Copying answers and steps is strictly forbidden. Evidence of copying results in zero for copied and copier. Working together is encouraged, share ideas not calculations. Explain your steps. This sheet must be printed and attached to your assignment as a cover sheet. The calculations and answers should be written neatly on one-side of paper which is attached and neatly stapled in the upper left corner. Box your answers where appropriate. Please do not fold. Thanks!

Note: In this assignment, R denotes a commutative ring.

Problem 1 Your signature below indicates you have:

(a.) I read up to §2.5 of Cook's lecture notes: ____________________.

(b.) I read Chapter 1 of Curtis' text: ____________________.

Problem 2 Let $S = \{a, b, c\}$ where a, b, c are **distinct** elements of S.

(a.) if we view S as a **set** then $S \cup \{a, b\}$ is:

(b.) if we view S as a **multiset** then $S \cup \{a, b\}$ is:

Problem 3 Let $z = 3 + i$ and $w = -3 + 4i$. Put the following into **Cartesian form**

(a) $\dfrac{1}{z} + \dfrac{1}{w}$

(b) $\dfrac{z^2 + 3wz - 4w^2}{z^2 - w^2}$

Problem 4 Following the example in my notes where I show the multiplication and addition table for $\mathbb{Z}/7\mathbb{Z}$, create the addition and multplication tables for $\mathbb{Z}/5\mathbb{Z}$. Use your table to find the multiplicative inverse for each nonzero element in $\mathbb{Z}/5\mathbb{Z}$. Given that $\mathbb{Z}/5\mathbb{Z}$ is a ring, are your observations sufficient to prove $\mathbb{Z}/5\mathbb{Z}$ is a field ?

Problem 5 Following the example in my notes where I show the multiplication and addition table for $\mathbb{Z}/7\mathbb{Z}$, create the addition and multplication tables for $\mathbb{Z}/8\mathbb{Z}$. Use your table to find multiplicative inverses for each unit in $\mathbb{Z}/8\mathbb{Z}$. Also, list all zero divisors in $\mathbb{Z}/8\mathbb{Z}$. Explain why $\mathbb{Z}/8\mathbb{Z}$ is **not** a field.

Problem 6 Show that if $a \in R$ is a zero divisor then a cannot have a multiplicative inverse.

Problem 7 Complete Problem 1(a) on page 14 of Curtis' text (basic induction problem).

Problem 8 Let $c, A_i \in R$ for all $i \in \mathbb{N}$. Use induction to prove that $\sum_{i=1}^{n}(cA_i) = c\sum_{i=1}^{n} A_i$.

Problem 9 Complete Problem 3 on page 15 of Curtis' text (binomial theorem).

Problem 10 Prove (3.) of Prop. 2.2.6 in my notes; that is, show $c_1(A + B) = c_1A + c_1B$ for $c_1 \in R$ and $A, B \in R^{m\times n}$.

Problem 11 Prove (1.) of Theorem 2.3.11 in my notes; that is, show $(AX)Z = A(XZ)$ for multipliable matrices A, X, Z over R.

Problem 12 Let $A \in R^{m\times n}$ and $B \in R^{n\times p}$. Prove that $(AB)^T = B^T A^T$.

Problem 13 The socks-shoes identity is easily generalized. Let $A_i \in R^{n\times n}$ for each $i \in \mathbb{N}$. Use induction to prove $(A_1 A_2 \cdots A_n)^T = A_n^T \cdots A_2^T A_1^T$ for all $n \in \mathbb{N}$

Problem 14 Let us define the set of component-wise nonzero 2×2 matrices over $\mathbb{Z}/3\mathbb{Z}$ by

$$M = \{A \in (\mathbb{Z}/3\mathbb{Z})^{2\times 2} \mid A_{ij} \neq 0 \text{ for all } (i,j) \in \mathbb{N}_2 \times \mathbb{N}_2\}$$

List all the matrices in M.

Problem 15 Find which of the matrices in M of the previous problem are invertible. Find the inverse of each invertible matrix in M and express them in the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a, b, c, d \in \{\overline{0}, \overline{1}, \overline{2}\}$. Notice, the formula in Example 2.4.6 works for any field $\mathbb{F}$ and $\mathbb{Z}/3\mathbb{Z}$ is a field.

