MATH 321: ON MATRIX NOTATION, ALGEBRA AND SYSTEMS OF EQUATIONS — MISSION 1

Please follow the format which was announced in Blackboard. Thanks!
Your PRINTED NAME indicates you have read through Chapters 1 and 2 of my notes: ________________.

Problem 1 Let $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$. Calculate the following if possible (otherwise, explain why it can't be done)

(a.) $A + A^2$

(b.) $B + B^2$

(c.) $A + BB^T$

(d.) $A + B^T B$

Problem 2 Let $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$. Prove $A^n = \begin{bmatrix} 3^n & 3^n - 2^n \\ 0 & 2^n \end{bmatrix}$ for all $n \in \mathbb{N}$.

Problem 3 A standard notation is that $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ in context. We could write $I = I_2$ to emphasize the size if needed. Calculate $I(a, b)$ and $[a, b]I$.

Problem 4 Let $M = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and $N = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}$ where $a, b, x, y \in \mathbb{R}$.

(a.) Calculate MN. Relate your result to the product of the complex numbers $(a + ib)(x + iy)$.

(b.) Calculate M^{-1} given that at least one of a or b is nonzero. Relate your result to cartesian form of $1/(a + ib)$.

Problem 5 Let $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$. Choose appropriate sized standard basis and $\wedge$ matrix units for the following calculations to make sense and calculate:

square

(a.) Ae_2

(b.) $e_1^T A$

(c.) AE_{12}

(d.) $E_{12}A$

Problem 6 Suppose A is a symmetric matrix and B is nilpotent of order 3. Let $M = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$. Is M symmetric? Is M^2 symmetric? Is M^3 symmetric. Comment on M^k, for which k can we be certain M^k a symmetric matrix?

Problem 7 Let A, B, N, S be invertible square matrices. Solve the following equation for X:

$$(BA)^{-1} X S^{-1} = (NA)^2$$

Problem 8 Prove Proposition 2.2.6 part (2.) in my notes.

Problem 9 Prove the column-by-column multiplication rule for $A \in R^{m\times n}$ and $B \in R^{n\times p}$. That is, if $B = [B_1|B_2|\cdots|B_p]$ then show

$$AB = [AB_1|AB_2|\cdots|AB_p].$$

Problem 10 Let A, B be matrices over R which can be multiplied. Prove $(AB)^T = B^T A^T$.

Problem 11 For each system below, find a matrix A and a vector of variables v such that the systems below are equivalent to $Av = b$. Also, state the augmented coefficient matrix for each system.

a.) $x + y = 2, \quad x - y = 1.$

b.) $a - b = 1 + c, \quad b = 3 - c.$

c.) $x_1 + x_2 = 1, \quad x_2 + x_3 = 0, \quad x_3 + x_4 = 1, \quad x_4 + x_5 = 0.$

Problem 12 Solve each system in the previous problem and write the real solution **set** with proper **set** notation. Feel free to use any reasonable method to find the solution.

Problem 13 Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Define $X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$. Order variables in the order x, y, z, w and find the augmented coefficient matrix for the linear system $AX + XB = I$.

I realize some of you have not taken Math 231, but, that is no reason to avoid questions of geometry! If anything, that is a reason we should do more of it in here. Towards that end, here are two geometry problems which I'd like you to solve with algebra. Do not take a geometric approach to the following two problems. Thanks!

Problem 14 The equation of a plane has the form $Ax + By + Cz = D$ when the normal to the plane is $\langle A, B, C\rangle$. Suppose the points $(1, 2, 2), (2, 3, 3)$ and $(0, 0, 2)$ are on a given plane. Find the possible normals for the given plane.

Problem 15 Consider the points $(1, 2), (3, 4), (8, 6)$. Show these points are not on a line. We define a **line** in the plane to be the solution set of $Ax + By = C$ where $\underbrace{A, B, C}$ are constants.

A, B not both zero.

Problem 16 Let $A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 3 & 1 & 1 & 0 \\ 0 & 4 & 4 & 0 & 0 \end{bmatrix}$. Use row reduction to calculate rref(A).

Problem 17 Let $B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 2 \\ 3 & 5 & 0 \end{bmatrix}$. Use row reduction to calculate rref(B).

Problem 18 Let $M = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & -1 \end{bmatrix}$. Calculate rref$(M)$ and rref(M^T) via row reduction.

