\to V$ is a linear transformation on a two-dimensional vector space V with basis $\beta=\{v_1,v_2\}$. You are given that the complexification of T has:

$$T_{\mathbb{C}}(v_1 + iv_2) = 6v_1 + 2v_2 + i(8v_1 + 4v_2)$$

Find the matrix of T in the β basis; that is, find $[T]_{\beta,\beta}$.

Problem 25 Let (V, g) be a real geometry. Show that the composition of g-orthogonal transformations on V is once more a g-orthogonal transformation on V.



Problem 27 Suppose $\det[u|v|w] = 2$. Furthermore, suppose A is a 3×3 matrix for which Au = u, Av = 3v and Aw = 5w. Calculate $\det(A)$ and $\operatorname{trace}(A)$.

Problem 28 Let R be a 3×3 rotation matrix. Is it possible for $\operatorname{tr}(R) = a$. What condition is needed for $a \in \mathbb{R}$? By what angle does R rotate most vectors?

Problem 29 Suppose that If $A = \operatorname{diag} \left(\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}, \begin{bmatrix} 4 & 1 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \begin{bmatrix} 2 & -3 \\ 3 & 2 \end{bmatrix} \right)$ where this notation indicates

that A is block-diagonal with the diagonal blocks as given. Find the eigenvalues of A and state the algebraic and geometric multiplicity of each eigenvalue. We use the notation λ_j has algebraic multiplicity a_j and geometric multiplicity g_j (we consider $A \in \mathbb{C}^{8\times 8}$). Also, find the characteristic and minimal polynomials of A (no need to multiply out!, please leave in factored form)

Problem 30 Define a metric on $V = P_1(\mathbb{R}) = \text{span}\{1, x\}$ by the rule $g(f(x), g(x)) = \int_0^1 f(x)g(x) dx$.

- (a.) show g is a metric.
- **(b.)** find G with respect to $\{1, x\}$
- (c.) Let $\alpha(a+bx)=a-b$ define $\alpha\in V^*$. Find $\sharp\alpha$ (aka find the Riesz' vector of α)

Problem 31 Suppose $T \in L(V, V)$ has $Y \leq V$ as an invariant subspace. Let $S: V/Y \to V/Y$ be defined by S(x+Y) = T(x) + Y. Show that S is well-defined.