Problem 16 Let $A = \begin{bmatrix} \overline{0} & \overline{1} \\ \overline{6} & \overline{4} \end{bmatrix}$ over $\mathbb{Z}/8\mathbb{Z}$. Calculate the following:

(a) $A + A$

(b) $A + 7A$

(c) A^2

(d) A^4

Problem 17 If $v = [1, 2, 3]$ and $B = \begin{bmatrix} 4 & 5 \\ 1 & 1 \\ 0 & 6 \end{bmatrix}$ then calculate the following if possible (or explain why it is not possible)

(a) vB

(b) Bv

(c) vBv^T

(d) BB^T

(e) $B^T B$

Problem 18 If $e_i^T A e_j = i^2 + j^2$ for all $(i,j) \in \mathbb{N}_3 \times \mathbb{N}_2$ then find A explitly as an array of numbers.

Mission 1 (Homework 1) Solution

Note: due to a decrease in Naruto knowledge the assignment of missions has been retired. Indeed, this may be the final mission. $\#$

Problem 2

(a.) as sets, $\{a,b,c\} \cup \{a,b\} = \{a,b,c\}$

(b.) as multisets, $\{a,b,c\} \cup \{a,b\} = \underbrace{\{a,a,b,b,c\}}_{\text{ordering can be altered.}}$

Problem 3

$z = 3+i$, $w = -3+4i$

(a.) $$\frac{1}{z} + \frac{1}{w} = \frac{3-i}{10} + \frac{-3-4i}{25} = \frac{15-5i-6-8i}{50} = \boxed{\frac{9-13i}{50}}$$

(b.) $$\frac{z^2+3wz-4w^2}{z^2-w^2} = \frac{\cancel{(z-w)}(z+4w)}{(z-w)\cancel{(z+w)}}$$

$$= \frac{3+i-12+16i}{3+i-(-3+4i)}$$

$$= \left(\frac{-9+17i}{6-3i}\right)\left(\frac{6+3i}{6+3i}\right)$$

$$= \frac{-54+102i-27i-51}{36+9}$$

$$= \frac{-105+75i}{45} = \boxed{\frac{-7}{3} + \frac{5}{3}i} = \frac{-7+5i}{3}$$

PROBLEM 4 $\mathbb{Z}/5\mathbb{Z} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$

$+$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{0}$
$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{0}$	$\bar{1}$
$\bar{3}$	$\bar{4}$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{4}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$

$\cdot$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\bar{2}$	$\bar{4}$	$\bar{1}$	$\bar{3}$
$\bar{3}$	$\bar{1}$	$\bar{4}$	$\bar{2}$
$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$

We find that $\bar{2}\bar{3} = \bar{1}$ and $\bar{4}\bar{4} = \bar{1}$ thus,

$\frac{1}{\bar{2}} = \bar{3}$ and $\frac{1}{\bar{4}} = \bar{4}$. (*)

of course, $\frac{1}{\bar{1}} = \bar{1}$ as well so every nonzero element in $\mathbb{Z}/5\mathbb{Z}$ has a multiplicative inverse

Since we know $\mathbb{Z}/5\mathbb{Z}$ is a commutative ring it follows that $\mathbb{Z}/5\mathbb{Z}$ is indeed a field.

(YES * suffice to show $\mathbb{Z}/5\mathbb{Z}$ a field given its a ring)

PROBLEM 5 $\mathbb{Z}/8\mathbb{Z} = \{\bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}, \bar{5}, \bar{6}, \bar{7}\}$

$+$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$	$\bar{7}$
$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$	$\bar{7}$	$\bar{0}$
$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$	$\bar{7}$	$\bar{0}$	$\bar{1}$
$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$	$\bar{7}$	$\bar{0}$	$\bar{1}$	$\bar{2}$
$\bar{4}$	$\bar{5}$	$\bar{6}$	$\bar{7}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$
$\bar{5}$	$\bar{6}$	$\bar{7}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$
$\bar{6}$	$\bar{7}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$
$\bar{7}$	$\bar{0}$	$\bar{1}$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$

$\cdot$	$\bar{2}$	$\bar{3}$	$\bar{4}$	$\bar{5}$	$\bar{6}$	$\bar{7}$
$\bar{2}$	$\bar{4}$	$\bar{6}$	$\bar{0}$	$\bar{2}$	$\bar{4}$	$\bar{6}$
$\bar{3}$	$\bar{6}$	$\bar{1}$	$\bar{4}$	$\bar{7}$	$\bar{2}$	$\bar{5}$
$\bar{4}$	$\bar{0}$	$\bar{4}$	$\bar{0}$	$\bar{4}$	$\bar{0}$	$\bar{4}$
$\bar{5}$	$\bar{2}$	$\bar{7}$	$\bar{4}$	$\bar{1}$	$\bar{6}$	$\bar{3}$
$\bar{6}$	$\bar{4}$	$\bar{2}$	$\bar{0}$	$\bar{6}$	$\bar{4}$	$\bar{2}$
$\bar{7}$	$\bar{6}$	$\bar{5}$	$\bar{4}$	$\bar{3}$	$\bar{2}$	$\bar{1}$