MATH 321 : MISSION 1 SOLUTION

P1 $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}$

(a.) $A + A^2 = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 9 & 9 \\ 0 & 4 \end{bmatrix} = \boxed{\begin{bmatrix} 12 & 10 \\ 0 & 6 \end{bmatrix}}$

(b.) $B + B^2$ does not make sense since B^2 is not defined. B is 2×3 and $(2 \times 3)(2 \times 3)$ multiplication is not defined (in our cource ☺)

(c.) $A + BB^T = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 3 \\ 3 & 9 \end{bmatrix} = \boxed{\begin{bmatrix} 6 & 4 \\ 3 & 11 \end{bmatrix}}$

(d.) $A + B^T B$ is not calculable as A is 2×2 whereas $B^T B$ is $(3\times2)(2\times3) : 3\times3$. Can't add matrices with different dimensions.

P2 when asked to prove something $\forall n \in \mathbb{N}$ it is often the case induction is the right proof technique.

Let $A = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}$ and notice $A^1 = \begin{bmatrix} 3^1 & 3^1 - 2^1 \\ 0 & 2^1 \end{bmatrix} = \begin{bmatrix} 3^n & 3^n - 2^n \\ 0 & 2^n \end{bmatrix}$ for $n = 1$. Suppose that $A^n = \begin{bmatrix} 3^n & 3^n - 2^n \\ 0 & 2^n \end{bmatrix}$ for some $n \in \mathbb{N}$.

Consider

$$A^{n+1} = AA^n = \begin{bmatrix} 3 & 1 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 3^n & 3^n - 2^n \\ 0 & 2^n \end{bmatrix} \quad : \text{by induction hypothesis.}$$

$$= \begin{bmatrix} 3^{n+1} & 3(3^n - 2^n) + 2^n \\ 0 & 2^{n+1} \end{bmatrix}$$

$$= \begin{bmatrix} 3^{n+1} & 3^{n+1} - 2 \cdot 2^n \\ 0 & 2^{n+1} \end{bmatrix}$$

$$= \begin{bmatrix} 3^{n+1} & 3^{n+1} - 2^{n+1} \\ 0 & 2^{n+1} \end{bmatrix}.$$

Thus $A^n = \begin{bmatrix} 3^n & 3^n - 2^n \\ 0 & 2^n \end{bmatrix}$ $\forall n \in \mathbb{N}$ by induction on n.

$\boxed{P3}$ $I(a,b) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} a \\ b \end{bmatrix} = (a,b)$

$$[a,b]\, I = [a,b]\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = [a\cdot 1 + b\cdot 0,\ a\cdot 0 + b\cdot 1] = [a,b].$$

$\boxed{P4}$ Let $M = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ and $N = \begin{bmatrix} x & -y \\ y & x \end{bmatrix}$ for $a,b,x,y \in \mathbb{R}$.

(a.) $$MN = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}\begin{bmatrix} x & -y \\ y & x \end{bmatrix}$$
$$= \begin{bmatrix} ax-by & -ay-bx \\ bx+ay & -by+ax \end{bmatrix}$$
$$= \begin{bmatrix} ax-by & -(bx+ay) \\ bx+ay & ax-by \end{bmatrix}.$$

If $M = \mathrm{Mat}(a+ib)$ then $N = \mathrm{Mat}(x+iy)$
and since $(a+ib)(x+iy) = ax-by+i(bx+ay)$
we've shown $\mathrm{Mat}(a+ib)\,\mathrm{Mat}(x+iy) = \mathrm{Mat}((a+ib)(x+iy))$.

(b.) we learned $\underline{\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}}^{*}$ provided $ad-bc \neq 0$.
Applying the above formula to $\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ we find,

$$M^{-1} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}^{-1} = \frac{1}{a^2+b^2}\begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \mathrm{Mat}\left(\frac{1}{a+ib}\right)$$

since
$$\frac{1}{a+ib} = \frac{a-ib}{(a+ib)(a-ib)} = \frac{a-ib}{a^2+b^2}$$

<u>Remark</u>: another way to look at this is that we used $*$ to derive the formula for $\frac{1}{a+ib}$.

P5 $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix}$

(a.) $Ae_2 = \begin{bmatrix} 2 \\ 6 \end{bmatrix}$ where $e_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

(b.) $e_1^T A = [1 \;\; 2 \;\; 3 \;\; 4]$ where $e_1^T = [1, 0, 0, 0]$.

