Same rules as Homework 1. **Assume** all vector spaces are **finite dimensional** for ease of mind. There are many results here which do transfer to the world of infinite dimensional vector spaces, but, I'll leave that for another day. We denote $V^* = L(V, \mathbb{F})$.

Problem 161 Your signature below indicates you have:

(a.) I read the handout from Berberian: ____________________.

(b.) I read Chapter 10 of Cook's Lecture Notes: ____________________.

Problem 162 Consider $T \in L(V, W)$. If $U \leq V$ then we can attempt to define a linear transformation $S : V/U \to W$ by the rule:

$$S(x + U) = T(x).$$

(i.) is S a function? If this is not generally true then what condition do we need to place on U in order that S be a function?

(ii.) if need be apply the condition found in part (i.), if T is injective does this imply S is injective? What condition is needed to make injectivity of T transfer to S ?

(iii.) if need be apply the condition found in part (i.), if T is surjective does this imply S is surjective? What condition is needed to make surjectivity of T transfer to S ?

Problem 163 Let V be a vector space over $\mathbb{F}$ and $M, N \leq V$. Prove the **2nd isomorphism theorem:**

$$M/(M \cap N) \approx (M + N)/N$$

Hint: consider the restriction of $\pi : V \to V/N$ to M. Find the kernel and range of $\pi|_M$.

Problem 164 Prove the **3rd isomorphism theorem:** If V is a vector space over $\mathbb{F}$ such that $U \leq N \leq V$ then $\dfrac{V/U}{N/U} \approx \dfrac{V}{N}$

Problem 165 Let V and W be vector spaces over $\mathbb{F}$ and $M \leq V$ and $N \leq W$. Prove

$$\frac{V \times W}{M \times N} \approx \frac{V}{M} \times \frac{W}{N}.$$

Hint: Consider $T : V \times W \to V/M \times V/N$ defined by $T(x, y) = (x + M, y + N)$.

Problem 166 The notation $\boxplus$ denotes the **external direct product** of vector spaces; given V, W vector spaces the point-set $V \times W$ with the usual operations $c(v_1, w_1) + (v_2, w_2) = (cv_1 + v_2, cw_1 + w_2)$ is denoted $V \boxplus W$. **Prove:** If $V = V_1 \oplus V_2$ and $S = S_1 \oplus S_2$ such that $S_1 \leq V_1$ and $S_2 \leq V_2$ then

$$\frac{V}{S} = \frac{V_1 \oplus V_2}{S_1 \oplus S_2} \approx \frac{V_1}{S_1} \boxplus \frac{V_2}{S_2}.$$

Problem 167 Prove the following: for V a vector space over a field $\mathbb{F}$:

(a.) for any nonzero vector $v \in V$ there exists a linear functional $\alpha \in V^*$ for which $\alpha(v) \neq 0$

(b.) a vector $v \in V$ is zero if and only if $\alpha(v) = 0$ for all $\alpha \in V^*$.

Problem 168 (continuation of last problem) Prove the following: for V a vector space over a field $\mathbb{F}$:

(c.) if $\alpha \in V^*$ and $\alpha(x) \neq 0$ then $V = \text{span}(x) \oplus \ker(\alpha)$

(c.) if $\alpha, \beta \in V^*$ are nonzero then $\ker(\alpha) = \ker(\beta)$ iff $\alpha = k\beta$ for some $k \in \mathbb{F}$

Problem 169 Given $V = S \oplus T$, prove $\text{ann}(S) \oplus \text{ann}(T) = (S \oplus T)^*$.
Incidentally, another common notation for the annihilator is given by $\text{ann}(S) = S^0$.

Problem 170 Show the first isomorphism theorem implies the rank nullity theorem for $T : V \to W$. That is show the first isomorphism theorem implies $\dim(\ker(T)) + \dim(\text{range}(T)) = \dim(V)$. (you are free to use Proposition 10.1.22 of page 341 in my notes)

Problem 171 Suppose (V, g) forms a geometry and β is a basis for V for which G is the matrix of g. Furthermore, suppose the linear mapping $L : V \to V$ is a g-orthogonal map such that A is its matrix; $[L(x)]_\beta = A[x]_\beta$ or simply $[L]_{\beta,\beta} = A$. Show $A^T G A = G$.

Problem 172 **(Gwyneth's Musical Morphism Problem)** Let $g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ be a metric with matrix $G = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}$. If $v = \sum_{i=1}^3 v^i e_i = (a, b, c)$ then calculate v_i for $i = 1, 2, 3$.
Also, show $g(v, v) = \sum_{i=1}^3 v^i v_i$ (you can show it in terms of a, b, c or v^1, v^2, v^3 whatever you prefer)

Problem 173 Let M be a symmetric matrix and define $\Upsilon(A, B) = AB + BA$ for all $A, B \in \mathbb{R}^{n\times n}$ show Υ is a symmetric, bilinear form.

Problem 174 Suppose (V, g) is a real geometry. Show (V^*, g^*) is also a real geometry given we define $g^*(\alpha, \beta) = g(\sharp\alpha, \sharp\beta)$.