We find zero divisors are $\bar{2}, \bar{4}, \bar{6}$ as $\bar{2}\bar{4} = \bar{0}$ and $\bar{4}\bar{6} = \bar{0}$.

the units of $\mathbb{Z}/8\mathbb{Z}$ are $\bar{3}, \bar{5}, \bar{7}$ as $\bar{3}\bar{3} = \bar{1}$, $\bar{5}\bar{5} = \bar{1}$ and $\bar{7}\bar{7} = \bar{1}$.

$\mathbb{Z}/8\mathbb{Z}$ is not a field since $\bar{2}, \bar{4}, \bar{6}$ have no multiplicative inverses.

PROBLEM 6 Show if $a \in R$ is zero divisor then a cannot have a multiplicative inverse (that is; a not a unit)

Suppose $a \in R$ is a zero divisor. It follows $a \neq 0$ and $\exists\, b \neq 0$ such that $ab = 0$. Next, suppose $a^{-1} \in R$ exists such that $a^{-1}a = 1$.

Consider, $ab = 0 \Rightarrow a^{-1}ab = a^{-1}(0) = 0$

Thus, $1 \cdot b = 0$ or $b = 0$. But, $b = 0$ contradicts $b \neq 0$ hence $\nexists\, a^{-1} \in R$. That is, a does not have a multiticative inverse. //

PROBLEM 7 Show $1+3+5+\cdots+(2k-1) = k^2 \;\; \forall k \in \mathbb{N}$

Let $\mathcal{P}(k): 1+3+5+\cdots+(2k-1) = k^2$. Consider, $\mathcal{P}(1)$ is true since $1 = 1^2$. Suppose inductively for $k \in \mathbb{N}$ $\mathcal{P}(k)$ true; $1+3+5+\cdots+(2k-1) = k^2$. Consider

$$1+3+5+\cdots+(2k-1)+[2(k+1)-1] =$$
$$= k^2 + [2(k+1)-1] \quad : \text{ by induction hypothesis } \mathcal{P}(k).$$
$$= k^2 + 2k + 1 \quad : \text{ algebra.}$$
$$= (k+1)^2$$

Thus $\mathcal{P}(k+1)$ is true. Hence, by Proof By Mathematical Induction (or we could just say "by induction") $\mathcal{P}(k)$ true $\forall k \in \mathbb{N}$. //

PROBLEM 8 Let $c, A_i \in R \ \forall i \in \mathbb{N}$.

Observe $\sum_{i=1}^{1} (cA_i) = cA_1 = c(A_1) = c\left(\sum_{i=1}^{1} A_i\right)$ hence $\sum_{i=1}^{n} cA_i = c\sum_{i=1}^{n} A_i$ for $n=1$. Suppose inductively that $\sum_{i=1}^{n} cA_i = c\sum_{i=1}^{n} A_i$ for some $n \in \mathbb{N}$. Consider,

$$\begin{aligned}
\sum_{i=1}^{n+1} cA_i &= cA_{n+1} + \sum_{i=1}^{n} cA_i && : \text{def}^{\underline{n}} \text{ of } \textstyle\sum \\
&= cA_{n+1} + c\sum_{i=1}^{n} A_i && : \text{by induction hypothesis.} \\
&= c\left(A_{n+1} + \sum_{i=1}^{n} A_i\right) && : \text{dist. prop. of ring multiplication.} \\
&= c\sum_{i=1}^{n+1} A_i && : \text{def}^{\underline{n}} \text{ of } \textstyle\sum.
\end{aligned}$$

Hence, $\sum_{i=1}^{n} cA_i = c\sum_{i=1}^{n} A_i \ \forall n \in \mathbb{N}$ by induction.//

$\boxed{\text{PROBLEM 9}}$ (Binomial Th$^{\underline{m}}$). Define $\binom{n}{k}$ recursively by:

$$\binom{1}{0} = \binom{1}{1} = 1, \quad \binom{n}{0} = \binom{n}{1} = 1 \quad \text{for } n \in \mathbb{N}$$

and $$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \quad \text{for } k = 1, 2, \ldots, n-1.$$