(c.) $\underset{2\times 4}{A}\;\underset{4\times 4}{E_{12}} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 5 & 0 & 0 \end{bmatrix}$

could reasonably put $4 \times m$ for any m, I made announcement to assume square E_{12}.

Likewise 2 would do E_{12} is $n \times 2$, but I added square comment.

(d.) $E_{12} A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \end{bmatrix} = \begin{bmatrix} 5 & 6 & 7 & 8 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

__Remark__: AE_{12} takes column 1 of A and moves it to column 2 while zeroing all other columns. In contrast, $E_{12}A$ takes row 2 to row 1 and zeroes the remaing rows. If E_{ij} is not square then $E_{ij}A$ and AE_{ij} would differ in size from A.

P6 Given $A^T = A$ and $B, B^2 \neq 0$ yet $B^3 = 0$ we let $M = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ hence $M^2 = \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = \begin{bmatrix} A^2 & 0 \\ 0 & B^2 \end{bmatrix}$ by block multiplication. Likewise, $M^3 = \begin{bmatrix} A^3 & 0 \\ 0 & B^3 \end{bmatrix}$.

Consider $B = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ then $B^2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $B^3 = 0$

yet $B^T = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \neq B$ and $(B^2)^T = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \neq B^2$ thus

$$M^T = \begin{bmatrix} A^T & 0 \\ 0 & B^T \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & B^T \end{bmatrix} \neq M \text{ and } (M^2)^T = \begin{bmatrix} (A^2)^T & 0 \\ 0 & (B^2)^T \end{bmatrix} \neq M^2$$

shows M and M^2 need not be symmetric. However, P6 continued

If $k \geq 3$ then $B^k = 0$ by assumption and $(A^k)^T = (AA\cdots A)^T = A^T A^T \cdots A^T = A\cdot A \cdots A = A^k$ (the k^{th} power of a symmetric matrix is symmetric)

Thus, once more using block notation,

$$(M^k)^T = \begin{bmatrix} A^k & 0 \\ 0 & B^k \end{bmatrix}^T = \begin{bmatrix} (A^k)^T & 0 \\ 0 & (B^k)^T \end{bmatrix} = \begin{bmatrix} A^k & 0 \\ 0 & 0 \end{bmatrix}$$

and $M^k = \begin{bmatrix} A^k & 0 \\ 0 & B^k \end{bmatrix} = \begin{bmatrix} A^k & 0 \\ 0 & 0 \end{bmatrix}$ thus $(M^k)^T = M^k$

for $k = 3, 4, 5, \ldots$.

In summary, we cannot be sure M, M^2 are symmetric since we've shown a counter-example. In contrast, M^k is symmetric for $k \geq 3$ by the general calculation given above.

Remark: to be careful, some authors would insist on induction to prove $(A^k)^T = (A^T)^k \ \forall k \in \mathbb{N}$. I think you could provide such proof is asked, it's almost the same as what I gave here with the ... notation.

P7 A, B, N, S invertible square matrices. Solve for X,

$$(BA)^{-1} X S^{-1} = (NA)^2$$

$$\Rightarrow (BA)(BA)^{-1} X S^{-1} S = BA(NA)^2 S$$

$$\Rightarrow I X I = BA(NA)^2 S$$

$$\therefore \boxed{X = BANANAS}$$

P8 Prove Prop. 2.2.6 part (2) from my notes.

Let $A, B \in R^{m\times n}$ where R is commutative ring with unity. Suppose $1 \leq i \leq m$ and $1 \leq j \leq n$,

$$\begin{aligned}
(A+B)_{ij} &= A_{ij} + B_{ij} && : \text{def}^{\underline{n}} \text{ of } + \text{ for matrices.} \\
&= B_{ij} + A_{ij} && : \text{since } R \text{ is commutative ring with } A_{ij}, B_{ij} \in R. \\
&= (B+A)_{ij} && : \text{def}^{\underline{n}} \text{ of } + \text{ for matrices.}
\end{aligned}$$

Thus $A + B = B + A$ as we've shown <u>all</u> their components align.

<u>Remark</u>: to show $M = N$ we have several options.

1.) show $M_{ij} = N_{ij}$ $\forall i, j$ (our choice for P8 argument)

2.) show $\text{col}_j(M) = \text{col}_j(N)$ for each j.

3.) show $\text{row}_i(M) = \text{row}_i(N)$ for each i.