Problem 175 Let (V, g) be a real geometry. Prove $\sharp \circ \flat = Id_V$ and $\flat \circ \sharp = Id_{V^*}$. See my notes for the necessary definitions.

Problem 176 Prove property (ii.) of Theorem 10.4.4.

Problem 177 Let V be a real vector space and $x, y \in V$. Define $x \otimes y : V^* \times V^* \to \mathbb{R}$ according to the rule $(x \otimes y)(\alpha, \beta) = \alpha(x)\beta(y)$. Show $x \otimes y$ is a bilinear mapping on $V^* \times V^*$.

Problem 178 Continuing the construction in the last problem, if V has basis $\beta = \{v_1, \dots, v_n\}$ show $\Upsilon = \{v_i \otimes v_j \mid 1 \leq i, j \leq n\}$ serves as a basis for $\mathcal{B}(V^*)$. That is, show Υ is LI and that any bilinear mapping $V^* \times V^* \to \mathbb{R}$ can be expressed as a linear combination of the Υ maps.

Problem 179 Suppose A, N are square matrices and N is nilpotent of order k. Show $A \otimes N$ is nilpotent of order k.

Problem 180 Let $T : V \times V \times V^* \to \mathbb{R}$ be a multilinear mapping. Determine if $S = T \circ (\sharp, \sharp, \flat)$ is also a multilinear mapping. Also, find the coordinate transformation rules for T and S. Here we assume (V, g) is a real geometry.

MISSION 10 SOLUTION

P162 $T: V \longrightarrow W$ linear and $U \leq V$.
define $S: V/U \longrightarrow W$ by $S(x+U) = T(x)$.

(a.) S is into W since $T(x) \in W$ so the codomain of S is fine. However, for S to be single-valued we need $x_1 + U = x_2 + U \Rightarrow S(x_1+U) = S(x_2+U)$

But, $x_1 + U = x_2 + U \Rightarrow x_2 - x_1 \in U$ hence $x_2 = x_1 + u$ for some $u \in U$ and thus,

$$S(x_1+U) = T(x_1)$$

vs.

$$S(x_2+U) = T(x_2) = T(x_1+u) = T(x_1) + T(u)$$

We need $T(u) = 0$ in order that S be single-valued.

Hence $\boxed{U \subset \text{Ker}(T) \text{ is a necessary condition}}$
(for S to be a function)

Assume $U \subset \text{Ker}(T)$,

(b.) Suppose T is injective. Consider $S(x+U) = S(y+U)$
$\Rightarrow T(x) = T(y) \Rightarrow x = y$ by injectivity of T
thus $x + U = y + U$ and we find S is injective.

Remark: seems like less than injectivity of T will also work, ..., how much less...

(c.) Suppose T is surjective. Let $w \in W$ then $\exists v \in V$ for which $T(v) = w$. Observe for $w \in W$ we also have $S(v+U) = T(v) = w \therefore S$ surjective. //

$\boxed{P163}$ Let $\varphi : M+N \longrightarrow \frac{M}{M\cap N}$

be defined by $\varphi(m+n) = m + M\cap N$

observe φ is linear and into $M/M\cap N$. It remains to show φ is well-defined. Suppose $m_1+n_1 = m_2+n_2$ then $m_2 - m_1 = n_1 - n_2$. Is $m_1 + M\cap N = m_2 + M\cap N$?

Notice $m_2 - m_1 = n_1 - n_2 \Rightarrow m_2 - m_1 \in M\cap N$

thus $m_2 + M\cap N = m_1 + M\cap N \Rightarrow \varphi(m_1+n_1) = \varphi(m_2+n_2)$.

Thus φ is well-defined.

//

Let $m + M\cap N$ then $\varphi(m) = m + M\cap N$ $\therefore$ φ is onto.

Calculate, $\text{Ker}(\varphi) = \{ m+n \mid \varphi(m+n) = m + M\cap N = M\cap N\}$

$$= \{ m+n \mid m \in M\cap N\}$$

$$= \{ m+n \mid m \in N \text{ and } n \in N\}$$

$$= \{ \tilde{n} \mid \tilde{n} \in N\}$$

$$= N.$$

Thus $\dfrac{M+N}{N} \approx \dfrac{M}{M\cap N}$ by 1^{st} Isomorphism Th^{m}.

P164 $U \le N \le V$ show $\dfrac{V/U}{N/U} \approx \dfrac{V}{N}$

Let $\varphi : V/U \longrightarrow V/N$ be defined by $\varphi(v+U) = v+N$ or $\varphi(x+U) = x+N$. Once more φ is linear and into. I'll show linearity (this does make sense in part by the well-def$^{=}$ argument I make)

$$\varphi(c(x+U) + y+U) = \varphi(cx+y+U)$$
$$= cx+y+N$$
$$= c(x+N) + y+N$$
$$= c\varphi(x+U) + \varphi(y+U).$$

Thus, if φ is a function, then φ is linear transformation.