Show: for $\alpha, \beta \in \mathbb{R}$ ($\alpha, \beta \in \mathbb{F}$ in CURTIS, BUT, I'll be greedier here)

$$\underbrace{(\alpha+\beta)^n = \binom{n}{0}\alpha^n + \binom{n}{1}\alpha^{n-1}\beta + \cdots + \binom{n}{n}\beta^n = \sum_{k=0}^{n} \binom{n}{k}\alpha^{n-k}\beta^k}_{\mathcal{P}(n).}$$

Observe $\mathcal{P}(1)$ true as $(\alpha+\beta)^1 = \alpha+\beta = \binom{n}{0}\alpha^n + \binom{n}{n}\beta^n$.

Suppose inductively $\mathcal{P}(n-1)$ true for some $(n-1) \in \mathbb{N}$.

Consider,

$$\begin{aligned}
(\alpha+\beta)^n &= (\alpha+\beta)(\alpha+\beta)^{n-1} \\
&= (\alpha+\beta)\sum_{k=0}^{n-1}\binom{n-1}{k}\alpha^{n-1-k}\beta^k \\
&= \sum_{k=0}^{n-1}\binom{n-1}{k}\alpha^{n-k}\beta^k + \sum_{k=0}^{n-1}\binom{n-1}{k}\alpha^{n-1-k}\beta^{k+1} \\
&= \binom{n-1}{0}\alpha^n + \sum_{k=1}^{n-1}\binom{n-1}{k}\alpha^{n-k}\beta^k + \sum_{k=0}^{n-2}\binom{n-1}{k}\alpha^{n-1-k}\beta^{k+1} + \beta^n \\
&= \alpha^n + \sum_{k=1}^{n-1}\binom{n-1}{k}\alpha^{n-k}\beta^k + \sum_{j=1}^{n-1}\binom{n-1}{j-1}\alpha^{n-j}\beta^j + \beta^n \qquad (j = k+1) \\
&= \alpha^n + \sum_{k=1}^{n-1}\left(\binom{n-1}{k} + \binom{n-1}{k-1}\right)\alpha^{n-k}\beta^k + \beta^n \qquad \text{swap } j \text{ to } k \text{ to add sums.} \\
&= \alpha^n + \sum_{k=1}^{n-1}\binom{n}{k}\alpha^{n-k}\beta^k + \beta^n
\end{aligned}$$

Continuing PROBLEM 9,

$$(\alpha+\beta)^n = \alpha^n + \sum_{k=1}^{n-1} \binom{n}{k} \alpha^{n-k}\beta^k + \beta^n$$

$$= \binom{n}{0}\alpha^{n-0}\beta^0 + \sum_{k=1}^{n-1}\binom{n}{k}\alpha^{n-k}\beta^k + \binom{n}{n}\alpha^{n-n}\beta^n$$

$$= \sum_{k=0}^{n} \binom{n}{k}\alpha^{n-k}\beta^k$$

Thus we've shown $(n-1) \Rightarrow n$ true for the Binomial Thᵐ
Hence, by induction, $\sum_{k=0}^{n}\binom{n}{k}\alpha^{n-k}\beta^k = (\alpha+\beta)^n \;\forall n \in \mathbb{N}_0$ //

$\boxed{\text{PROBLEM 10}}$ Prove $c_1(A+B) = c_1A + c_1B \;\; \forall c_1 \in R, \forall A, B \in R^{m \times n}$

Let $(i,j) \in \mathbb{N}_m \times \mathbb{N}_n$ and $c_1 \in R$ and $A_{ij}, B_{ij} \in R \;\forall (i,j)$.
Consider, the following argument holds $\forall (i,j) \in \mathbb{N}_m \times \mathbb{N}_n$,

$(c_1(A+B))_{ij} = c_1(A+B)_{ij}$: defᵐ of scalar mult.

$= c_1(A_{ij} + B_{ij})$: defᵐ of matrix add.

$= c_1A_{ij} + c_1B_{ij}$: prop. of ring add and multiplication : the distributive prop for R.

$= (c_1A)_{ij} + (c_1B)_{ij}$: defᵐ of scalar mult.

$= (c_1A + c_1B)_{ij}$: defᵐ of matrix add.

