Your PRINTED NAME indicates you read Chapter 1 and §2.1, 2.2 of the notes: ____________________.

Assume R is a commutative ring with identity throughout this homework. Also, $\mathbb{N} = \{1, 2, \dots\}$.

Problem 1 Consider the system of equations in $\mathbb{Z}_7$:

$$\begin{aligned} x + 2y &= 1 \\ -2x + 7y &= 2 \end{aligned}.$$

Find the solution set by writing this system as $Av = b$ and then multiplying by A^{-1} to obtain $v = A^{-1}b$. *Hint: you ought to be able to calculate* A^{-1} *as we did in lecture*

Problem 2 Consider the system of equations in $\mathbb{Z}_{17}$:

5

$$\begin{aligned} x + 4y &= a \\ -2x + 7y &= b \end{aligned}.$$

Find the solution set through row-reduction or another technique. In the following cases:

(a.) $a = 1$, $b = 2$

(b.) $a = b = 0$.

Problem 3 Suppose $a_i, b_i, c \in R$ for $i \in \mathbb{N}$. Prove $\sum_{i=1}^{n}(ca_i + b_i) = c\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$ for all $n \in \mathbb{N}$.

Problem 4 Prove Theorem 1.3.11.1; prove matrix multiplication is associative for matrices over R.

Problem 5 Let A, B be matrices over R of the same size and let $c \in R$. Prove $(cA + B)^T = cA^T + B^T$.

Problem 6 The **trace** is defined by $tr(A) = \sum_{i=1}^{n} A_{ii}$ for $A \in R^{n \times n}$. Prove that:

(a.) $tr(cA + B) = ctr(A) + tr(B)$ for $c \in R$ and $A, B \in R^{n \times n}$

(b.) $tr(AB) = tr(BA)$ for $A \in R^{m \times n}$ and $B \in R^{n \times m}$.

Problem 7 A diagonal matrix $D = diag(a_1, \dots, a_n)$ can be written as $D = \sum_{i=1}^{n} a_i E_{ii}$. Give a proof my mathematical induction that $D^k = \sum_{i=1}^{n} a_i^k E_{ii}$ for all $k \in \mathbb{N}$.
Hint: I would like for you to use the formulas developed for E_{ij} *in Chapter 1 to aid in this proof. Please do not just quote Proposition 1.4.20, however, perhaps the proof I give there will be helpful*

Problem 8 Find all $n \times n$ square matrices over R which commute with **all** other square matrices.

Problem 9 Friedberg, Insel and Spence 5th edition, §1.2#19, page 16.
(check with my book to make sure you're doing the right problem)

Problem 10 Let $W = \{(x, y) \mid x + y \geq 0\}$. Prove or disprove that W is a subspace of $\mathbb{R}^2$.

Problem 11 Let $W = \{(x_2 - x_4, x_2, x_4, x_4) \mid x_2, x_4 \in \mathbb{Q}\}$. Prove or disprove that W is a subspace of $\mathbb{Q}^4$.

Problem 12 A square matrix is said to be **antisymmetric** if $A^T = -A$. Let W be the set of all antisymetric $n \times n$ matrices over a field $\mathbb{F}$. Show $W \leq \mathbb{F}^{n \times n}$.

Problem 13 Let $W = \{f(t) \in \mathbb{R}[t] \mid f''(3) = 1\}$. Prove or disprove that W is a subspace of real polynomials in t.

Problem 14 Friedberg, Insel and Spence 5th edition, §1.3#12, page 21.
(check with my book to make sure you're doing the right problem)

Problem 15 Friedberg, Insel and Spence 5th edition, §1.3#13, page 21.
(check with my book to make sure you're doing the right problem)

Problem 16 Friedberg, Insel and Spence 5th edition, §1.3#15, page 22.
(check with my book to make sure you're doing the right problem)

MATH 321 : MISSION 1 SOLUTION

P1 Solve $x + 2y = 1$, $-2x + 7y = 2$ in $\mathbb{Z}_7$ by multiplication by inverse.

$$\underbrace{\begin{bmatrix} 1 & 2 \\ -2 & 7 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{v} = \underbrace{\begin{bmatrix} 1 \\ 2 \end{bmatrix}}_{b}$$

$$A^{-1} = \begin{bmatrix} 1 & 2 \\ -2 & 7 \end{bmatrix}^{-1}$$

$$= (7+4)^{-1} \begin{bmatrix} 7 & -2 \\ 2 & 1 \end{bmatrix}$$

$$= 4^{-1} \begin{bmatrix} 7 & -2 \\ 2 & 1 \end{bmatrix}$$

$4 \cdot 2 = 8 = 1$
$\therefore 4^{-1} = 2$.