Suppose $x_1 + U = x_2 + U \Rightarrow x_2 - x_1 \in U \le N$

thus $x_2 - x_1 \in N$ and $x_1 + N = x_2 + N$

but, this is precisely $\varphi(x_1+U) = \varphi(x_2+U)$

$\therefore$ φ is well-defined.

Let $x+N \in V/n$ and observe $\varphi(x+U) = x+N$ thus φ is a surjection onto V/N. Moreover,

$$\operatorname{Ker}\varphi = \{x+U \mid \varphi(x+U) = x+N = N\}$$
$$= \{x+U \mid x \in N\}$$
$$= N/U$$

Thus, by 1st isomorphism Th^{m}, $\dfrac{V/U}{N/U} \approx \dfrac{V}{N}$.

$\boxed{P165}$ Let $T: V \times W \longrightarrow \frac{V}{M} \times \frac{W}{N}$ be defined by $T(x, y) = (x + M, y + N)$.

Observe T is linear. Calculate the kernel,

$$\begin{aligned} \text{Ker}(T) &= \{ (x,y) \in V \times W \mid (x+M, y+N) = (M,N) \} \\ &= \{ (x,y) \mid x \in M,\ y \in N \} \\ &= M \times N \end{aligned}$$

$$\therefore \quad \frac{V \times W}{M \times N} \approx \frac{V}{M} \times \frac{W}{N}$$

as the map T is clearly surjective.
(Surjectivity Proof: if $(x + M, y + N) \in \frac{V}{M} \times \frac{W}{N}$ then $T(x, y) = (x + M, y + N) \therefore T$ onto.)

$\boxed{P166}$ Let $V = V_1 \oplus V_2$ with $S_1 \leq V_1$ and $S_2 \leq V_2$ and $S = S_1 \oplus S_2$. Define $T: V \longrightarrow \frac{V_1}{S_1} \times \frac{V_2}{S_2} = \underbrace{\frac{V_1}{S_1} \boxplus \frac{V_2}{S_2}}_{\text{external direct sum.}}$

by $T(x_1 + x_2) = (x_1 + S_1, x_2 + S_2)$ for $x_1 \in V_1$, $x_2 \in V_2$

observe T is linear and

$$\begin{aligned} \text{Ker}\, T &= \{ x_1 + x_2 \in V_1 \oplus V_2 \mid (x_1 + S_1, x_2 + S_2) = \underbrace{(S_1, S_2)}_{\text{Zero in } \frac{V_1}{S_1} \boxplus \frac{V_2}{S_2}} \} \\ &= \{ x_1 + x_2 \mid x_1 \in S_1,\ x_2 \in S_2 \} \\ &= S_1 + S_2 \\ &= S_1 \oplus S_2 \quad (\text{given } S_1 \cap S_2 = \{0\}.) \end{aligned}$$

Thus, as T is a surjection (left to reader) we find

$$\frac{V_1 \oplus V_2}{S_1 \oplus S_2} \approx \frac{V_1}{S_1} \boxplus \frac{V_2}{S_2} \quad \text{by } 1^{st} \text{ isomorphism Th}^{m}.$$

P167 Let $V(\mathbb{F})$ be non trivial vector space.

(a.) if $v \neq 0$ is vector in V then $\exists \alpha \in V^*$ for which $\alpha(v) \neq 0$

Extend v to basis $\beta = \{v_1, v_2, \dots, v_n\}$ (set $v_1 = v$)
then $\beta^* = \{v^1, v^2, \dots, v^n\} \subset V^*$ and $\alpha = v^1$
fits the demands; $\alpha(v) = v^1(v_1) = 1$.

(b.) Suppose $v = 0$ then $\alpha(0) = 0 \quad \forall \alpha \in V^*$.
Suppose $\alpha(v) = 0 \quad \forall \alpha \in V^*$. If $v = \sum c_i v_i$
then $\alpha(v) = \sum_{i=1}^{n} c_i \alpha(v_i) = 0 \quad \forall \alpha \in V^*$
choose $\alpha = v^j$ to obtain $v^j(v) = \sum_{i=1}^{n} c_i \underbrace{v^j(v_i)}_{\delta_{ij}} = c_j$
hence $c_j = 0$ for each $j = 1, 2, \dots, n \Rightarrow v = 0$.
We conclude $\alpha(v) = 0 \quad \forall \alpha \in V^* \iff v = 0$.