There are other methods, but usually one of the above suffices. (ignoring larger theoretical techniques... later)

$\boxed{P9}$ Let $A \in R^{m \times n}$ and $B \in R^{n \times p}$ where R is commutative ring with unity. Consider,

$$
\begin{aligned}
(AB)_{ij} &= \sum_{k=1}^{n} A_{ik} B_{kj} \\
&= \sum_{k=1}^{n} A_{ik} \left(\mathrm{col}_j(B)\right)_k \\
&= (AB_j)_i \\
&= [\,AB_1 \,|\, AB_2 \,|\cdots|\, AB_p\,]_{ij}
\end{aligned}
$$

Thus $AB = [AB_1 | AB_2 | \cdots | AB_p]$.

Alternatively,

$$(AB)_{ij} = \mathrm{row}_i(A) \cdot \mathrm{col}_j(B)$$

$$[AB_1|\cdots|AB_p]_{ij} = (AB_j)_i = \left[\begin{array}{c} \mathrm{row}_1(A)\cdot B_j \\ \hline \mathrm{row}_2(A)\cdot B_j \\ \hline \vdots \\ \mathrm{row}_m(A)\cdot B_j \end{array}\right]_i = \mathrm{row}_i(A)\cdot \mathrm{col}_j(B).$$

There are many ways to convincingly argue this important identity.

$\boxed{P10}$ Let $A \in R^{m \times n}$, $B \in R^{n \times p}$ for R a commutative ring. Consider, for $1 \leq i \leq m$ and $1 \leq j \leq p$

$$
\begin{aligned}
\left((AB)^T\right)_{ij} &= (AB)_{ji} && : \text{ def}^{\underline{n}} \text{ of transpose} \\
&= \sum_{k=1}^{n} A_{jk} B_{ki} && : \text{ def}^{\underline{n}} \text{ of } AB. \\
&= \sum_{k=1}^{n} B_{ki} A_{jk} && : \ A_{jk}, B_{ki} \in R \text{ and thus commute.} \\
&= \sum_{k=1}^{n} (B^T)_{ik} (A^T)_{kj} && : \text{ def}^{\underline{n}} \text{ of transpose.}
\end{aligned}
$$

Thus $(AB)^T = B^T A^T$.

$\boxed{P11}$ (the answers below are not unique due to labling ambiguity)

(a.)
$$
\begin{aligned} x+y &= 2 \\ x-y &= 1 \end{aligned}
\implies
\underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{A}
\underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{v}
=
\underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{b}
$$

the augmented coefficient matrix $\underline{[A|b] = \left[\begin{array}{cc|c} 1 & 1 & 2 \\ 1 & -1 & 1 \end{array}\right]}$.

(b.)
$$
\begin{aligned} a-b-c &= 1 \\ b+c &= 3 \end{aligned}
\implies
\underbrace{\begin{bmatrix} 1 & -1 & -1 \\ 0 & 1 & 1 \end{bmatrix}}_{A}
\underbrace{\begin{bmatrix} a \\ b \\ c \end{bmatrix}}_{v}
=
\underbrace{\begin{bmatrix} 1 \\ 3 \end{bmatrix}}_{b}
$$

$\therefore$ aug. coeff. matrix is $\underline{\left[\begin{array}{ccc|c} 1 & -1 & -1 & 1 \\ 0 & 1 & 1 & 3 \end{array}\right]}$.

(c.)
$$
\left.\begin{aligned} x_1+x_2 &= 1 \\ x_2+x_3 &= 0 \\ x_3+x_4 &= 1 \\ x_4+x_5 &= 0 \end{aligned}\right\}
\underbrace{\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}}_{A}
\underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}}_{v}
=
\underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}}_{b}
\quad \therefore \quad
\underbrace{[A|b] = \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]}_{\text{aug. coeff. matrix.}}
$$

P 12

(a.)
$$\begin{array}{r} x+y=2 \\ +\;(x-y=1) \\ \hline 2x=3 \end{array}$$
$\therefore\ x = 3/2$ and $y = 2 - x = 2 - \frac{3}{2} = \frac{1}{2}$

$\therefore$ $\boxed{\text{Sol}^{n}\text{ Set} = \{(3/2, 1/2)\}}$

(b.)
$$\begin{array}{r} a-b-c=1 \\ +\;(b+c=3) \\ \hline \underline{a=4} \end{array}$$
(bothe eq^{n}'s then say $b+c=3$ essentially) so, $b = 3-c$ and c is free.)