$$= 2 \begin{bmatrix} 7 & -2 \\ 2 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 14 & -4 \\ 4 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 3 \\ 4 & 2 \end{bmatrix}.$$

$v = A^{-1}b$ thus,

$$v = \begin{bmatrix} 0 & 3 \\ 4 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 8 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \end{bmatrix}$$

$\therefore$ $\boxed{(x, y) = (6, 1)}$

P2 $x + 5y = a$, $-2x + 7y = b$ over $\mathbb{Z}_{17}$

$$\left[\begin{array}{cc|c} 1 & 5 & a \\ -2 & 7 & b \end{array}\right] \xrightarrow{r_2 + 2r_1} \left[\begin{array}{cc|c} 1 & 5 & a \\ 0 & 17 & b+2a \end{array}\right] = \left[\begin{array}{cc|c} 1 & 5 & a \\ 0 & 0 & b+2a \end{array}\right]$$

(a.) Solve for $a = 1$, $b = 2$

Notice $b + 2a = 2 + 2 = 4 \neq 0$ thus the given system is inconsistent by the row-reduction above. That is, solution set $= \emptyset$.

(b.) $a = b = 0$

$$\text{rref} \left[\begin{array}{cc|c} 1 & 5 & 0 \\ -2 & 7 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & 5 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$$\begin{aligned} \text{Solution Set} &= \{ (x, y) \in \mathbb{Z}_{17}^2 \mid x + 5y = 0 \} \\ &= \{ (-5y, y) \mid y \in \mathbb{Z}_{17} \} \\ &= \text{span}_{\mathbb{Z}_{17}} \{ (-5, 1) \} \\ &= \text{span}_{\mathbb{Z}_{17}} \{ (12, 1) \}. \end{aligned}$$

$$\begin{aligned} \#(\text{Solution Set}) &= 17 \\ &= \{ (0,0), (12,1), (7,2), \dots, \end{aligned}$$

P3 Suppose $a_i, b_i, c \in \mathbb{R}$ for $i \in \mathbb{N}$.

Let $\mathcal{P}_n$ be the claim $\sum_{i=1}^{n}(ca_i + b_i) = c\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$.

Observe $\mathcal{P}_1$ is true since

$$\sum_{i=1}^{1} ca_i + b_i = ca_1 + b_1 = c\sum_{i=1}^{1} a_i + \sum_{i=1}^{1} b_i .$$

Suppose $\mathcal{P}_n$ is true for some $n \in \mathbb{N}$. Consider,

$$\begin{aligned}\sum_{i=1}^{n+1}(ca_i + b_i) &= ca_{n+1} + b_{n+1} + \sum_{i=1}^{n}(ca_i + b_i) && \text{by def}^{\text{n}} \text{ of } \Sigma \\ &= ca_{n+1} + b_{n+1} + c\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i && \text{: by Induction hypothesis.} \\ &= c\left(a_{n+1} + \sum_{i=1}^{n} a_i\right) + b_{n+1} + \sum_{i=1}^{n} b_i && \text{: arithmetic in } \mathbb{R} \\ &= c\sum_{i=1}^{n+1} a_i + \sum_{i=1}^{n+1} b_i && \text{: def}^{\text{n}} \text{ of } \Sigma\end{aligned}$$

Hence $\mathcal{P}_{n+1}$ is true and we conclude that

$\sum_{i=1}^{n}(ca_i + b_i) = c\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$ for all $n \in \mathbb{N}$

by proof by mathematical induction.