P168 if $\alpha \in V^*$ and $\alpha(x) \neq 0$ then $V = \text{span}\{x\} \oplus \text{Ker}(\alpha)$

Notice, if $y \in \text{span}\{x\} \cap \text{Ker}(\alpha)$ then $\alpha(y) = 0$ and
$y = kx$ for some $k \in \mathbb{F} \Rightarrow \alpha(y) = k\alpha(x) = 0 \Rightarrow \underline{k = 0}$
thus $\text{span}\{x\} \cap \text{Ker}(\alpha) = \{0\}$. It remains to show $V = \text{span}\{x\} + \text{Ker}(\alpha)$.
Notice, for $v \in V$ we have the following decomposition,

$$\underline{v = \frac{\alpha(v)}{\alpha(x)} x + \left(v - \frac{\alpha(v)}{\alpha(x)} x \right)}_{*}$$

and note $\alpha\left(v - \frac{\alpha(v)}{\alpha(x)} x\right) = \alpha(v) - \frac{\alpha(v)}{\alpha(x)} \alpha(x) = 0$
thus $v - \frac{\alpha(v)}{\alpha(x)} x \in \text{Ker}(\alpha) \therefore v \in \text{span}\{x\} + \text{Ker}(\alpha)$ by $*$
hence $V = \text{span}\{x\} \oplus \text{Ker}(\alpha)$.

e way in which the expression V/S

theorem) *Let V be a vector space and* *f V. Then*

$$\approx \frac{V}{T}$$

y $\tau(v+S) = v+T$. We leave it to the surjective linear transformation whose first isomorphism theorem.□

ne way in which the expression V/S

and let S be a subspace of V. Suppose *$S_i \subseteq V_i$. Then*

$$\approx \frac{V_1}{S_1} \boxplus \frac{V_2}{S_2}$$

e defined by

$$+ S_1, v_2 + S_2)$$

$V = V_1 \oplus V_2$ is direct. We leave it to linear transformation, whose kernel is somorphism theorem.□

se field F (thought of as a vector space

e over F. A linear transformation se field F is called a **linear functional** *thors use the term* linear function.) *The* *V is denoted by V^* and is called the*

since there is another type of dual space tor spaces, where continuity of linear discuss the so-called *continuous dual* until then, the term "dual space" will

To help distinguish linear functionals from other types of linear transformations, we will usually denote linear functionals by lowercase italic letters, such as f, g and h.

Example 3.1 The map $f\colon F[x] \to F$ defined by $f(p(x)) = p(0)$ is a linear functional, known as **evaluation at** 0.□

Example 3.2 Let $C[a,b]$ denote the vector space of all continuous functions on $[a,b] \subseteq \mathbb{R}$. Let $f\colon C[a,b] \to \mathbb{R}$ be defined by

$$f(\alpha(x)) = \int_a^b \alpha(x)\,dx$$

Then $f \in C[a,b]^*$.□

For any $f \in V^*$, the rank plus nullity theorem is

$$\dim(\ker(f)) + \dim(\operatorname{im}(f)) = \dim(V)$$

But since $\operatorname{im}(f) \subseteq F$, we have either $\operatorname{im}(f) = \{0\}$, in which case f is the zero linear functional, or $\operatorname{im}(f) = F$, in which case f is surjective. In other words, a nonzero linear functional is surjective. Moreover, if $f \neq 0$, then

$$\operatorname{codim}(\ker(f)) = \dim\left(\frac{V}{\ker(f)}\right) = 1$$

and if $\dim(V) < \infty$, then

$$\dim(\ker(f)) = \dim(V) - 1$$

Thus, in dimensional terms, the kernel of a linear functional is a very "large" subspace of the domain V.

The following theorem will prove very useful.

Theorem 3.10

1) *For any nonzero vector $v \in V$, there exists a linear functional $f \in V^*$ for which $f(v) \neq 0$.*
2) *A vector $v \in V$ is zero if and only if $f(v) = 0$ for all $f \in V^*$.*
3) *Let $f \in V^*$. If $f(x) \neq 0$, then*

$$V = \langle x \rangle \oplus \ker(f)$$

4) *Two nonzero linear functionals $f, g \in V^*$ have the same kernel if and only if there is a nonzero scalar λ such that $f = \lambda g$.*

Proof. For part 3), if $0 \neq v \in \langle x \rangle \cap \ker(f)$, then $f(v) = 0$ and $v = ax$ for $0 \neq a \in F$, whence $f(x) = 0$, which is false. Hence, $\langle x \rangle \cap \ker(f) = \{0\}$ and the direct sum $S = \langle x \rangle \oplus \ker(f)$ exists. Also, for any $v \in V$ we have

$$v = \frac{f(v)}{f(x)}x + \left(v - \frac{f(v)}{f(x)}x\right) \in \langle x \rangle + \ker(f)$$

and so $V = \langle x \rangle \oplus \ker(f)$.

For part 4), if $f = \lambda g$ for $\lambda \neq 0$, then $\ker(f) = \ker(g)$. Conversely, if $K = \ker(f) = \ker(g)$, then for $x \notin K$ we have by part 3),

$$V = \langle x \rangle \oplus K$$

Of course, $f|_K = \lambda g|_K$ for any λ. Therefore, if $\lambda = f(x)/g(x)$, it follows that $\lambda g(x) = f(x)$ and hence $f = \lambda g$.□

Dual Bases

Let V be a vector space with basis $\mathcal{B} = \{v_i \mid i \in I\}$. For each $i \in I$, we can define a linear functional $v_i^* \in V^*$ by the orthogonality condition

$$v_i^*(v_j) = \delta_{i,j}$$

where $\delta_{i,j}$ is the **Kronecker delta function**, defined by

$$\delta_{i,j} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Then the set $\mathcal{B}^* = \{v_i^* \mid i \in I\}$ is linearly independent, since applying the equation

$$0 = a_{i_1} v_{i_1}^* + \cdots + a_{i_n} v_{i_n}^*$$

to the basis vector v_{i_k} gives

$$0 = \sum_{j=1}^{n} a_{i_j} v_{i_j}^*(v_{i_k}) = \sum_{j=1}^{n} a_{i_j} \delta_{i_j, i_k} = a_{i_k}$$

for all i_k.

