Same rules as Homework 1. **Assume** all vector spaces are **finite dimensional** for ease of mind. There are many results here which do transfer to the world of infinite dimensional vector spaces, but, I'll leave that for another day. We denote $V^* = L(V, \mathbb{F})$.

**Problem 161** Your signature below indicates you have:

**(a.)** I read the handout from Berberian: ____________________.

**(b.)** I read Chapter 10 of Cook's Lecture Notes: ____________________.

**Problem 162** Consider $T \in L(V, W)$. If $U \leq V$ then we can attempt to define a linear transformation $S : V/U \to W$ by the rule:

$$S(x + U) = T(x).$$

**(i.)** is $S$ a function? If this is not generally true then what condition do we need to place on $U$ in order that $S$ be a function?

**(ii.)** if need be apply the condition found in part (i.), if $T$ is injective does this imply $S$ is injective? What condition is needed to make injectivity of $T$ transfer to $S$ ?

**(iii.)** if need be apply the condition found in part (i.), if $T$ is surjective does this imply $S$ is surjective? What condition is needed to make surjectivity of $T$ transfer to $S$ ?

**Problem 163** Let $V$ be a vector space over $\mathbb{F}$ and $M, N \leq V$. Prove the **2nd isomorphism theorem:**

$$M/(M \cap N) \approx (M + N)/N$$

*Hint: consider the restriction of $\pi : V \to V/N$ to $M$. Find the kernel and range of $\pi|_M$.*

**Problem 164** Prove the **3rd isomorphism theorem:** If $V$ is a vector space over $\mathbb{F}$ such that $U \leq N \leq V$ then $\dfrac{V/U}{N/U} \approx \dfrac{V}{N}$

**Problem 165** Let $V$ and $W$ be vector spaces over $\mathbb{F}$ and $M \leq V$ and $N \leq W$. Prove

$$\frac{V \times W}{M \times N} \approx \frac{V}{M} \times \frac{W}{N}.$$

*Hint: Consider $T : V \times W \to V/M \times V/N$ defined by $T(x, y) = (x + M, y + N)$.*

**Problem 166** The notation $\boxplus$ denotes the **external direct product** of vector spaces; given $V, W$ vector spaces the point-set $V \times W$ with the usual operations $c(v_1, w_1) + (v_2, w_2) = (cv_1 + v_2, cw_1 + w_2)$ is denoted $V \boxplus W$. **Prove:** If $V = V_1 \oplus V_2$ and $S = S_1 \oplus S_2$ such that $S_1 \leq V_1$ and $S_2 \leq V_2$ then

$$\frac{V}{S} = \frac{V_1 \oplus V_2}{S_1 \oplus S_2} \approx \frac{V_1}{S_1} \boxplus \frac{V_2}{S_2}.$$

**Problem 167** Prove the following: for $V$ a vector space over a field $\mathbb{F}$:

**(a.)** for any nonzero vector $v \in V$ there exists a linear functional $\alpha \in V^*$ for which $\alpha(v) \neq 0$

**(b.)** a vector $v \in V$ is zero if and only if $\alpha(v) = 0$ for all $\alpha \in V^*$.

**Problem 168** (continuation of last problem) Prove the following: for $V$ a vector space over a field $\mathbb{F}$:

**(c.)** if $\alpha \in V^*$ and $\alpha(x) \neq 0$ then $V = \text{span}(x) \oplus \ker(\alpha)$

**(c.)** if $\alpha, \beta \in V^*$ are nonzero then $\ker(\alpha) = \ker(\beta)$ iff $\alpha = k\beta$ for some $k \in \mathbb{F}$

**Problem 169** Given $V = S \oplus T$, prove $\text{ann}(S) \oplus \text{ann}(T) = (S \oplus T)^*$.
Incidentally, another common notation for the annihilator is given by $\text{ann}(S) = S^0$.

**Problem 170** Show the first isomorphism theorem implies the rank nullity theorem for $T : V \to W$. That is show the first isomorphism theorem implies $\dim(\ker(T)) + \dim(\text{range}(T)) = \dim(V)$. (you are free to use Proposition 10.1.22 of page 341 in my notes)

**Problem 171** Suppose $(V, g)$ forms a geometry and $\beta$ is a basis for $V$ for which $G$ is the matrix of $g$. Furthermore, suppose the linear mapping $L : V \to V$ is a $g$-orthogonal map such that $A$ is its matrix; $[L(x)]_\beta = A[x]_\beta$ or simply $[L]_{\beta,\beta} = A$. Show $A^T G A = G$.

**Problem 172** (**Gwyneth's Musical Morphism Problem**) Let $g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be a metric with matrix $G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$. If $v = \sum_{i=1}^3 v^i e_i = (a, b, c)$ then calculate $v_i$ for $i = 1, 2, 3$. Also, show $g(v, v) = \sum_{i=1}^3 v^i v_i$ (you can show it in terms of $a, b, c$ or $v^1, v^2, v^3$ whatever you prefer)

**Problem 173** Let $M$ be a symmetric matrix and define $\Upsilon(A, B) = AB + BA$ for all $A, B \in \mathbb{R}^{n \times n}$ show $\Upsilon$ is a symmetric, bilinear form.

**Problem 174** Suppose $(V, g)$ is a real geometry. Show $(V^*, g^*)$ is also a real geometry given we define $g^*(\alpha, \beta) = g(\sharp\alpha, \sharp\beta)$.

**Problem 175** Let $(V, g)$ be a real geometry. Prove $\sharp \circ \flat = Id_V$ and $\flat \circ \sharp = Id_{V^*}$. See my notes for the necessary definitions.

**Problem 176** Prove property (ii.) of Theorem 10.4.4.

**Problem 177** Let $V$ be a real vector space and $x, y \in V$. Define $x \otimes y : V^* \times V^* \to \mathbb{R}$ according to the rule $(x \otimes y)(\alpha, \beta) = \alpha(x)\beta(y)$. Show $x \otimes y$ is a bilinear mapping on $V^* \times V^*$.

**Problem 178** Continuing the construction in the last problem, if $V$ has basis $\beta = \{v_1, \dots, v_n\}$ show $\Upsilon = \{v_i \otimes v_j \mid 1 \leq i, j \leq n\}$ serves as a basis for $\mathcal{B}(V^*)$. That is, show $\Upsilon$ is LI and that any bilinear mapping $V^* \times V^* \to \mathbb{R}$ can be expressed as a linear combination of the $\Upsilon$ maps.

**Problem 179** Suppose $A, N$ are square matrices and $N$ is nilpotent of order $k$. Show $A \otimes N$ is nilpotent of order $k$.

**Problem 180** Let $T : V \times V \times V^* \to \mathbb{R}$ be a multilinear mapping. Determine if $S = T \circ (\sharp, \sharp, \flat)$ is also a multilinear mapping. Also, find the coordinate transformation rules for $T$ and $S$. Here we assume $(V, g)$ is a real geometry.