Thus, $c_1(A+B) = c_1A + c_1B$. //

PROBLEM 11 Let $A \in R^{m \times n}$, $B \in R^{n \times p}$, $C \in R^{p \times q}$

Consider, for $(i,j) \in \mathbb{N}_m \times \mathbb{N}_q$

$$((AB)C)_{ij} = \sum_{k=1}^{p} (AB)_{ik} C_{kj} \quad : \text{def}^{\underline{n}} \text{ of mat. mult.}$$

$$= \sum_{k=1}^{p} \left(\sum_{l=1}^{n} A_{il} B_{lk} \right) C_{kj} \quad : \text{def}^{\underline{n}} \text{ of mat. mult.}$$

$$= \sum_{k=1}^{p} \left(\sum_{l=1}^{n} (A_{il} B_{lk}) C_{kj} \right) \quad : \text{prop. of } \sum$$

$$= \sum_{k=1}^{p} \left(\sum_{l=1}^{n} A_{il} (B_{lk} C_{kj}) \right) \quad : \text{associativity of } R\text{-multiplication}$$

$$= \sum_{l=1}^{n} \left(\sum_{k=1}^{p} A_{il} (B_{lk} C_{kj}) \right) \quad : \text{prop of finite sums. Can swap order.}$$

$$= \sum_{l=1}^{n} A_{il} \left(\sum_{k=1}^{p} B_{lk} C_{kj} \right) \quad : \text{using PROBLEM 10 actually. Notice } \nexists \text{ a } k\text{-dependence on } A_{il}.$$

$$= \sum_{l=1}^{n} A_{il} (BC)_{lj} \quad : \text{def}^{\underline{n}} \text{ of matrix mult.}$$

$$= (A(BC))_{ij} \quad : \text{def}^{\underline{n}} \text{ of mat. mult.}$$

Hence, $(AB)C = A(BC)$, $\forall A, B, C$ multipliable over the commutative ring R (although, I don't see as we needed commutative here...)

Of course, $(AX)Z = A(XZ)$ follows. //

PROBLEM 12 Let $A \in R^{m \times n}$ and $B \in R^{n \times p}$. Consider,

$$\begin{aligned}
((AB)^T)_{ij} &= (AB)_{ji} && : \text{def}^{\underline{n}} \text{ of transpose} \\
&= \sum_{k=1}^{n} A_{jk} B_{ki} && : \text{def}^{\underline{n}} \text{ of matrix mult.} \\
&= \sum_{k=1}^{n} (A^T)_{kj} (B^T)_{ik} && : \text{def}^{\underline{n}} \text{ of transpose} \\
&= \sum_{k=1}^{n} (B^T)_{ik} (A^T)_{kj} && : R \text{ is } \underline{\text{commutative}} \text{ ring, } ab = ba \ \forall a, b \in R. \\
&= (B^T A^T)_{ij} && : \text{def}^{\underline{n}} \text{ of matrix mult.}
\end{aligned}$$

Hence $((AB)^T)_{ij} = (B^T A^T)_{ij} \quad \forall (i,j) \in \mathbb{N}_m \times \mathbb{N}_p$

thus $(AB)^T = B^T A^T \quad \forall A, B$ multipliable over R. //

PROBLEM 13 Let $A_i \in R^{n \times n}$ for each $i \in \mathbb{N}$. Consider,

$n = 1$ and $n = 2$ are notation and PROBLEM 12 for the following claim:

$$(A_1 A_2 \cdots A_n)^T = A_n^T \cdots A_2^T A_1^T. \quad (*)$$

Suppose inductively $(*)$ is true for some $n \in \mathbb{N}$. Consider,

$$\begin{aligned}
(A_1 A_2 \cdots A_n A_{n+1})^T &= (B A_{n+1})^T && : \text{for } B = A_1 \cdots A_n \in R^{n \times n} \\
&= A_{n+1}^T B^T && : \text{by PROBLEM 12.} \\
&= A_{n+1}^T A_n^T \cdots A_2^T A_1^T && : \text{by induct. hypo.}
\end{aligned}$$

Hence $(*)$ holds for $(n+1)$ and we conclude $(*)$ is true $\forall n \in \mathbb{N}$ by induction on n. //

$\boxed{\text{PROBLEM 14}}$ $M = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \;\middle|\; a, b, c, d \in \{\bar{1}, \bar{2}\} \right\}$

this gives 16 possible matrices in M:

$\begin{bmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{bmatrix}$, $\underline{\begin{bmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{2} \end{bmatrix}}$, $\underline{\begin{bmatrix} \bar{1} & \bar{1} \\ \bar{2} & \bar{1} \end{bmatrix}}$, $\begin{bmatrix} \bar{1} & \bar{1} \\ \bar{2} & \bar{2} \end{bmatrix}$