Theorem 3.11 *Let V be a vector space with basis $\mathcal{B} = \{v_i \mid i \in I\}$.*
1) The set $\mathcal{B}^* = \{v_i^* \mid i \in I\}$ is linearly independent.
2) If V is finite-dimensional, then $\mathcal{B}^*$ is a basis for V^*, called the **dual basis** *of $\mathcal{B}$.*

Proof. For part 2), for any $f \in V^*$, we have

$$\sum_j f(v_j) v_j^*(v_i) = \sum_j f(v_j) \delta_{i,j} = f(v_i)$$

and so $f = \sum f(v_j) v_j^*$ is in the span of $\mathcal{B}^*$. Hence, $\mathcal{B}^*$ is a basis for V^*.□

P169 If $V = S \oplus T$ then show $ann(S) \oplus ann(T) = (S \oplus T)^*$

Recall, $\dim V = \dim(S) + \dim(ann(S))$
and $\dim V = \dim(T) + \dim(ann(T))$ } by P153

We also know $\dim V = \dim S + \dim T$. Observe,

$$\dim(ann(S)) + \dim(ann(T)) = (\dim V - \dim S) + (\dim V - \dim T)$$

$$\Rightarrow \underline{\dim(ann(S)) + \dim(ann(T)) = \dim V = \dim V^*}.$$

However, if $\alpha \in ann(S) \cap ann(T)$ then

$$\begin{aligned} \alpha(x) &= \alpha(s+t) \quad : \text{since } x \in V = S \oplus T \ \exists\, s \in S,\ t \in T \text{ such that } x = s+t. \\ &= \alpha(s) + \alpha(t) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

Thus $\alpha(x) = 0 \ \forall x \in V \Rightarrow \underline{ann(S) \cap ann(T) = \{0\}}$. $*$

Recall, $\dim(U+W) = \dim(U) + \dim(W) - \dim(U \cap W)$
apply to our current problem,

$$\dim(ann(S) + ann(T)) = \dim(ann(S)) + \dim(ann(T)) - 0$$

$$\therefore \dim(ann(S) + ann(T)) = \dim V^*$$

$$\Rightarrow ann(S) + ann(T) = V^* \quad \text{\& by } *$$

we conclude $\underline{ann(S) \oplus ann(T) = V^*}$.

P170 Consider $T: V \longrightarrow W$ linear
then 1st isomorphism Th^{m} says $V/_{Ker(T)} \approx T(V)$

thus $\underbrace{\dim(V/_{Ker\,T})} = \dim(T(V))$

$\dim V - \dim Ker\,T = \dim(T(V))$ $\therefore$ $\boxed{\dim V = \dim(Ker\,T) + \dim(T(V))}$ Rank-Nullity Th^{m}!

(by Prop 10.1.22)

$\boxed{P171}$ Let (V, g) form a geometry and β a basis of V for which G is the matrix of g. Suppose $L : V \to V$ is a g-orthogonal map with matrix A $\left(\underbrace{[L(x)]_\beta = A[x]_\beta}_{[L]_{\beta\beta} = A}\right)$

Observe, $g(L(x), L(y)) = g(x, y)$

$$\Rightarrow [L(x)]_\beta^T G [L(y)]_\beta = [x]_\beta^T G [y]_\beta$$

$$\Rightarrow (A[x]_\beta)^T G A [y]_\beta = [x]_\beta^T G [y]_\beta$$

$$\Rightarrow \underline{[x]_\beta^T A^T G A [y]_\beta = [x]_\beta^T G [y]_\beta} \; *$$

Since L is g-orthogonal we have $*$ $\forall x, y \in V$.
If $\beta = \{v_1, \dots, v_n\}$ then setting $x = v_i$ and $y = v_j$ yields $[x]_\beta = e_i$ and $[y]_\beta = e_j$ and $*$ shows $(A^T G A)_{ij} = G_{ij}$ thus $\underline{A^T G A = G}$.