P4 Let $A \in R^{m\times n}$, $X \in R^{n\times p}$, $Z \in R^{p\times q}$. Consider,

$$((AX)Z)_{ij} = \sum_{k=1}^{p} (AX)_{ik} Z_{kj} \quad : \text{def}^{n} \text{ of matrix mult.}$$

$$= \sum_{k=1}^{p} \left(\sum_{l=1}^{n} A_{il} X_{lk} \right) Z_{kj} \quad : \text{def}^{n} \text{ of mat. mult.}$$

$$= \sum_{k=1}^{p} \sum_{l=1}^{n} (A_{il} X_{lk}) Z_{kj} \quad : \text{property of } \Sigma \text{ (distributivity)}$$

$$= \sum_{k=1}^{p} \sum_{l=1}^{n} A_{il} (X_{lk} Z_{kj}) \quad : \text{associativity of } R\text{-mult.}$$

$$= \sum_{l=1}^{n} \sum_{k=1}^{p} A_{il} (X_{lk} Z_{kj}) \quad : \Sigma\text{'s commute}$$

$$= \sum_{l=1}^{n} A_{il} \left(\sum_{k=1}^{p} X_{lk} Z_{kj} \right) \quad : A_{il} \text{ constant w.r.t. } \sum_{k}$$

$$= \sum_{l=1}^{n} A_{il} (XZ)_{lj} \quad : \text{def}^{n} \text{ of mat. mult.}$$

$$= (A(XZ))_{ij} \quad : \text{def}^{n} \text{ of matrix mult.}$$

Thus $(AX)Z = A(XZ)$ as the above calculation holds for all $1 \le i \le m$ and $1 \le j \le q$. //

P5 Suppose $A, B \in R^{m \times n}$ and $c \in R$,

$$((cA+B)^T)_{ij} = (cA+B)_{ji} \quad : \text{def}^{\underline{n}} \text{ of transpose}$$
$$= (cA)_{ji} + B_{ji} \quad : \text{def}^{\underline{n}} \text{ of matrix add.}$$
$$= cA_{ji} + B_{ji} \quad : \text{def}^{\underline{n}} \text{ of scalar mult.}$$
$$= ((cA)^T)_{ij} + (B^T)_{ij} \quad : \text{def}^{\underline{n}} \text{ of transpose}$$
$$= (cA^T + B^T)_{ij} \quad : \text{def}^{\underline{n}} \text{ of matrix add.}$$

Thus $(cA+B)^T = cA^T + B^T$ as the above holds $\forall i,j$. //

P6 Let $\text{trace}(A) = \text{tr}(A) = \sum_{i=1}^{n} A_{ii}$ for $A \in R^{n \times n}$.

(a.) Let $A, B \in R^{n \times n}$ and $c \in R$,

$$\text{tr}(cA+B) = \sum_{i=1}^{n} (cA+B)_{ii} \quad : \text{def}^{\underline{n}} \text{ of tr.}$$
$$= \sum_{i=1}^{n} cA_{ii} + B_{ii} \quad : \text{def}^{\underline{n}} \text{ of matrix add and scalar mult.}$$
$$= c\sum_{i=1}^{n} A_{ii} + \sum_{i=1}^{n} B_{ii} \quad : \text{properties of } \Sigma$$
$$= \underline{c\,\text{tr}(A) + \text{tr}(B)}. \quad : \text{def}^{\underline{n}} \text{ of tr.}$$

(b.) Let $A \in R^{m \times n}$ and $B \in R^{n \times m}$ then,

$$\text{tr}(AB) = \sum_{i=1}^{m} (AB)_{ii} \quad : \text{def}^{\underline{n}} \text{ of tr.}$$
$$= \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} B_{ji} \quad : \text{def}^{\underline{n}} \text{ of mat. mult.}$$
$$= \sum_{j=1}^{n} \sum_{i=1}^{m} B_{ji} A_{ij} \quad : \text{mult. in } R \text{ commutes and } \Sigma\text{'s commute}$$
$$= \sum_{j=1}^{n} (BA)_{jj} = \underline{\text{tr}(BA)}. // \quad (\text{again using def}^{\underline{n}} \text{ of tr and mat. mult.})$$

P7 <u>Claim</u>: If $D = \sum_{i=1}^{n} a_i E_{ii}$ then $D^k = \sum_{i=1}^{n} a_i^k E_{ii}$ for all $k \in \mathbb{N}$.

Observe the claim is true for $k=1$ since $D^1 = D$ & $a_i^1 = a_i$ thus $D^1 = D = \sum_{i=1}^{n} a_i E_{ii} = \sum_{i=1}^{n} a_i^1 E_{ii}$.

Suppose inductively the claim is true for $k \in \mathbb{N}$.

