Math 321: index proofs, solution sets, matrix calculation	Mission 1 [50pts]

Your printed name indicates you read Chapter 1 of the notes: ____________________.

Note: I assume you know elementary linear algebra including definitions of LI, span and rref from Math 221. You are free to use calculational techniques from the previous course. However, proofs must be given in terms of the definitions and theory presented within the present context. See my Lecture Notes for definitions if in doubt.

Note: please write your solutions on one side of the paper neatly(this problem sheet should be the only double-sided page in the solution, it is to be used as a coversheet). Solutions should be understandable without consulting the problem statements given below.

Assume $\mathbb{F}$ is a field and $n \in \mathbb{N}$ in this assignment.

Problem 1 Solve $\left\{ \begin{array}{c} x + 2y = 0 \\ -2x + 7y = 3 \end{array} \right\}$ in $\mathbb{Z}_7$ via multiplication by inverse matrix.

Problem 2 How many solutions are there to $x_1 + 2x_2 + 3x_3 = 1$ over $\mathbb{Z}_{11}$?

Problem 3 Let $S = \{(1,1,1,1), (0,0,-3,2), (-1,-1,2,0)\} \subseteq \mathbb{R}^4$.

(a.) show S is linearly independent.

(b.) find the form of a typical vector in $\mathbb{R}^4 - \text{span}(S)$.

(c.) for which i is $S \cup \{e_i\}$ a basis for $\mathbb{R}^4$?

Problem 4 Consider $M = \begin{bmatrix} 1 & 1 & 2 & 0 & 1 \\ 2 & 2 & 3 & -4 & 2 \end{bmatrix}$

(a.) calculate $\text{rref}(M)$

(b.) find the basis for $Null(M) = \{x \in \mathbb{R}^5 \mid Mx = 0\}$

(c.) solve $\left\{ \begin{array}{c} x_1 + x_2 + 2x_3 = 1 \\ 2x_1 + 2x_2 + 3x_3 - 4x_4 = 2 \end{array} \right\}$

Problem 5 Suppose $A = [A_1|A_2|A_3|A_4|A_5]$ is matrix for which

$$\text{rref}[A] = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & -1 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}$$

Suppose $A_3 = (1,0,1)$ and $A_4 = (1,3,2)$ and $A_5 = (1,-3,1)$ then

(a.) find A explicitly with the aid of the CCP (should not require much calculation if you're doing it correctly)

(b.) find invertible matrix E for which $EA = \text{rref}[A]$ (you can construct E as the product of appropriate elementary matrices)

(c.) is A invertible ?

(d.) does $Ax = 0$ have solutions ? What is the basis for $Null(A)$?

Problem 6 Suppose $A, B, N \in \mathbb{F}^{n\times n}$ where A is symmetric, B is antisymmetric, and N is nilpotent with degree three. Prove the following:

(a.) $(ABN)^T(ABN)$ is symmetric

(b.) If $AN = NA$ then AN is nilpotent degree three.

(c.) If $M = \left[\begin{array}{c|c} A & 0 \\ \hline 0 & B \end{array}\right]$ then M^2 is symmetric.

Problem 7 Let $A, B \in \mathbb{F}^{m\times n}$ and $X \in \mathbb{F}^{n\times p}$ and $c_1, c_2 \in \mathbb{F}$. Prove the following:

(a.) $c_1(A + B) = c_1A + c_1B$,

(b.) $(c_1 + c_2)A = c_1A + c_2A$,

(c.) $(A + B)X = AX + BX$.

Problem 8 Suppose $A, B \in \mathbb{F}^{n\times n}$ are lower triangular matrices. Prove AB is also a lower triangular matrix. Note: to say A is lower triangular is to say that all the entries above the diagonal are zero; that is, A is lower triangular if $A_{ij} = 0$ whenever $j > i$.

Problem 9 The **trace** is defined by $tr(A) = \sum_{i=1}^{n} A_{ii}$ for $A \in \mathbb{F}^{n\times n}$. Prove that:

(a.) $tr(cA + B) = ctr(A) + tr(B)$ for $c \in R$ and $A, B \in R^{n\times n}$

(b.) $tr(AB) = tr(BA)$ for $A \in R^{m\times n}$ and $B \in R^{n\times m}$.

(c.) matrices X, Y are **similar** if there exists an invertible matrix P for which $Y = P^{-1}XP$. Given X, Y similar, prove $tr(X) = tr(Y)$.

Problem 10 Suppose $E_{ij} \in \mathbb{F}^{n\times n}$ where $(E_{ij})_{kl} = \delta_{ik}\delta_{jl}$.

(a.) prove $E_{ij} = e_i e_j^T$ where $e_1, \dots, e_n$ is the usual standard basis of $\mathbb{F}^n$

(b.) let A be an $n \times n$ matrix with entries in $\mathbb{F}$ calculate AE_{ij} and $E_{ij}A$. Explain how these matrices relate to A in terms of rows and columns and their interchange.

(c.) find the set of all $n \times n$ square matrices with entries in $\mathbb{R}$ which commute with all other $n \times n$ square matrices.