$\boxed{P172}$ $g : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}$ with $g(v, w) = v^T \underbrace{\begin{bmatrix} 1 & & \\ & -2 & \\ & & -3 \end{bmatrix}}_{G} w$

Let $v = (v^1, v^2, v^3) = (a, b, c)$ then

$$v_i = \sum_{j=1}^{3} g_{ij} v^j$$

$$= g_{i1} v^1 + g_{i2} v^2 + g_{i3} v^3$$

$$\Rightarrow \boxed{(v_1, v_2, v_3) = (a, -2b, -3c)}$$

$$G^{-1} = \begin{bmatrix} 1 & & \\ & 1/-2 & \\ & & 1/-3 \end{bmatrix}$$

But, didn't need it, Btw,

if $\alpha = ae^1 + be^2 + ce^3$ then $(\alpha^1, \alpha^2, \alpha^3) = (a, \frac{-b}{2}, \frac{-c}{3})$

Observe,

$$g(v, v) = [v^1, v^2, v^3] \begin{bmatrix} 1 & & \\ & -2 & \\ & & -3 \end{bmatrix} \begin{bmatrix} v^1 \\ v^2 \\ v^3 \end{bmatrix} = \underline{(v^1)^2 - 2(v^2)^2 - 3(v^3)^2}$$

Hence, $\underline{g(v, v) = a^2 - 2b^2 - 3c^2}$ whereas

$$\sum_{i=1}^{3} v^i v_i = v^1 v_1 + v^2 v_2 + v^3 v_3 = \underline{a^2 - 2b^2 - 3c^2}. //$$

P172 generally,

$$\sum_{i=1}^{n} V^i V_i = \sum_{i=1}^{n} V^i \sum_{j=1}^{n} g_{ij} V^j = \sum_{i,j=1}^{n} V^i g_{ij} V^j \underset{\text{by } *}{=} g(V,V)$$

and, for $\beta = \{f_1, .., f_n\}$ where $g_{ij} = g(f_i, f_j)$

$$g(V,W) = g\left(\sum_i V^i f_i, \sum_j W^j f_j\right) = \sum_{i,j} V^i W^j g(f_i, f_j) = \sum_{i,j} V^i W^j g_{ij} \quad *$$

Remark: Notice V^i, V_i are #'s in this problem. Elsewhere, we have used $V_1, .., V_n \in V$ & $V^1, .., V^n \in V^*$

P173 $\Upsilon(A,B) = AB + BA \quad \forall A, B \in \mathbb{R}^{n\times n}$

Observe $\Upsilon(B,A) = BA + AB = AB + BA = \Upsilon(A,B) \therefore \Upsilon$ symmetric.

Then
$$\begin{aligned}
\Upsilon(c_1 A + C, B) &= (c_1 A + C)B + B(c_1 A + C) \\
&= c_1 AB + CB + c_1 BA + BC \\
&= c_1 (AB + BA) + CB + BC \\
&= c_1 \Upsilon(A,B) + \Upsilon(C,B)
\end{aligned}$$

Moreover, by symmetry $\Upsilon(B, c_1 A + C) = c_1 \Upsilon(B,A) + \Upsilon(B,C)$ thus $\Upsilon : \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times n} \longrightarrow \mathbb{R}^{n\times n}$ is a symmetric bilinear form.

Suppose $g : V \times V \to \mathbb{R}$ is a metric for g.

P174 Let $g^* : V^* \times V^* \longrightarrow \mathbb{R}$ be defined by $g^*(\alpha, \beta) = g(\#\alpha, \#\beta)$ for all $(\alpha, \beta) \in V^* \times V^*$ where $\# : V^* \longrightarrow V$ is the usual musical morphism (in coordinates $\#(\sum_i \alpha_i V^i) = \sum_{i,j} \alpha_i g^{ij} V_j$)

Observe, $g^*(\alpha, \beta) = g(\#\alpha, \#\beta) = g(\#\beta, \#\alpha) = g^*(\beta, \alpha) \therefore$ symmetric.

Also,
$$\begin{aligned}
g^*(c\alpha + \gamma, \beta) &= g(\#(c\alpha + \gamma), \#\beta) \\
&= g(c\#\alpha + \#\gamma, \#\beta) \\
&= c\, g(\#\alpha, \#\beta) + g(\#\gamma, \#\beta) \\
&= c\, g^*(\alpha, \beta) + g^*(\gamma, \beta) \quad \therefore \quad g^* \text{ bilinear}
\end{aligned}$$

it remains to show g^* is nondegenerate ↴

P 174 continued

to prove nondegeneracy we can calculate the matrix of g^* and relate it to the matrix of g. Let $\{v_1, \dots, v_n\}$ have dual basis $\{v^1, \dots, v^n\}$ then $g_{ij} = g(v_i, v_j)$. Calculate $v_i = \sum_{j=1}^{n} \delta_{ij} v_j$

thus $(v_i)^j = \delta_{ij}$ describe components of v_i. Thus,

$$b(v_i) = \sum_{k,l} (v_i)^k g_{kl} v^l = \sum_{k,l} \delta_{ik} g_{kl} v^l$$

$$\therefore \quad \underline{b(v_i) = \sum_l g_{il} v^l} \; . \; *$$

Oh, well, that's not what I need (for this problem)

Observe $v^i = \sum_{j=1}^{n} \delta_{ij} v^j \;\therefore\; (v^i)_j = \delta_{ij}$ thus

$$\#(v^i) = \sum_{k,l} \delta_{ik} g^{kl} v_l = \underline{\sum_l g^{il} v_l = \#(v^i)} \; **$$

Thus,

$$\begin{aligned} g^*(v^i, v^j) &= g(\#(v^i), \#(v^j)) \\ &= g\Big(\sum_l g^{il} v_l, \sum_k g^{jk} v_k\Big) \\ &= \sum_{l,k} g^{il} g^{jk} \underbrace{g(v_l, v_k)}_{g_{lk}} \\ &= \sum_l g^{il} \delta_{jl} \\ &= g^{ij} \Rightarrow \end{aligned}$$

g^* has matrix G^{-1} when g has matrix G (provided we use dual basis for V^* relative to given basis of V) $\therefore$ g^* nondegenerate

$\therefore$ (V^*, g^*) forms a real geometry.