Consider

$$\begin{aligned}
D^{k+1} &= D D^k && : \text{def}^{\text{n}} \text{ of matrix power} \\
&= \left(\sum_{i=1}^{n} a_i E_{ii}\right)\left(\sum_{j=1}^{n} a_j^k E_{jj}\right) && : \text{by induction hypothesis.}^{*} \\
&= \sum_{i,j=1}^{n} a_i a_j^k E_{ii} E_{jj} && : \text{properties of } \textstyle\sum \\
&= \sum_{i,j=1}^{n} a_i a_j^k \delta_{ij} E_{ij} && : \left(\text{property of } E_{ij}:\ E_{ij} E_{kl} = \delta_{jk} E_{il}\right) \\
&= \sum_{i=1}^{n} a_i a_i^k E_{ii} && : \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \text{ collapses } \textstyle\sum_j \text{ to single term.} \\
&= \sum_{i=1}^{n} a_i^{k+1} E_{ii}
\end{aligned}$$

Thus the claim is true for $k+1$ and we conclude the claim holds for all $k \in \mathbb{N}$ by P.M.I.

$*$: can't use $\sum_i$ since we already used it. I had to introduce $\sum_j$ to avoid confusing the $\sum$'s.

P8 Find all $n \times n$ square matrices over $\mathbb{R}$ which commute with <u>all</u> other square matrices.

Let us call the desired set $Z(\mathbb{R}^{n\times n})$ then $A \in Z(\mathbb{R}^{n\times n})$ means $AB = BA \ \forall \ B \in \mathbb{R}^{n\times n}$.

The most convenient choice to gain information concerning A is $B = E_{ij}$. Recall

$$A = \sum_{k,l} A_{kl} E_{kl} \quad \text{thus}$$

$$AB = \sum_{k,l}{}' A_{kl} E_{kl} E_{ij} = \sum_{k,l} A_{kl} \delta_{li} E_{kj}$$

$$BA = E_{ij} \sum_{k,l} A_{kl} E_{kl} = \sum_{k,l}{}' A_{kl} E_{ij} E_{kl} = \sum_{k,l}{}' A_{kl} \delta_{jk} E_{il}$$

Next, I'll look at the r,s component of AB vs BA

$$(AB)_{rs} = \sum_{k,l} A_{kl} \delta_{li} \underbrace{(E_{kj})_{rs}}_{\delta_{kr}\delta_{js}} = \sum_{k,l} A_{kl} \delta_{li} \delta_{kr} \delta_{js}$$

$$(BA)_{rs} = \sum_{k,l} A_{kl} \delta_{jk} (E_{il})_{rs} = \sum_{k,l}{}' A_{kl} \delta_{jk} \delta_{ir} \delta_{ls}$$

But, $(AB)_{rs} = (BA)_{rs}$.

<u>(1.) If $r = s$</u> then $\sum_{k,l} A_{kl} \delta_{li} \delta_{kr} \delta_{jr} = \sum_{k,l}{}' A_{kl} \delta_{jk} \delta_{ir} \delta_{lr}$

Thus, $\boxed{A_{ri} \delta_{jr} = A_{jr} \delta_{ir}}$ ☆

Hence if $j \neq r$ we have $\delta_{jr} = 0$ and so $0 = A_{jr} \delta_{ir}$ and if $i = r$ we obtain $A_{jr} \cdot 1 = \underline{A_{jr} = 0 \text{ for } j \neq r}$.

P8 continued (nothing happens on this page, skip ahead for interesting case 2)

We have show $A_{jr} = 0$ for $j \neq r$. It remains to determine conditions on the diagonal entries. Return to ✩ once more and suppose $j = r$

$$A_{ri}\,\delta_{rr} = A_{rr}\,\delta_{ir}$$

$$A_{ri} = A_{rr}\,\delta_{ir}$$

Well, $i \neq r$ gives $0 = A_{rr}(0)$ thus $0 = 0$. Great. However, if $i = r$ then $A_{rr} = A_{rr}\,\delta_{rr} = A_{rr}$. Yep. I guess ✩ has told us all it's secrets. Moving on,

(2.) If $r \neq s$ (for $(AB)_{rs} = (BA)_{rs}$). Study,

$$\underbrace{\sum_{k,l} A_{kl}\,\delta_{li}\,\delta_{kr}\,\delta_{js} = \sum_{k,l} A_{kl}\,\delta_{jk}\,\delta_{ir}\,\delta_{ls}}_{i,j,r,s \text{ are free indices, (we consider } r\neq s)}$$