$\underline{\begin{bmatrix} \bar{1} & \bar{2} \\ \bar{1} & \bar{1} \end{bmatrix}}$, $\begin{bmatrix} \bar{1} & \bar{2} \\ \bar{1} & \bar{2} \end{bmatrix}$, $\begin{bmatrix} \bar{1} & \bar{2} \\ \bar{2} & \bar{1} \end{bmatrix}$, $\underline{\begin{bmatrix} \bar{1} & \bar{2} \\ \bar{2} & \bar{2} \end{bmatrix}}$

$\underline{\begin{bmatrix} \bar{2} & \bar{1} \\ \bar{1} & \bar{1} \end{bmatrix}}$, $\begin{bmatrix} \bar{2} & \bar{1} \\ \bar{1} & \bar{2} \end{bmatrix}$, $\begin{bmatrix} \bar{2} & \bar{1} \\ \bar{2} & \bar{1} \end{bmatrix}$, $\underline{\begin{bmatrix} \bar{2} & \bar{1} \\ \bar{2} & \bar{2} \end{bmatrix}}$

$\begin{bmatrix} \bar{2} & \bar{2} \\ \bar{1} & \bar{1} \end{bmatrix}$, $\underline{\begin{bmatrix} \bar{2} & \bar{2} \\ \bar{1} & \bar{2} \end{bmatrix}}$, $\underline{\begin{bmatrix} \bar{2} & \bar{2} \\ \bar{2} & \bar{1} \end{bmatrix}}$, $\begin{bmatrix} \bar{2} & \bar{2} \\ \bar{2} & \bar{2} \end{bmatrix}$

Remark: the underlining stems from PROB. 15. The underlined matrices are invertible.

$\boxed{\text{PROBLEM 15}}$ We have $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \dfrac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ provided $ad - bc \neq \bar{0}$. If $ad - bc = \bar{0}$ then no inverse for $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ exists. ((You can prove this a vareity of ways, for instance

$$\underbrace{\begin{bmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{bmatrix}}_{A} \begin{bmatrix} \bar{1} \\ -\bar{1} \end{bmatrix} = \begin{bmatrix} \bar{0} \\ \bar{0} \end{bmatrix} \quad \text{thus } Ax = 0 \not\Leftrightarrow x = 0 \;\therefore A^{-1} \text{ d.n.e.} \;))$$

① Observe, $\begin{bmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{1} \end{bmatrix}$, $\begin{bmatrix} \bar{1} & \bar{1} \\ \bar{2} & \bar{2} \end{bmatrix}$, $\begin{bmatrix} \bar{1} & \bar{2} \\ \bar{1} & \bar{2} \end{bmatrix}$, $\begin{bmatrix} \bar{2} & \bar{1} \\ \bar{2} & \bar{1} \end{bmatrix}$, $\begin{bmatrix} \bar{2} & \bar{2} \\ \bar{1} & \bar{1} \end{bmatrix}$, $\begin{bmatrix} \bar{2} & \bar{2} \\ \bar{2} & \bar{2} \end{bmatrix}$ are not invertible, you can check $ad - bc = 0$ and for each of these we can solve $Ax = 0$ for $x \neq 0$.

② $\underline{\begin{bmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{2} \end{bmatrix}^{-1} = \dfrac{1}{\bar{2} - \bar{1}} \begin{bmatrix} \bar{2} & -\bar{1} \\ -\bar{1} & \bar{1} \end{bmatrix} = \begin{bmatrix} \bar{2} & \bar{2} \\ \bar{2} & \bar{1} \end{bmatrix}}_{①}$ $\therefore$ $\underline{\begin{bmatrix} \bar{2} & \bar{2} \\ \bar{2} & \bar{1} \end{bmatrix}^{-1} = \begin{bmatrix} \bar{1} & \bar{1} \\ \bar{1} & \bar{2} \end{bmatrix}}_{②}$

while we're at it. $A^{-1} = A \Rightarrow (A^{-1})^{-1} = A$.

continued ↪

PROBLEM 15 continued : $-\bar{1} = \overline{-1+3} = \bar{2}$, $-\bar{2} = \overline{-2+3} = \bar{1}$ etc.