P 175 we'll use $*$ and $**$ to prove $\# \circ b = id_V$, $b \circ \# = id_{V^*}$

$\boxed{P175}$ <u>following notation of P174</u>

$$(\# \circ \flat)(v_i) = \#(\flat(v_i))$$
$$= \#\left(\sum_l g_{il} v^l\right)$$
$$= \sum_l g_{il}\ \#(v^l)$$
$$= \sum_l g_{il} \sum_k g^{lk} v_k$$
$$= \sum_{l,k} g_{il} g^{lk} v_k$$
$$= \sum_k \delta_{ik} v_k$$
$$= v_i \implies \underline{\# \circ \flat = id_V}. \quad \text{as } \{v_1, .., v_n\} \text{ is basis}$$

Likewise,

$$(\flat \circ \#)(v^i) = \flat(\#(v^i))$$
$$= \flat\left(\sum_k g^{ik} v_k\right)$$
$$= \sum_k g^{ik}\ \flat(v_k)$$
$$= \sum_k g^{ik} \sum_l g_{kl} v^l$$
$$= \sum_{k,l} g^{ik} g_{kl} v^l$$
$$= \sum_l \delta_{il} v^l$$
$$= v^i \implies \flat \circ \# = id_{V^*}$$

as we shown the identity holds for the <u>basis</u> $\{v^1, .., v^n\}$ for V^*.

$\boxed{P176}$ prove $x \otimes (z + w) = x \otimes z + x \otimes w$ for $x \in V$ and $z, w \in W$ where $V \otimes W = L(V^*, W)$

$$(x \otimes (z+w))(\alpha) = \alpha(x)(z+w) \quad : \text{by def}^{n}\ (x\otimes y)(\alpha) = \alpha(x)y \text{ etc.}$$
$$= \alpha(x)z + \alpha(x)w \quad : \text{distributing in } W$$
$$= (x \otimes z)(\alpha) + (x \otimes w)(\alpha) \quad : \text{def}^{n} \text{ of } \otimes \text{ on vectors here.}$$
$$= (x \otimes z + x \otimes w)(\alpha) \quad : \text{addition of maps.}$$

Thus $x \otimes (z+w) = x \otimes z + x \otimes w$ as claimed. //

$\boxed{P177}$ $x \otimes y : V^* \times V^* \longrightarrow \mathbb{R}$ is defined by $(x \otimes y)(\alpha, \beta) = \alpha(x)\beta(y)$. Consider,

$$(x \otimes y)(c\alpha_1 + \alpha_2, \beta) = (c\alpha_1 + \alpha_2)(x)\beta(y) \quad : \text{def}^{n}$$
$$= (c\alpha_1(x) + \alpha_2(x))\beta(y) \quad : \text{def}^{n} \text{ of adding functions}$$
$$= c\alpha_1(x)\beta(y) + \alpha_2(x)\beta(y)$$
$$= c(x \otimes y)(\alpha_1, \beta) + (x \otimes y)(\alpha_2, \beta)$$

Likewise,

$$(x \otimes y)(\alpha, c\beta_1 + \beta_2) = \alpha(x)(c_1\beta_1 + \beta_2)(y)$$
$$= \alpha(x)(c\beta_1(y) + \beta_2(y))$$
$$= c\,\alpha(x)\beta_1(y) + \alpha(x)\beta_2(y)$$
$$= c(x \otimes y)(\alpha, \beta_1) + (x \otimes y)(\alpha, \beta_2)$$

Thus $\underbrace{x \otimes y}_{\text{bilinear map on } V^* \times V^*} \in \mathcal{B}(V^*) = \mathcal{B}(V^* \times V^*, \mathbb{R})$.

$\boxed{P178}$ V has basis $\beta = \{V_1, \dots, V_n\}$ then prove

$\Upsilon = \{ V_i \otimes V_j \mid 1 \le i,j \le n \}$ serves as basis for $\mathcal{B}(V^*)$