I'll put $i = j = r$ for specificity,

$$\sum_{k,l} A_{kl}\,\delta_{lr}\,\delta_{kr}\,\underset{\downarrow 0}{\delta_{rs}} = \sum_{k,l} A_{kl}\,\delta_{rk}\,\underset{\downarrow 1}{\delta_{rr}}\,\delta_{ls}$$

$$0 = A_{rs} \quad (\text{for } r \neq s)$$

YES, we know this already. Continuing

P8 continued

(2.) If $r \neq s$ for $(AB)_{rs} = (BA)_{rs}$ then

$$\sum_{k,l} A_{kl}\,\delta_{li}\,\delta_{kr}\,\delta_{js} = \sum_{k,l} A_{kl}\,\delta_{jk}\,\delta_{ir}\,\delta_{ls}$$

Consider $i = r$,

$$\underbrace{\sum_{k,l} A_{kl}\,\delta_{lr}\,\delta_{kr}\,\delta_{js}}_{A_{rr}\,\delta_{js}} = \underbrace{\sum_{k,l} A_{kl}\,\delta_{jk}\,\delta_{rr}\,\delta_{ls}}_{A_{js}}$$

$$A_{rr}\,\delta_{js} = A_{js}$$

If $j = s$ then $A_{rr}\,\delta_{ss} = A_{ss} \Rightarrow \underline{A_{rr} = A_{ss}}$ <u>for $r \neq s$</u>.

We find $A = \text{diag}(A_{11}, A_{11}, \dots, A_{11}) = A_{11}\, I$

$$\therefore\ Z(\mathbb{R}^{n \times n}) = \{ cI \mid c \in \mathbb{R} \}$$

$$= \text{span}_{\mathbb{R}}(I).$$

P9 §1.2 #19

Let $V = \{(a_1, a_2) \mid a_1, a_2 \in \mathbb{R}\}$ for $(a_1, a_2), (b_1, b_2) \in V$ and $c \in \mathbb{R}$ define addition as usual $(a_1, a_2) + (b_1, b_2) = (a_1 + a_2, b_1 + b_2)$ and

$$c(a_1, a_2) = \begin{cases} (0, 0) & \text{for } c = 0 \\ (ca_1, \frac{1}{c} a_2) & \text{for } c \neq 0 \end{cases}$$

is V a vector space over $\mathbb{R}$ with the operations defined above?

Almost. So, no, V not a vector space.

Using the Axioms in my Lecture Notes we may affirm all Axioms hold <u>except</u> A8

$$(c_1 + c_2) \cdot a \neq c_1 \cdot a + c_2 \cdot a$$

For example (and in counter-"exampling" this is all that is needed, and what is expected)

$$(1 + 2) \cdot (3, 4) = 3 \cdot (3, 4) = (3 \cdot 3, \tfrac{1}{3} \cdot 4) = (9, 4/3)$$

$$\begin{aligned} 1 \cdot (3, 4) + 2(3, 4) &= (3, 4) + (2 \cdot 3, \tfrac{1}{2} \cdot 4) \\ &= (3, 4) + (6, 2) \\ &= (9, 8) \neq (9, 4/3). \end{aligned}$$

Hence V is not a vector space.

($(c_1 + c_2) \cdot a = \left((c_1 + c_2) a_1, \frac{1}{c_1 + c_2} \cdot a_2\right)$ the issue is $\frac{1}{c_1 + c_2} \neq \frac{1}{c_1} + \frac{1}{c_2}$)

P10 $W = \{(x,y) \mid x+y \geq 0\}$

Notice $(1,2) \in W$ since $1+2 = 3 \geq 0$. However, $-1 \cdot (1,2) = (-1,-2) \notin W$ since $-1-2 = -3 \ngeq 0$

Thus $W \nleq \mathbb{R}^2$ since it is not closed under scalar multiplication.

<u>Remark</u>: obviously my sol^{n} to P10 is far from unique.

P11 $W = \{(x_2 - x_4, x_2, x_4, x_4) \mid x_2, x_4 \in \mathbb{Q}\}$

$= \{x_2(1,1,0,0) + x_4(-1,0,1,1) \mid x_2, x_4 \in \mathbb{Q}\}$

$= \text{span}_{\mathbb{Q}}\{(1,1,0,0), (-1,0,1,1)\} \leq \mathbb{Q}^4$

as we proved $\text{span}(S) \leq V$ whenever $S \subseteq V$.