$$\begin{bmatrix}\bar1 & \bar1\\ \bar2 & \bar1\end{bmatrix}^{-1} = \frac{1}{\bar1-\bar2}\begin{bmatrix}\bar1 & -\bar1\\ -\bar2 & \bar1\end{bmatrix} = \frac{1}{\bar2}\begin{bmatrix}\bar1 & \bar2\\ \bar1 & \bar1\end{bmatrix} \qquad \bar2\,\bar2 = \bar4 = \bar1 \quad \therefore \frac{1}{\bar2} = \bar2$$

$$= \bar2\begin{bmatrix}\bar1 & \bar2\\ \bar1 & \bar1\end{bmatrix}$$

$$= \begin{bmatrix}\bar2 & \bar1\\ \bar2 & \bar2\end{bmatrix}$$

Hence, $\underline{\begin{bmatrix}\bar1 & \bar1\\ \bar2 & \bar1\end{bmatrix}^{-1} = \begin{bmatrix}\bar2 & \bar1\\ \bar2 & \bar2\end{bmatrix}}$ (3) & $\underline{\begin{bmatrix}\bar2 & \bar1\\ \bar2 & \bar2\end{bmatrix}^{-1} = \begin{bmatrix}\bar1 & \bar1\\ \bar2 & \bar1\end{bmatrix}}$. (4)

Next,

$$\begin{bmatrix}\bar1 & \bar2\\ \bar1 & \bar1\end{bmatrix}^{-1} = \frac{1}{\bar1-\bar2}\begin{bmatrix}\bar1 & -\bar2\\ -\bar1 & \bar1\end{bmatrix} = \bar2\begin{bmatrix}\bar1 & \bar1\\ \bar2 & \bar1\end{bmatrix} = \begin{bmatrix}\bar2 & \bar2\\ \bar1 & \bar2\end{bmatrix}$$

Hence, $\underline{\begin{bmatrix}\bar1 & \bar2\\ \bar1 & \bar1\end{bmatrix}^{-1} = \begin{bmatrix}\bar2 & \bar2\\ \bar1 & \bar2\end{bmatrix}}$ (5) and $\underline{\begin{bmatrix}\bar2 & \bar2\\ \bar1 & \bar2\end{bmatrix}^{-1} = \begin{bmatrix}\bar1 & \bar2\\ \bar1 & \bar1\end{bmatrix}}$ (6)

Next,

$$\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}^{-1} = \frac{1}{\bar1-\bar4}\begin{bmatrix}1 & -2\\ -2 & 1\end{bmatrix} \quad \text{oops!} \quad \bar1-\bar4 = -\bar3 = \bar0$$

We see $\underline{\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}}$ and, now that I see it, $\underline{\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}}$ are not invertible! (thus 8 total non invertible)

Next,

$$\begin{bmatrix}1 & 2\\ 2 & 2\end{bmatrix}^{-1} = \frac{1}{\bar2-\bar4}\begin{bmatrix}\bar2 & -\bar2\\ -\bar2 & \bar1\end{bmatrix} = \begin{bmatrix}\bar2 & \bar1\\ \bar1 & \bar1\end{bmatrix}$$

$\therefore$ $\underline{\begin{bmatrix}1 & 2\\ 2 & 2\end{bmatrix}^{-1} = \begin{bmatrix}\bar2 & \bar1\\ \bar1 & \bar1\end{bmatrix}}$ (7) and $\underline{\begin{bmatrix}\bar2 & \bar1\\ \bar1 & \bar1\end{bmatrix}^{-1} = \begin{bmatrix}1 & 2\\ 2 & 2\end{bmatrix}}$ (8)

$\boxed{\text{PROBLEM 16}}$ over $\mathbb{Z}/8\mathbb{Z}$. $A = \begin{bmatrix} \bar{0} & \bar{1} \\ \bar{6} & \bar{4} \end{bmatrix}$

(a.) $$A + A = \left[\begin{array}{c|c} \bar{0}+\bar{0} & \bar{1}+\bar{1} \\ \hline \bar{6}+\bar{6} & \bar{4}+\bar{4} \end{array}\right] = \underline{\begin{bmatrix} \bar{0} & \bar{2} \\ \bar{4} & \bar{0} \end{bmatrix}}.$$

(b.) $$A + 7A = \bar{8}A = \begin{bmatrix} \bar{8}\cdot\bar{0} & \bar{8}\bar{1} \\ \bar{8}\bar{6} & \bar{8}\bar{4} \end{bmatrix} = \begin{bmatrix} \bar{0} & \bar{0} \\ \bar{0} & \bar{0} \end{bmatrix} = \underline{0}.$$