$\boxed{\text{Sol}^{n}\text{ Set} = \{(4, 3-c, c) \mid c \in \mathbb{R}\}}$ admittedly, could put $\mathbb{Q}$, $\mathbb{C}$ or many things here. So I made announcement to say "real" sol^{n}s.

(c.)
$$\underbrace{\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & -1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right] \sim \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & 2 \\ 0 & 1 & 0 & 0 & 1 & -1 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \end{array}\right]}$$

Gaussian Elimination, actually slow compared to

Thus translating back to eq^{n}s,

$x_1 - x_5 = 2 \quad \therefore \quad x_1 = 2 + x_5$

$x_2 + x_5 = -1 \quad \therefore \quad x_2 = -x_5 - 1$

$x_3 - x_5 = 1 \quad \therefore \quad x_3 = 1 + x_5$

$x_4 + x_5 = 0 \quad \therefore \quad x_4 = -x_5$

---

alternatively,

$x_4 + x_5 = 0 \Rightarrow x_4 = -x_5$

$x_3 + x_4 = 1 \Rightarrow x_3 = 1 - x_4 = 1 + x_5$

$x_2 + x_3 = 0 \Rightarrow x_2 = -x_3 = -1 - x_5$

$x_1 + x_2 = 1 \Rightarrow x_1 = 1 - x_2 = 1 + 1 + x_5$

back-substitution is probably fastest here.

Thus,

$\boxed{\text{Sol}^{n}\text{ Set} = \{(2+x_5, -1-x_5, 1+x_5, -x_5, x_5) \mid x_5 \in \mathbb{R}\}}$

$\boxed{P13}$ $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, $X = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$

$$AX + XB = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} x & y \\ z & w \end{bmatrix} + \begin{bmatrix} x & y \\ z & w \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{c|c} x + 2z & y + 2w \\ \hline 3y + 4w & 3y + 4w \end{array}\right] + \begin{bmatrix} y & x \\ w & z \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\left[\begin{array}{c|c} x + y + 2z & x + y + 2w \\ \hline 3y + 5w & 3y + z + 4w \end{array}\right] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 & 2 & 0 \\ 1 & 1 & 0 & 2 \\ 0 & 3 & 0 & 5 \\ 0 & 3 & 1 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \Rightarrow \underbrace{\boxed{\left[\begin{array}{cccc|c} 1 & 1 & 2 & 0 & 1 \\ 1 & 1 & 0 & 2 & 0 \\ 0 & 3 & 0 & 5 & 0 \\ 0 & 3 & 1 & 4 & 1 \end{array}\right]}}_{\text{aug. coeff. matrix}}$$

–(not unique, could order eq$^{\underline{n}}$'s differently)–

$\boxed{P14}$ For $(1,2,2), (2,3,3)$ and $(0,0,2)$ to be on the plane with normal $\langle A, B, C\rangle$ we need these to solve $Ax + By + Cz = D$ for some D.

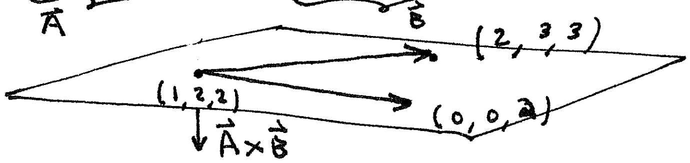
$$\left.\begin{aligned} A + 2B + 2C &= D \\ 2A + 3B + 3C &= D \\ 2C &= D \end{aligned}\right] \longrightarrow A + 2B + D = D \quad \therefore \quad \underline{A = -2B}.$$

Then $2A + 3B + 3C = 2C \Rightarrow C = -2A - 3B = 4B - 3B = \underline{B}$.

Thus $\langle A, B, C\rangle = \langle -2B, B, B\rangle$ serves as a normal for any choice of $B \in \mathbb{R}$ with $B \neq 0$.

<u>Remark:</u> calculus III typical sol$^{\underline{n}}$

$$\underbrace{\langle 1, 1, 1\rangle}_{\vec{A}} \times \underbrace{\langle -1, -2, 0\rangle}_{\vec{B}} = \langle 2, -1, -1\rangle \Rightarrow$$ any scalar multiple of this serves as normal



P15 Consider pts. $(1,2), (3,4), (8,6)$ if these are on line with eq^{n} $Ax + By = C$ then,

$$\left.\begin{aligned} A + 2B &= C \\ 3A + 4B &= C \\ 8A + 6B &= C \end{aligned}\right\} \longrightarrow$$

Solving both for C yields

$$3A + 4B = A + 2B$$
$$\Rightarrow 2A = -2B$$
$$\Rightarrow \underline{A = -B}.$$

But, $3A + 4B = -3B + 4B = B = C$

whereas $8A + 6B = -8B + 6B = -2B = C \Rightarrow B = -2B$

thus $B = 0$ and hence $A = -0 = 0$. Also, $B = C = 0$.