P12 <u>Let $W = \{A \in \mathbb{F}^{n \times n} \mid A^T = -A\}$ show $W \leq \mathbb{F}^{n \times n}$</u>

Observe $W \subseteq \mathbb{F}^{n \times n}$ and $0^T = -0 = 0$ thus $W \neq \emptyset$.

Let $A, B \in W$ and let $c \in \mathbb{F}$. Consider,

$$
\begin{aligned}
(cA + B)^T &= cA^T + B^T && : \text{property of transpose} \\
&= c(-A) + (-B) && : \text{since } A, B \in W \\
&= -(cA + B)
\end{aligned}
$$

Thus $cA + B \in W$ and thus $cA, A+B \in W$ and we conclude $W \leq \mathbb{F}^{n \times n}$ by subspace test.//

P13 Let $W = \{ f(t) \in \mathbb{R}[t] \mid f''(3) = 1 \}$

Observe $z(t) = 0 \ \forall t \in \mathbb{R}$ has $z''(3) = 0 \neq 1$

thus $0 \notin W$ hence $\underbrace{W \not\leq \mathbb{R}[t]}_{W \text{ not a subspace.}}$.

P14 § 1.3 # 12, p. 21

Let $W = \{ A \in \mathbb{F}^{m \times n} \mid \underbrace{A_{ij} = 0 \text{ for } i > j}_{\text{this forces all entries below diagonal}} \}$ to be zero, it makes A upper-Δ

Clearly $W \subseteq \mathbb{F}^{m \times n}$ and $O_{ij} = 0 \ \forall i,j$ defines the zero matrix and we note $O \in W \neq \emptyset$.

Suppose $c \in \mathbb{F}$ and $A, B \in W$ then for $i > j$,

$$
\begin{aligned}
(cA + B)_{ij} &= cA_{ij} + B_{ij} && \text{: def}^{\underline{n}} \text{ of matrix add. and scalar mult.} \\
&= c(0) + 0 && \text{: since } A, B \in W \\
&= 0
\end{aligned}
$$

Thus $cA, A + B \in W$ and we conclude $W \leq \mathbb{F}^{m \times n}$ by the subspace test.//

P15 §1.3 #13 page 21

Let $S \neq \emptyset$ and suppose $\mathbb{F}$ is a field. Let $s_o \in S$ and define $W = \{ f \in \mathcal{F}(S, \mathbb{F}) \mid f(s_o) = 0 \}$.

Observe the zero function $z(x) = 0 \ \forall x \in S$ has $z(s_o) = 0$ thus $z \in W \neq \emptyset$. Suppose $f, g \in W$ and $c \in \mathbb{F}$. Consider,

$$(cf + g)(s_o) = cf(s_o) + g(s_o) \quad : \text{def}^{\underline{n}} \text{ of add \& scalar mult. in } W$$
$$= c(0) + 0 \quad : \text{since } f, g \in W$$
$$= 0$$

thus $cf, f+g \in W$ and we conclude $W \leq \mathcal{F}(S, \mathbb{F})$ by the subspace test.

P16 §1.3 #15. Let $W = \{ f \in \mathcal{F}(\mathbb{R}, \mathbb{R}) \mid f'(x) \text{ exists } \forall x \in \mathbb{R} \}$ and $C(\mathbb{R})$ is the set of continuous functions on $\mathbb{R}$.

By Caratheodory, for $f \in W$ and $a \in \mathbb{R}$ there exists $\phi_a(x)$ continuous at $x = a$ such that $f(x) = f(a) + \phi_a(x)(x-a)$ then $\lim_{x \to a} f(x) = \lim_{x \to a} [f(a) + \phi_a(x)(x-a)] = f(a) + f'(a)(a-a) = f(a)$.

Thus f continuous at $x = a$ for any $a \in \mathbb{R}$ $\therefore f \in C(\mathbb{R})$ and I've shown $W \subseteq C(\mathbb{R})$. Notice $f(x) = c$ is differentiable for any $c \in \mathbb{R}$ thus $W \neq \emptyset$ (just to be different ☺)

Let $f, g \in W$ and $c \in \mathbb{R}$ then by calculus I,

$(cf + g)'(x) = cf'(x) + g'(x) \quad \therefore cf, f+g \in W$

$\Rightarrow W \leq C(\mathbb{R})$ by subspace test.//