(c.) $$A^2 = \begin{bmatrix} \bar{0} & \bar{1} \\ \bar{6} & \bar{4} \end{bmatrix}\begin{bmatrix} \bar{0} & \bar{1} \\ \bar{6} & \bar{4} \end{bmatrix} = \left[\begin{array}{c|c} \bar{0}+\bar{6} & \bar{0}+\bar{4} \\ \hline \bar{0}+\bar{24} & \bar{6}+\bar{16} \end{array}\right] = \left[\begin{array}{c|c} \bar{6} & \bar{4} \\ \hline \bar{24} & \bar{22} \end{array}\right] = \underline{\begin{bmatrix} \bar{6} & \bar{4} \\ \bar{0} & \bar{6} \end{bmatrix}}.$$

(d.) $$A^4 = A^2A^2 = \begin{bmatrix} -\bar{2} & \bar{4} \\ \bar{0} & -\bar{2} \end{bmatrix}\begin{bmatrix} -\bar{2} & \bar{4} \\ \bar{0} & -\bar{2} \end{bmatrix} = \underline{\begin{bmatrix} \bar{4} & \bar{0} \\ \bar{0} & \bar{4} \end{bmatrix}}.$$

<u>Remark</u>: sometimes using $\mathbb{Z}/8\mathbb{Z} = \{-\bar{3}, -\bar{2}, -\bar{1}, \bar{0}, \bar{1}, \bar{2}, \bar{3}, \bar{4}\}$ makes for easier calculation. Same goes for $\mathbb{Z}/n\mathbb{Z}$.

$\boxed{\text{PROBLEM 17}}$ $v = [1 \;\; 2 \;\; 3]$, $B = \begin{bmatrix} 4 & 5 \\ 1 & 1 \\ 0 & 6 \end{bmatrix}$

(a.) $$vB = \underset{(1\times 3)}{[1, 2, 3]}\underset{(3\times 2)}{\begin{bmatrix} 4 & 5 \\ 1 & 1 \\ 0 & 6 \end{bmatrix}} = \underset{(1\times 2)}{\underline{[6, 7]}}.$$

(b.) Bv is $(3\times 2)(1\times 3)$ <u>not</u> <u>multipliable</u>.

(c.) $vBv^T = [6, 7]\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ again, not multipliable. (dimension mismatch)

(d.) $$BB^T = \begin{bmatrix} 4 & 5 \\ 1 & 1 \\ 0 & 6 \end{bmatrix}\begin{bmatrix} 4 & 1 & 0 \\ 5 & 1 & 6 \end{bmatrix} = \left[\begin{array}{c|c|c} 4^2+5^2 & 9 & 30 \\ \hline 4+5 & 2 & 6 \\ \hline 30 & 6 & 6^2 \end{array}\right]$$

$(3\times 2)(2\times 3)$

PROBLEM 17 continued

(d.) $BB^T = \begin{bmatrix} 41 & 9 & 30 \\ 9 & 2 & 6 \\ 30 & 6 & 36 \end{bmatrix}$.

Remark: $(BB^T)^T = BB^T$.

(e.) $B^TB = \begin{bmatrix} 4 & 1 & 0 \\ 5 & 1 & 6 \end{bmatrix}\begin{bmatrix} 4 & 5 \\ 1 & 1 \\ 0 & 6 \end{bmatrix} = \underbrace{\begin{bmatrix} 17 & 21 \\ 21 & 62 \end{bmatrix}}_{\text{again symmetric}}$.

Remark: If $\text{trace}(M) = M_{11} + M_{22} + \cdots + M_{nn}$ for $M \in \mathbb{R}^{n\times n}$ then we can also appreciate

$$\text{trace}(BB^T) = \text{trace}(B^TB) = 41+2+36 = 17+62 = 79.$$

Observe $4^2+1^2+0^2+5^2+6^2 = 17+25+36 = 79$. Curious.

PROBLEM 18

$$e_i^T A e_j = A_{ij} = i^2 + j^2 \quad \forall (i,j) \in \mathbb{N}_3 \times \mathbb{N}_2$$

$$\therefore \quad A = \begin{bmatrix} 1^2+1^2 & 1^2+2^2 \\ 2^2+1^2 & 2^2+2^2 \\ 3^2+1^2 & 3^2+2^2 \end{bmatrix} = \begin{bmatrix} 2 & 5 \\ 5 & 8 \\ 10 & 13 \end{bmatrix}.$$