Suppose $\sum_{i,j=1}^{n} C_{ij} V_i \otimes V_j = 0$ then evaluate on (V^k, V^ℓ)

where $V^k, V^\ell \in \beta^* \subset V^*$ defined as usual $V^k(V_i) = \delta_{ki}$

and $V^\ell(V_j) = \delta_{\ell j}$ etc. Thus,

$$0 = \left(\sum_{i,j} C_{ij} V_i \otimes V_j \right)(V^k, V^\ell) = \sum_{i,j} C_{ij} (V_i \otimes V_j)(V^k, V^\ell)$$

$$= \sum_{i,j} C_{ij} V^k(V_i) V^\ell(V_j)$$

$$= \sum_{i,j} C_{ij} \delta_{ki} \delta_{\ell j}$$

$$= C_{k\ell} \Rightarrow C_{ij} = 0 \quad \forall i,j$$

$$\Rightarrow \underline{\Upsilon \text{ is LI}}.$$

If $b : V^* \times V^* \longrightarrow \mathbb{R}$ is bilinear then

$$b(\alpha, \beta) = b\left(\sum_i \alpha_i V^i, \sum_j \beta_j V^j \right)$$

$$= \sum_{i,j} \alpha_i \beta_j \, b(V^i, V^j)$$

$$= \sum_{i,j} b(V^i, V^j) \alpha(V_i) \beta(V_j)$$

$$= \left(\sum_{i,j} b(V^i, V^j) V_i \otimes V_j \right)(\alpha, \beta) \quad \forall \alpha, \beta \in V^*$$

$\therefore$ $b = \sum_{i,j} b(V^i, V^j) V_i \otimes V_j$ $\therefore$ $\underline{\text{span}(\Upsilon) = \mathcal{B}(V^*)}$.

(we already showed $\Upsilon \subseteq \mathcal{B}(V^*)$ in P177 hence span $\Upsilon \subseteq \mathcal{B}(V^*)$.) (the calculation here shows $\mathcal{B}(V^*) \subseteq \text{span}\, \Upsilon$)

$\boxed{P179}$ Suppose A, N are square matrices for which $N^k = 0$ yet $N^{k-1} \neq 0$. Consider,

$$(A \otimes N)^p = \underbrace{(A \otimes N)(A \otimes N) \cdots (A \otimes N)}_{p\text{-fold copies}}$$
$$= (A^2 \otimes N^2)\underbrace{(A \otimes N) \cdots (A \otimes N)}_{(p-2)\text{-copies}}$$
$$\vdots$$
$$= A^p \otimes N^p$$

can make rigorous by induction on p if you wish.

Thus, $(A \otimes N)^{k-1} = A^{k-1} \otimes N^{k-1} \neq 0$ (provided $A^{k-1} \neq 0$)

and $(A \otimes N)^k = A^k \otimes N^k = A^k \otimes 0 = 0$.

Thus, $(A \otimes N)^k = 0$ and $(A \otimes N)^{k-1} \neq 0$ provided $A^{k-1} \neq 0$.

This much I can say, $A \otimes N$ is nilpotent of degree at <u>most</u> k. (if $A^2 = 0$ & $A \neq 0$ then $(A \otimes N)^2 = A^2 \otimes N^2 = 0$)

<u>Generally</u> $A \otimes N$ is nilpotent if either A or N is nilpotent and the degree of $A \otimes N$ will be the smallest degree of nilpotent A or N.

$\boxed{P180}$ Let $T : V \times V \times V^* \longrightarrow \mathbb{R}$ be multilinear

$$T = \sum_{i,j,k} T_{ij}{}^{k}\, v^i \otimes v^j \otimes v_k$$

follows from calculations much like I've shown for bilinear case in this sol^{n}

where $T_{ij}{}^{k} = T(v_i, v_j, v^k)$.

If $\bar{v}^i = \sum \Lambda^i{}_j v^j$ then $\bar{v}_i = \sum (\Lambda^{-1})_i{}^j v_j$

and thus,

$$\begin{aligned}
\bar{T}_{ij}{}^{k} &= T(\bar{v}_i, \bar{v}_j, \bar{v}^k) \\
&= T\Big(\sum_l (\Lambda^{-1})_i{}^l v_l,\ \sum_m (\Lambda^{-1})_j{}^m v_m,\ \sum_n \Lambda^k{}_n v^n\Big) \\
&= \sum_{l,m,n} (\Lambda^{-1})_i{}^l (\Lambda^{-1})_j{}^m \Lambda^k{}_n \underbrace{T(v_l, v_m, v^n)}_{T_{lm}{}^{n}} \\
&= \underline{\sum_{l,m,n} (\Lambda^{-1})_i{}^l (\Lambda^{-1})_j{}^m \Lambda^k{}_n T_{lm}{}^{n}}
\end{aligned}$$

coord. change rule for components of T

Yes, $S = T \circ (\#, \#, b)$ is multilinear as it is composite of linear & multilinear map and $S : V^* \times V^* \times V \xrightarrow[(\#,\#,b)]{} V \times V \times V^* \xrightarrow[T]{} \mathbb{R}$

$$S = \sum_{i,j,k} S^{ij}{}_{k}\, v_i \otimes v_j \otimes v^k$$

and by a calculation much like that given for T,

$$\underline{S(\bar{v}^i, \bar{v}^j, \bar{v}_k) = \bar{S}^{ij}{}_{k} = \sum_{l,m,n} \Lambda^i{}_l \Lambda^j{}_m (\Lambda^{-1})_k{}^n S^{lm}{}_{n}}.$$

coord. change rule for components of S.