But, $A = B = C = 0 \Rightarrow Ax + By = C$ is not the equation of a line. (it's just $0 = 0$ which is true but not especially limiting for (x, y)).

P16

$$A = \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 2 & 3 & 1 & 1 & 0 \\ 0 & 4 & 4 & 0 & 0 \end{bmatrix} \xrightarrow[\frac{1}{4}r_3]{r_2 - 2r_1} \begin{bmatrix} 1 & 2 & 0 & 1 & -1 \\ 0 & -1 & 1 & -1 & 2 \\ 0 & 1 & 1 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow[r_3 + r_2]{r_1 + 2r_2} \begin{bmatrix} 1 & 0 & 2 & -1 & 3 \\ 0 & -1 & 1 & -1 & 2 \\ 0 & 0 & 2 & -1 & 2 \end{bmatrix} \xrightarrow{2r_2} \begin{bmatrix} 1 & 0 & 2 & -1 & 3 \\ 0 & -2 & 2 & -2 & 4 \\ 0 & 0 & 2 & -1 & 2 \end{bmatrix} \xrightarrow[r_2 - r_3]{r_1 - r_3} \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & -1 & 2 \\ 0 & 0 & 2 & -1 & 2 \end{bmatrix}$$

$$\xrightarrow[r_3/2]{r_2/-2} \boxed{\begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1/2 & -1 \\ 0 & 0 & 1 & -1/2 & 1 \end{bmatrix} = \text{rref}(A)}$$

P17

$$B = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 2 \\ 3 & 5 & 0 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 3 & 2 \\ 3 & 5 & 0 \end{bmatrix} \xrightarrow{r_4 - 3r_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 3 & 2 \\ 0 & 2 & -3 \end{bmatrix}$$

$$\begin{matrix} \xrightarrow{6r_1} \\ \xrightarrow{2r_3} \\ \xrightarrow{3r_4} \end{matrix} \begin{bmatrix} 6 & 6 & 6 \\ 0 & 0 & 1 \\ 0 & 6 & 4 \\ 0 & 6 & -9 \end{bmatrix} \begin{matrix} \xrightarrow{r_1 - r_3} \\ \\ \xrightarrow{r_4 - r_3} \end{matrix} \begin{bmatrix} 6 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 6 & 4 \\ 0 & 0 & -13 \end{bmatrix} \begin{matrix} \xrightarrow{r_1 - 2r_2} \\ \xrightarrow{r_3 - 4r_2} \\ \xrightarrow{r_4 + 13r_2} \end{matrix} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 6 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{r_2 \leftrightarrow r_3} \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{matrix} \xrightarrow{r_1/6} \\ \xrightarrow{r_2/6} \end{matrix} \boxed{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{rref}(B)}$$

P18

$$M = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 3 & 2 \\ 1 & 0 & -1 \end{bmatrix} \begin{matrix} \xrightarrow{r_2 - r_1} \\ \xrightarrow{r_3 - r_1} \end{matrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 1 \\ 0 & -2 & -2 \end{bmatrix} \begin{matrix} \xrightarrow{r_1 - 2r_2} \\ \xrightarrow{r_3 + 2r_2} \end{matrix} \boxed{\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{rref}(M)}$$

$$M^T = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 0 \\ 1 & 2 & -1 \end{bmatrix} \begin{matrix} \xrightarrow{r_2 - 2r_1} \\ \xrightarrow{r_3 - r_1} \end{matrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & -2 \end{bmatrix} \begin{matrix} \xrightarrow{r_1 - r_2} \\ \xrightarrow{r_3 - r_2} \end{matrix} \boxed{\begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} = \operatorname{rref}(M^T)}$$

Remark: I can see $3\,\text{row}_1(M) - 2\,\text{row}_2(M) = \text{row}_3(M)$ whereas $-\text{col}_1(M) + \text{col}_2(M) = \text{col}_3(M)$ from the calculations above. (you should learn how later...)